Assignment 4

Introduction to Linear Algebra

Material from Lectures 13 - 16

Due Friday, November 5, 2021

(Exercise 13.1 in the lecture notes)

1. Consider the matrix $A = \begin{bmatrix} 6 & -5 \\ 5 & -2 \end{bmatrix}$.

   (a) Find the eigenvalues and eigenvectors of A. Be careful, there may be complex numbers!

   (b) If $\mathbf{v}_1$ and $\mathbf{v}_2$ are the eigenvectors, compute the dot product $\mathbf{v}_1 \cdot \mathbf{v}_2$. Is it a complex or a real number?

(Exercise 13.2 in the lecture notes)

2. Consider the values $\lambda_1 = -3$, $\lambda_2 = -2$, $\lambda_3 = 5$.

   (a) Construct two different 3×3 matrices with $\lambda_1, \lambda_2, \lambda_3$ as eigenvalues.

   (b) What are the eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ of the two matrices you created in part (a)?

   (c) If $\lambda_3 = -2$, explain why every linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$ is an eigenvector.

(Exercise 14.1 in the lecture notes)

3. Decopose both matrices below in their $X\Lambda X^{-1}$-decomposition, where Λ is a diagonal matrix with the eigenvalues, and X is the matrix with columns as eigenvectors.

$$A = \begin{bmatrix} 2 & 2 \\ 5 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

(Exercise 14.2 in the lecture notes)

4. Let $A \in \mathcal{M}_{3\times 3}$ with the eigenvectors $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$ and eigenvalues $-1, 2, -3$, respectively.

   (a) Construct the eigenvector matrix X and the eigenvalues matrix Λ.

   (b) Construct A by the diagonalization equation $A = X\Lambda X^{-1}$.

(Exercise 15.1 in the lecture notes)

5. Let $a \in \mathbf{R}$ be nonzero.

   (a) Find the eigenvalues of $\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$.

   (b) Find the eigenvalues of $\begin{bmatrix} 0 & 0 & a \\ 0 & ia & 0 \\ -a & 0 & 0 \end{bmatrix}$.

   (c) Using a, construct a 4×4 skew-symmetric matrix that has all imaginary eigenvalues.

   (d) Construct a 3×3 symmetric matrix that has three pivots a and no zero entries.

(Exercise 15.3 in the lecture notes)

6. Consider the two symmetric matrices below, for $a, b \in \mathbf{R}$:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} b & 2 & 0 \\ 2 & b & 3 \\ 0 & 3 & b \end{bmatrix}.$$

   (a) Find the pivots for both matrices. For what values of a, b will the pivots be positive?

   (b) Find the eigenvalues for both matrices. For what values of a, b will the eigenvalues be positive?

   (c) Find the upper left determinants for both matrices. For what values of a, b will the determinants be positive?

   (d) Choose some a, b so that pivots, eigenvalues, determinants are positive. Find the $Q\Lambda Q^T$-decomposition for both matrices.

(Exercise 16.2 in the lecture notes)

7. Let $a \in \mathbf{R}_{\neq 0}$, and consider the matrix

$$A = \begin{bmatrix} a & 0 & a & 0 \\ 0 & 0 & 0 & 2a \end{bmatrix}.$$

   (a) Compute the SVD of A by finding the eigenvalue / eigenvector pairs for AA^T and A^TA.

   (b) What are the dimensions of the four fundamental subspaces of A?

(Exercise 16.3 in the lecture notes)

8. (a) Construct a 3×4 matrix with singular values $1, 2, 3$.

   (b) Construct a 2×2 rank 1 matrix with right singular vectors $\begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}, \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix}$.

   (c) Find the rank 1 and rank 2 approximations for

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$