Assignment 3

Introduction to Linear Algebra

Material from Lectures 7 - 11
Due Friday, October 8, 2021

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1. Let U, V be subspaces of $\mathbf{R}^n$.
   (a) Show that $(U^\perp)^\perp = U$.
   (b) Show that $(U \cap V)^\perp = U^\perp + V^\perp$.
   (c) Suppose there exist matrices A, B with $U = \text{col}(A)$ and $V = \text{col}(B)$. Find a matrix C for which $\text{null}(C) = (U + V)^\perp$.
   Hint: construct C as a block matrix.

2. The set $U \subseteq \mathbf{R}^n$ is a subspace with basis $u_1, \ldots, u_k$. These basis vectors are the columns of the $n \times k$ matrix A. For any $\mathbf{v} \in \mathbf{R}^n$, define the *reflection* of $\mathbf{v}$ in U to be the vector

$$\text{refl}_U(\mathbf{v}) := \mathbf{v} - 2\text{proj}_{U^\perp}(\mathbf{v}).$$

   (a) Construct the matrix of refl_U.
   (b) Show that refl_U preserves length, that is, show that $\|\text{refl}_U(\mathbf{v})\| = \|\mathbf{v}\|$ for all $\mathbf{v} \in \mathbf{R}^n$.

3. The set $U \subseteq \mathbf{R}^n$ is a subspace with basis $u_1, \ldots, u_k$. These basis vectors are the columns of the $n \times k$ matrix A. Let $U \subseteq \mathbf{R}^n$ be a codimension 2 subspace, so that $\left[\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right]$ is a basis matrix for $\mathbf{R}^n$. For any $\mathbf{v} \in \mathbf{R}^n$, define the *rotation* by θ of $\mathbf{v}$ around U to be the vector

$$\text{rot}_U(\mathbf{v}, \theta) := \text{proj}_U(\mathbf{v}) + \begin{bmatrix} I & 0 \\ 0 & R_\theta \end{bmatrix} \text{proj}_{U^\perp}(\mathbf{v}),$$

   where $R_\theta = \left[\begin{smallmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{smallmatrix}\right]$ is the usual 2×2 rotation matrix.
   (a) Construct the matrix of rot_U.
   (b) Show that rot_U preserves length, that is, show that $\|\text{rot}_U(\mathbf{v}, \theta)\| = \|\mathbf{v}\|$ for all $\mathbf{v} \in \mathbf{R}^n$ and all $\theta \in [0, 2\pi]$.
   (c) **Bonus (1 point):** Define the rotation function without the assumption that $\left[\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right]$ is a basis matrix for $\mathbf{R}^n$.

4. Consider the following two planes, as subspaces of $\mathbf{R}^3$:

$$P_1 = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{R}^3 \ : \ 3x_1 - 4x_2 + x_3 = 0\},$$
$$P_2 = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{R}^3 \ : \ 5x_1 - 10x_3 = 0\}.$$

   (a) Find the normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ to the planes P_1 and P_2, respectively.
   (b) Find bases B_1 and B_2 for the planes P_1 and P_2, respectively.
   Hint: the basis of a plane is the nullspace of the defining equation.
   (c) Construct a 2×3 matrix A_1 whose row space is P_1. Show that the nullspace of A_1 is the span of $\mathbf{n}_1$.
   (d) Construct a 3×2 matrix A_2 whose column space is P_2. Show that the left nullspace of A_2 the span of $\mathbf{n}_2$.

5. Consider the set of six points $P = \{p_1, \ldots, p_6\} \subseteq \mathbf{R}^2$, with:

$$p_1 = (-1,3),\ p_2 = (4,6),\ p_3 = (3,1),\ p_4 = (-2,-3),\ p_5 = (6,-7),\ p_6 = (-6,4).$$

(a) Either using the projection matrix or partial derivatives, find the line $y = ax + b$ that is the least squares approximation to the points.

(b) Find a point $p_7 \in \mathbf{R}^2$ such that the least squares approximation to P is the same as to $P \cup \{p_7\}$.
Hint: Don't redo all your work! Use an observation from partial derivatives.

(c) **Bonus (1 point):** Let $c \in \mathbf{R}$. Find a point $p_8 \in \mathbf{R}^2$ such that the least squares approximation to $P \cup \{p_8\}$ has slope c.

6. Consider the following collection of four points $P = \{p_1, p_2, p_3, p_4\} \subseteq \mathbf{R}^3$:

$$p_1 = (1,-2,-4),\ p_2 = (0,5,5),\ p_3 = (-6,-7,2),\ p_4 = (1,4,-1).$$

(a) Generalize the least squares approach and find the closest plane H in $\mathbf{R}^3$ to the points in P (instead of the closest line in $\mathbf{R}^2$).

(b) Project the points in P onto the plane H from part (a).
Warning: The plane H will not go through the origin.

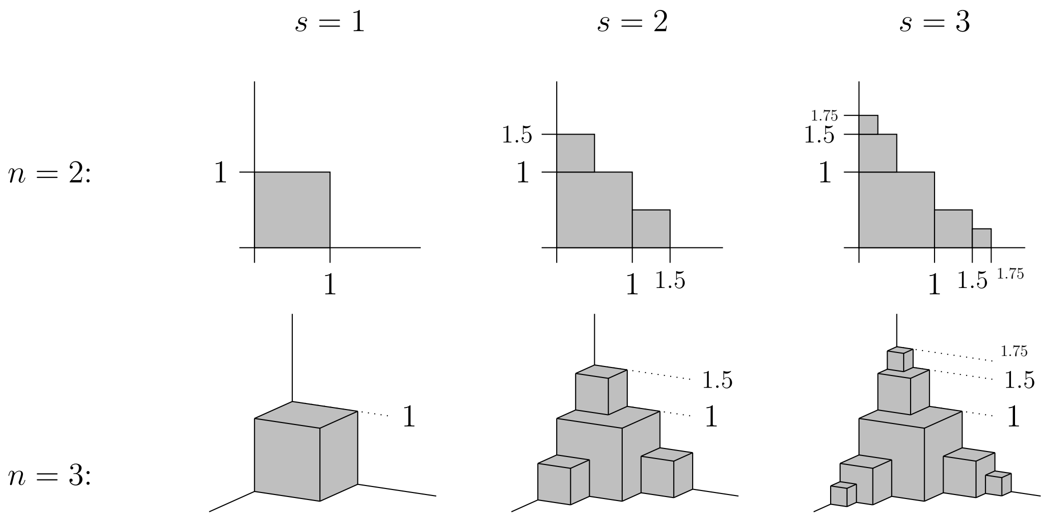
(c) **Bonus (2 points):** Find the least squares approximation line in H that approximates the points from part (b).

7. Consider $1, x, x^2, x^3$ as functions in $C[-1,1]$, the inner product space of continuous functions on $[-1,1]$, with inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\ dx$.

(a) Show that they are all linearly indpendent functions.

(b) Find an orthonormal basis for the subspace of $C[-1,1]$ that is the span of these four functions.

8. The *cube fractal* in dimension n has its first few steps given below.



(a) Express the volume of the n-cubes added in step $s > 1$ as a sum of determinants.
Hint: Use powers of 2.

(b) What is the total volume V_s of all the cubes after step s has been done?

(c) Compute the limit $\lim_{s\to\infty} V_s$.

(d) **Bonus (1 point):** Find the equation of the plane that intersects all the corners of cubes which do not have any zeros in their coordinates.

If you use row reduction for your solutions, you do not need to show the steps.