Assignment 6

Introduction to Linear Algebra

Material from Lectures 21 - 23

Due Friday, December 3, 2021

(Exercise 21.2 in the lecture notes)

1. Consider the function $f \in C[0, 2\pi]$ given by $f(x) = \left\{ \begin{smallmatrix} -1 & x<\pi \\ 1 & x\geqslant\pi \end{smallmatrix} \right.$.

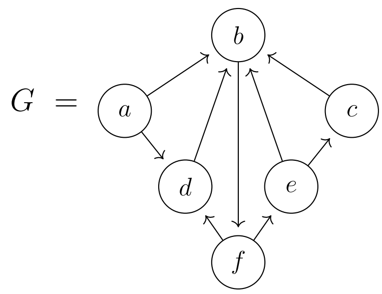
   (a) Compute the Fourier series of f up to $n = 1$, $n = 3$, and $n = 5$. Plot these three functions together with f.

   (b) Compute the discrete Fourier transform of f for $n = 4$, using evenly spaced samples $f(x_k)$ for $x_k = 2k\pi/4$, with $k = 0, 1, 2, 3$. Express it as a sum of sin and cos functions using Euler's formula.

   (c) Plot the real part of the discrete Fourier transform of f for $n = 4, 8, 12$ together with f. As above, take $4, 8, 12$ evenly spaced samples in the interval $[0, 2\pi]$, starting with 0. You do not need to show your computations.

(Exercise 22.3 in the lecture notes)

2. Consider the following directed graph:



   (a) Give the adjacency and incidence matrix for G.

   (b) Find all $k \in \mathbf{N}$ for which there are no walks of length k from f to f.

   (c) Find as many spanning trees as you can for G.

The preferred (but not required) order of the edges is the lexicographic order: $ab, ad, bf, cb, db, eb, ec, fd, fe$.

(Exercise 23.1 in the lecture notes)

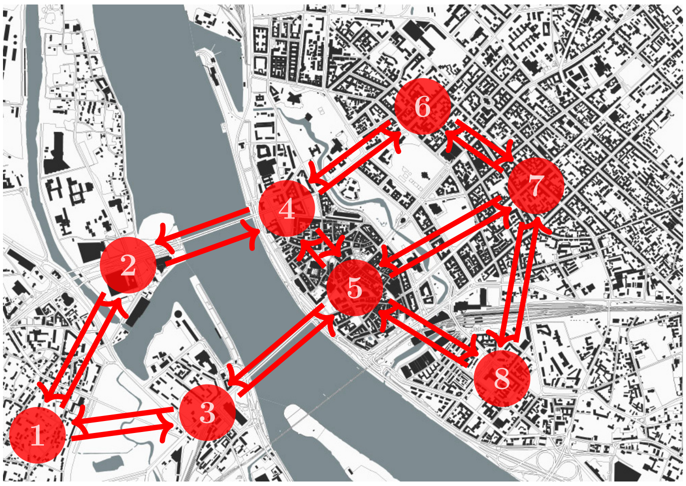
3. Let $M \in \mathcal{M}_{2\times 2}$.

   (a) Let $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ be Markov (right stochastic) with $d = 0$. Show that if M^3 has only positive entries, then M^2 has only positive entries.

   (b) Find and example of M (not Markov, but with $|M_{ij}| \leqslant 1$) for which M^3 has only positive entries, but M has at least one negative entry.

(Exercise 23.2 in the lecture notes)

4. Consider the map of Riga below, with vertices as marked.



   (a) Construct the transition probability matrix R for Riga.

   (b) Will the matrix have a steady state? If not, explain why. If yes, compute it.

   (c) Find the Fiedler eigenvector and give the clustering according to it.

You may use a computer to help solve Questions 1, 2(b), and 4.