Discrete Structures

Worksheet 14

Spring 2022
February 18

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1. **Warm up:**
   (a) Complete the Midterm evaluation for Kalvis and Jānis on ORTUS.
   (b) Which topics do you feel weakest at, of all the topics you've seen so far?

2. Find what is wrong with the following "proof" by induction.
   - **Claim:** All dogs have the same name.
   - **Setup:** Suppose there are $N \in \mathbf{N}$ dogs in the world. We proceed by induction on the size $n \in \mathbf{N}$ of a group of dogs.
   - **Base case:** For $n = 1$, it is obvious that every dog in a group of size 1 has the same name, as there is only one dog.
   - **Inductive hypothesis:** Suppose that every dog in any group of size $n > 1$ has the same name.
   - **Inductive step:** Consider a group of dogs $d_1, \ldots, d_n, d_{n+1}$. This is composed of two separate groups: $d_1, \ldots, d_n$, and $d_2, \ldots, d_{n+1}$. Both have size n, so all dogs in them have the same name. Since they have a non-empty intersection, the name must be the same. So every dog in the group $d_1, \ldots, d_{n+1}$ has the same name.
   - **Conclusion:** By induction, every dog in the world has the same name.

3. Find what is wrong with the following "proof" by induction.
   - **Claim:** The sum $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ converges.
   - **Setup:** We will proceed by adding individual terms, and showing that the sum $1 + \frac{1}{2} + \cdots + \frac{1}{n}$ stays finite.
   - **Base case:** For $n = 1$, it is clear that $\frac{1}{1} = 1 < \infty$ is finite.
   - **Inductive hypothesis:** Suppose that the sum $1 + \frac{1}{2} + \cdots + \frac{1}{n} < \infty$ is finite, for some $n > 1$.
   - **Inductive step:** Consider the sum $1 + \frac{1}{2} + \frac{1}{n} + \frac{1}{n+1}$. Clearly $\frac{1}{n+1}$ is finite, and $1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is finite by the inductive hypothesis. The sum of two finite things is finite, so $1 + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+1}$ is finite.
   - **Conclusion:** By induction, the infinite sum $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is finite.

4. Prove the following claim: The sum $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges to both 1 and -1.
   Hint: Observe that the sum consists of two sums: $1 + \frac{1}{3} + \frac{1}{5} + \cdots$ *and* $-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \cdots$*, one positive and one negative. Take as many terms from each so that your sum stays close to the number you want to get to.*

5. Complete the following tasks for next lab (Tuesday). They will be presented at the beginning of the lab.

(a) What is the relationship between the two sets? Is it $A \subseteq B$, or $B \subseteq A$, or $A = B$ or $A \neq B$?

$$A = \left\{ n \in \mathbf{Z} \ : \ \sin\left(\frac{\pi}{1+n^2}\right) = 0 \right\} \qquad B = \bigcap_{n=1}^{\infty} \{x \in \mathbf{R} \ : \ \cos(\pi n x)\}$$

(b) Let $r \in \mathbf{R}$ be an irrational number.

i. Construct a sequence $\{a_n\}_{n=1}^{\infty}$ of rational numbers that converges to r

ii. Construct a sequence $\{b_n\}_{n=1}^{\infty}$ of rational numbers that does not converge to r, but for which there is always a term as close as you want to r.

(c) Construct a bijection between the set of all square numbers and the set of all integers.

(d) Consider the set $S = \{1, 3, 5, 7, 9, 11\}$. Construct an equivalence relation on S which has two equivalence classes of the same size.