Assignment 5

Introduction to Linear Algebra

Material from Lectures 17 - 20

Due Wednesday, November 17, 2021

Questions 1 and 2 are about data points in *general position*, which is a simple condition to ensure that the data is not degenerate. A set of k points in $\mathbf{R}^n$ is in general position if every subset of size $n+1$ does not lie in a hyperplane (a codimension 1 space). For example,

- in $\mathbf{R}^2$, it means that no 3 points lie on a line,
- in $\mathbf{R}^3$, it means that no 4 points lie on a plane.

On s set of $n+1$ points $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbf{R}^n$ this condition can be checked by looking for solutions to the equation

$$\begin{bmatrix} | & | & & | \\ \mathbf{v}_1 - \mathbf{v}_0 & \mathbf{v}_2 - \mathbf{v}_0 & \cdots & \mathbf{v}_n - \mathbf{v}_0 \\ | & | & & | \end{bmatrix} \mathbf{x} = 0.$$

If there is a nonzero solution $\mathbf{x}$ to this equation, the points are not in general position. If the only solution is $\mathbf{x} = 0$, then they are in general position. This method is provided as a guaranteed way of having points in general position. For Questions 1 and 2, you may justify points being in general position by drawing a picture.

(Exercise 17.2 in the lecture notes)

1. Create a matrix of 2-dimensional data for which the first principal component of the data is a multiple of the eigenvector $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, for $a, b \in \mathbf{R}_{\neq 0}$. Make sure that:
   - the matrix has at least 3 columns (samples),
   - no 3 samples are colinear.

(Exercise 17.3 in the lecture notes)

2. (a) Create a matrix of 3-dimensional data for which first two principal components are the vectors $\left[\begin{smallmatrix} 1 & 0 & 0 \end{smallmatrix}\right]^T$ and $\left[\begin{smallmatrix} 0 & 1 & 0 \end{smallmatrix}\right]^T$. Make sure that:
   - the data is centered at 0,
   - the matrix has at least 4 columns (samples),
   - no 3 samples are colinear.

   (b) Do the same as in part (a), but change the last condition to "no 4 samples lie on a plane."

(Exercise 18.1 in the lecture notes)

3. Consider the following transformations T_i:

$$T_1 \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} w \\ y \\ z \\ x \end{bmatrix} \qquad T_2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2e^y \\ x \end{bmatrix} \quad T_3 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 \\ y^2 \end{bmatrix} \qquad T_4 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \sin(x^2+y^2) \\ \cos(x^2+y^2) \end{bmatrix}$$

$$T_5 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3y+x \\ 0 \\ x^2-y \end{bmatrix} \quad T_6 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad T_7 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3x \\ z+y \end{bmatrix} \quad T_8 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x+2y \\ y+z \\ 0 \end{bmatrix}$$

   (a) Which of the T_i are linear? For those that are not, give a counterexample in which one of the linearity conditions fail. For those that are, give the associated matrix.

   (b) Let $S\colon \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation for which

$$ST_5 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad ST_8 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad ST_8 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

   Construct the 3×3 matrix of S.

(Exercise 19.2 in the lecture notes)

4. Let $J \in \mathcal{M}_{6\times 6}$ be a matrix in Jordan form with two eigenvalues 3 (having algebraic multiplicity 4 and geometric multiplicity 2) and -3 (having algebraic multiplicity 2 and geometric multiplicity 1).

   (a) How many Jordan blocks will J have? Give the two possibilities for their sizes.

   (b) Suppose that the Jordan blocks of J all have the same size. Find a matrix B that is similar to J and has no zero entries.

   (c) For the matrix B from part (b), find all its generalized eigenvectors.

(Exercise 20.2 in the lecture notes)

5. Prove all the claims of Proposition 20.4, for $z = x + yi, w = a + bi \in \mathbf{C}$:

   (a) $\overline{z+w} = \overline{z} + \overline{w}$

   (b) $\overline{zw} = \overline{z}\overline{w}$

   (c) $\overline{\overline{z}} = z$

   (d) $z + \overline{z} = 2x$

   (e) $z - \overline{z} = 2yi$

   (f) $z^{-1} = \overline{z}/|z|^2$ for $z \neq 0$

   (g) $|z| = 0$ iff $z = 0$

   (h) $|\overline{z}| = |z|$

   (i) $|zw| = |z||w|$

   (j) $|z + w| \leqslant |z| + |w|$