Assignment 1

Introduction to Linear Algebra

Material from Lectures 1 and 2
Due Friday, September 10, 2021

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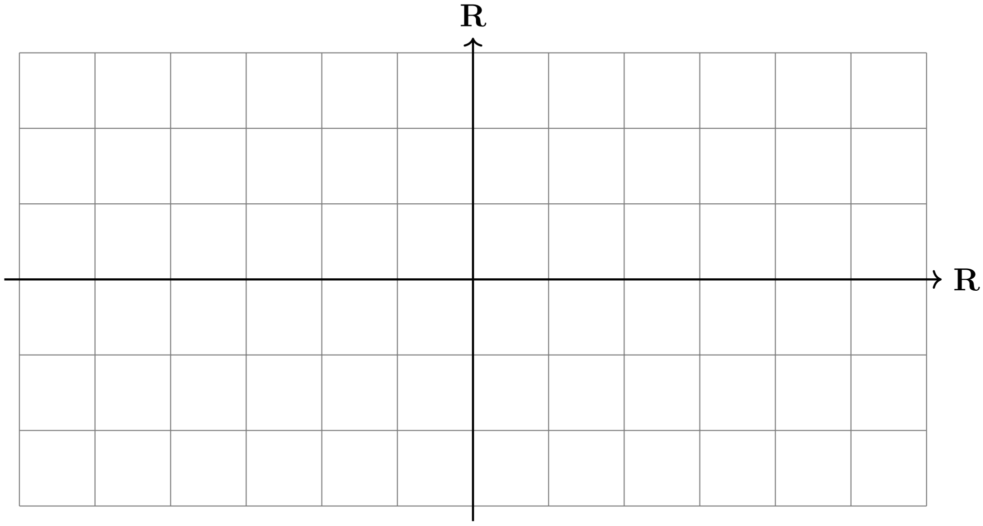
1. Consider the grid below, with squares of width and height 1.



   (a) Identify all the points that correspond to linear combinations of $a\left[\begin{smallmatrix}3\\0\end{smallmatrix}\right] + b\left[\begin{smallmatrix}-1\\2\end{smallmatrix}\right]$, for $a, b \in \mathbf{Z}$.

   (b) Which of the points from part (a) lie a distance of more than 2 but less than 3 from the origin?

2. The *unit n-cube* in $\mathbf{R}^n$ has corners $(v_1, v_2, \ldots, v_n)$, where $v_i \in \{0, 1\}$ for all i.

   (a) For $n = 2$ and 3, give the corners of the unit n-cube and add them together. What will be the sum of all the corners for any $n \in \mathbf{N}$?

   (b) Compute the length between the two opposite corners $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$. What is the limit of this length as $n \to \infty$?

3. Let $\mathbf{v} = (1, 1, 1)$, $\mathbf{w} = (2, -2, 0)$ and $\mathbf{z} = (-3, 1, 2)$ be vectors in $\mathbf{R}^3$.

   (a) Using a linear equation in three variables, describe the plane of points $\mathbf{R}^3$ that are equidistant from $\mathbf{v}$ and $\mathbf{w}$.

   (b) Using two equations, describe the line of points in $\mathbf{R}^3$ that are equidistant from $\mathbf{v}$, $\mathbf{w}$, $\mathbf{z}$. Hint: A line is the intersection of two planes.

4. This question is about the Cauchy–Schwarz inequality, $|\mathbf{v} \cdot \mathbf{w}| \leqslant \|\mathbf{v}\| \cdot \|\mathbf{w}\|$.

   (a) Suppose that there exists $c \in \mathbf{R} \setminus \{0\}$ with $\mathbf{w} = c \cdot \mathbf{v}$. Show that the Cauchy–Schwarz inerquality holds with equality.

   (b) Suppose that the Cauchy–Schwarz inequality holds with equality. Show that there exists $c \in \mathbf{R} \setminus \{0\}$ with $\mathbf{w} = c \cdot \mathbf{v}$.

5. Let A, B, C, D be $n \times n$ matrices that are invertible.

   (a) Find the inverses of the following block matrices.

i. $\begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix}$ ii. $\begin{bmatrix} I & B \\ 0 & D \end{bmatrix}$ iii. $\begin{bmatrix} A & 0 \\ I & D \end{bmatrix}$

(b) Do one step of Gaussian elimination on $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ to get a zero in the lower left corner.

6. Let $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, and consider the 4×4 matrix $A = 2 \cdot \mathbf{v} \cdot \mathbf{v}^T - 4I$.

(a) Use Gauss–Jordan elimination to find the inverse of A.

(b) What is the inverse of $-4I$? What is the inverse of $2 \cdot \mathbf{v} \cdot \mathbf{v}^T$?

7. (a) Find the pivots of the matrix $\left[\begin{smallmatrix} a & b & b \\ a & a & b \\ a & a & a \end{smallmatrix}\right]$. For what values of a and b is this matrix invertible?

(b) Find the pivots of the matrix $\left[\begin{smallmatrix} 2 & 3 & c \\ c & c & c \\ -1 & -4 & c \end{smallmatrix}\right]$. For what values of c is this matrix invertible?

8. Decompose the matrix $A = \begin{bmatrix} 0 & 0 & 3 & 4 \\ 1 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ as $PA = LDU$.