Assignment 2

Introduction to Linear Algebra

Material from Lectures 3 - 6
Due Friday, September 24, 2021

---

1. Consider the set X of all functions $f\colon \mathbf{R} \to \mathbf{R}$.

   (a) If addition on X is defined as usual, with $f + g = f(x) + g(x)$, but multiplication is defined as $cf = f(cx)$, show that X is not a vector space.

   (b) If multiplication is defined as usual, with $cf = cf(x)$, but addition is defined as $f + g = f(g(x))$, but show that X is not a vector space.

2. Let $A, B \in \mathcal{M}_{n\times n}$. Show that A and the block matrix $[A\ AB]$ have the same column space.

3. Let $A = \begin{bmatrix} 2 & 3 & 6 \\ 2/3 & 1 & 2 \\ 1 & -3/2 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -3 \\ -1 \\ -1/2 \end{bmatrix}$.

   (a) Find the complete solution to $A\mathbf{x} = \mathbf{b}$.

   (b) What is the rank of A?

   (c) Find a 3×1 vector $\mathbf{c}$ so that $A\mathbf{x} = \mathbf{c}$ does not have a solution.

4. (a) Find all values $a \in \mathbf{R}$ which make the set of vectors $\begin{bmatrix} a \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ a \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ a \end{bmatrix}$ linearly dependent in $\mathbf{R}^3$.

   (b) Explain why the set of vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ a \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ a \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is always linearly dependent in $\mathbf{R}^3$, for any $a \in \mathbf{R}$.

5. This question is about *vector spaces of matrices*, where matrix addition and scalar multiplication are defined as usual.

   (a) Give a basis for the space of diagonal 3×3 matrices and the space of skew-symmetric 3×3 matrices.

   (b) What is the dimension of the space of $n \times n$ diagonal matrices and the space of $n \times n$ skew-symmetric matrices?

   (c) Show by example that the set of all invertible 2×2 matrices does not form a vector space. Show that the span of all invertible 2×2 matrices is equal to $\mathcal{M}_{2\times 2}$.
   Hint: Construct the basis matrices of $\mathcal{M}_{2\times 2}$ as linear combinatons of invertible matrices.

6. Let $\mathbf{u} = \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$ and $\mathbf{v} = \left[\begin{smallmatrix} 3 \\ 4 \end{smallmatrix}\right]$.

   (a) Construct a 2×4 matrix A for which $\operatorname{col}(A) = \operatorname{span}(\{\mathbf{u}, \mathbf{v}\})$.

   (b) Find a basis for the column space and row space of $\mathbf{u}\mathbf{v}^T + (\mathbf{u}\mathbf{v}^T)^2$.

7. Find a basis for the column space, nullspace, row space, and left nullspace of

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$