Discrete Structures

Worksheet 11

Spring 2022
February 8

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1. **Warm up:** Find the range of each of the following functions.

$$\begin{aligned} f\colon \mathbf{R} &\to \mathbf{R} \\ x &\mapsto x^2 \end{aligned} \qquad \begin{aligned} k\colon \mathbf{R} &\to \mathbf{R} \\ x &\mapsto \arctan(x) \end{aligned}$$

$$\begin{aligned} g\colon \mathbf{R} &\to \mathbf{R} \\ x &\mapsto x^2+1 \end{aligned} \qquad \begin{aligned} \ell\colon \mathbf{R}_{>0} &\to \mathbf{R} \\ x &\mapsto \ln(x)+10^{10} \end{aligned}$$

$$\begin{aligned} h\colon \mathbf{R} &\to \mathbf{R} \\ x &\mapsto \sin^2(x)/2 \end{aligned} \qquad \begin{aligned} m\colon \mathbf{R} &\to \mathbf{R} \\ x &\mapsto 4x^5-3x^2+10x-\tfrac{1}{2} \end{aligned}$$

2. Trace out x, y, r in the Euclidean algorithm (Algorithm 1 on page 284) on the inputs $a = 31463$ and $b = 9782$.

3. Let $a, b \in \mathbf{Z}$.
   (a) What is the relationship between the gcd of a, b and the lcm of a, b? Express it as an equation.
   (b) If $c \in \mathbf{N}$, prove that $\gcd(ac, bc) = c \cdot \gcd(a, b)$.

4. Find all solutions, if any, to each of the following systems of congruences.
   (a) $x \equiv 5 \pmod 6$, $x \equiv 3 \pmod{10}$, $x \equiv 8 \pmod{15}$
   (b) $y \equiv 7 \pmod 9$, $y \equiv 4 \pmod{12}$, $y \equiv 16 \pmod{21}$

5. Let $A_n = \{1, \ldots, n\}$ and let $B = \{0, 1\}$, where $n \in \mathbf{N}$ is fixed.
   (a) How many functions are there from A_n to B?
   (b) How many injective functions are there from A_n to B?
   (c) How many surjective functions are there from A_n to B?

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6. Complete the following tasks for next lab (Friday). They will be presented at the beginning of the lab.
   (a) Let $m, n \in \mathbf{N}$ with $m \leqslant n$. Prove that $\dfrac{\gcd(m,n)}{n}\dbinom{n}{m}$ is an integer.
   (b) Using the Euclidean algorithm and Bezout's identity, find the values x, y that solve $27x + 339y = -12$.
   (c) Show that a positive integer n is divisible by 4 if and only if its last two digits are divisible by 4. For example, 516 is divisible by 4 because 16 is divisible by 4.
   (d) A *googolplex* is the number $10^{10^{100}}$. If today is Tuesday, what day of the week will it be a googolplex days from now?
   (e) Find all positive integers x such that $2^{2^x+1} + 2$ is divisible by 17.