Discrete Structures

Worksheet 22

Spring 2022
March 22

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1. For each of the statements below, find the $N \in \mathbf{N}$ and $c \in \mathbf{R}$ that satisfy the given asymptotic notation definition.

   (a) $200n + 30000$ is in $O(n^2)$

   (b) $6n^2 - 4n + 10$ is in $O(n^2)$

   (c) $\log_{10}(2^n) + 2020^{20}n^2$ is in $O(n^2)$

   (d) $3n^3 - 4n + 1$ is in $\Omega(n^3)$

   (e) $3n^3 - 4n + 1$ is in $\Theta(n^3)$

2. Let f, g be functions $\mathbf{Z}^+ \to \mathbf{R}$ from the positive integers to the real numbers. Define a relation R so that $(f, g) \in R$ iff f is in $\Theta(g)$.

   (a) Is the relation R reflexive, symmetric or transitive?

   (b) Is R an equivalence relation?

   (c) Consider the following two functions:

   $$f(n) = \sum_{i=1}^{n} i^2, \qquad g(n) = n^3 \cdot (1 + 0.99 \cdot \sin n).$$

   Is $(f, g) \in R$? Is $(g, f) \in R$?

3. Let $f(n), g(n)$ be any two functions.

   (a) Show that $f(n) + g(n)$ is in $\Theta(\max\{f(n), g(n)\})$.

   (b) What will be the Big-Oh and Big-Omega of $f(n) \cdot g(n)$?

4. Let $f(n) = \sum_{i=1}^{n} \frac{1}{i}$.

   (a) Show that $\int_1^{n+1} \frac{1}{x}\, dx \leqslant f(n) \leqslant 1 + \int_1^{n} \frac{1}{x}\, dx$.

   (b) Explain why part (a) above implies that $f(n)$ is in $\Theta(\ln(n))$.

   Hint: To show part (a), draw rectangles above and below the curve $\frac{1}{x}$.

5. Complete the following tasks for next lab (Tuesday). They will be presented at the beginning of the lab.

(a) Let $f(n) = \ln(n!)$.

i. Show that $\int_1^n \ln(n)\ dx \leqslant f(n) \leqslant \int_1^{n+1} \ln(x)\ dx$.

ii. Explain why part (a) above implies that $f(n)$ is in $\Theta(n \ln(n))$.

(b) Let $a, b, c \in \mathbf{R}$, with $a > 0$, $b > 0$, and $c > 1$.

i. Explain why $(\log(n))^a$ is in $O(n^b)$.

ii. Explain why n^b is in $O(c^n)$.

(c) Let A, B be two $n \times n$ matrices with real number entries. You may assume that adding or multiplying two real numbers takes the same amount of time.

i. What is the Big-Oh of matrix addition?

ii. What is the Big-Oh of matrix multiplication?