Discrete Structures

Worksheet 20

Spring 2022
March 15

1. Answer questions about groups of 3:
   (a) How many ways are there to split a group of 9 people into 3 groups, where each group has at least one member?
   (b) How many ways are there to split a group of 9 people into 3 groups, where each group has exactly 3 members?
   (c) What is the coefficient for $x^3y^3z^3$ in the expansion of $(x+y+z)^9$?

   Note. Polynomial expansions can also be verified by typing them into Wolfram Alpha.

2. Suppose that there are n people in a group, $n \geqslant 2$. Every person has a birthday; assume that it is one of the 365 calendar dates, and nobody has birthday on February 29. Assume that for any one person, any day of the year is equally likely to be their borthday.
   (a) When $n = 2$, what is the probability that both people have the same birthday?
   (b) When $n = 3$, what is the probability that at least two people out of the three have the same birthday?
   (c) Find the smallest n so that the possibility of at least two people in the group having the same birthday is more than 50%.

   Note. In this exercise a *probability* (of coinciding birthdays etc.) is a ratio m/n, where n is the total number of all ways how to assign birthdays to n people, but m is the number of those birthday assignments, where some people have the same birthday.

3. Suppose that there are 4 white and 2 black marbles. One way to arrange them is:

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   (a) List all 15 ways to arrange these marbles.
   (b) How many ways are there to arrange 4 white and n black marbles?
   (c) In which arrangements from part (a) is every white marble next to at least one other white marble?
   (d) List all the ways to arrange 2 black and 2 red marbles.
   (e) Explain why the lists in parts (c) and (d) have the same size.
   (f) How many ways are there to arrange 4 white and n black marbles so that every white marble is next to at least one other white marble?
   (g) If 21 black and 4 white marbles are arranged at random, what is the probability that every white marble is adjacent to at least one other white marble?

4. Let $n, m \in \mathbf{N}$ and consider a lattice with points (i, j) for $0 \leqslant i, j \leqslant n$. You start at $(0, 0)$, and from (i, j) you are allowed to "move" on this lattice only to $(i+1, j)$ or to $(i, j+1)$. Your goal is to get to (n, m).

   (a) Draw all the possible ways to get from $(0, 0)$ to (n, m) for:

   i. $n = 1, m = 1$ ii. $n = 2, m = 1$ iii. $n = 3, m = 1$ iv. $n = 2, m = 2$

   (b) Let $C(n, m)$ be the number of ways to get from $(0, 0)$ to (n, m), so your answers to part (a) give $C(1, 1)$, $C(2, 1)$, $C(3, 1)$, $C(2, 2)$, respectively. Express $C(3, 3)$ using these four expressions.

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5. Complete the following tasks for next lab (Friday). They will be presented at the beginning of the lab.

   (a) The password system `SillyPass` allows passwords that have at least one lowercase letter and at least one uppercase letter. In an alphabet of 26 letters, find the recurrence relation for allowed passwords of n letters.

   (b) Let $k \in \mathbf{N}$. Show that there are infintely Fibonacci numbers divisible by k.
   Hint: Consider the Fibonacci numbers modulo k.

   (c) For each of the two matrices A below, express the entries of A^n as a recurrence relation.

   i. $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ ii. $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

   (d) Suppose that $a_0 = 18$ and $a_{n+1} = \frac{10}{3} a_n - a_{n-1}$ for all $n > 0$. If the sequence $\{a_n\}_{n=0}^{\infty}$ converges to a real number, what is a_1?