Discrete Structures

Worksheet 9

Spring 2022
February 1

1. **Warm up:** Answer the following True / False questions.
   (a) If a relation is symmetric, it must be reflexive.
   (b) The sets $\{1, \ldots, n\}$ and $\{1, \ldots, n\} \times \{0\}$ have the same cardinality, for $n \in \mathbf{N}$.
   (c) Any relation on a set must be either reflexive, or symmetric, or transitive.
   (d) Every equivalence relation is a partial order.

2. Let $A = \{1, 2, 3, 4\}$. The following of subset of $A \times A$ is defined by the relation $a \geqslant b$.

	1	2	3	4
1	T	T	T	T
2	F	T	T	T
3	F	F	T	T
4	F	F	F	T

For each of the following subsets of $A \times A$, come up with relations that define them.

(a)

	1	2	3	4
1	T	F	T	F
2	F	T	F	T
3	T	F	T	F
4	F	T	F	T

(b)

	1	2	3	4
1	T	T	T	T
2	F	F	F	F
3	T	T	T	T
4	T	T	T	T

(c)

	1	2	3	4
1	T	F	F	T
2	T	F	F	T
3	F	T	T	F
4	F	F	F	F

3. Let $G = (V = \{v_1, \ldots, v_9\}, E = \emptyset)$ be a directed graph with 9 vertices and no edges. What is the largest number of edges that you can add to E so that G is still a Hasse diagram?

4. Complete the following tasks for next lab (Tuesday). They will be presented at the beginning of the lab.

(a) Let $\preccurlyeq$ be a relation on $\mathbf{N}$, defined by $a \preccurlyeq b$ iff $a' = b' - 1$, where $\leqslant$ is the usual order relation on $\mathbf{N}$ and $a \equiv a' \pmod 5$ and $b \equiv b' \pmod 5$.

i. Draw arrows on the graph below, with an arrow from i to j if $i \preccurlyeq j$.

ii. What is the transitive closure of this relation?

iii. How many equivalence classes does this relation split $\mathbf{N}$ into?

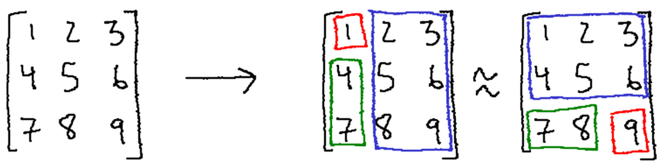
(b) Consider the set $S = \{(a, b) \ : \ a, b \in \mathbf{N}, a + b \leqslant 5\} \subseteq \mathbf{N} \times \mathbf{N}$, and the relation $(a_1, b_1) \leqslant (a_2, b_2)$ whenever $a_1 \leqslant a_2$ and $b_1 \leqslant b_2$.

i. Show that $\leqslant$ is a partial order on $\mathbf{N} \times \mathbf{N}$.

ii. By finding a counterexample, show that $\leqslant$ is not a total order on $\mathbf{N} \times \mathbf{N}$.

iii. Draw the Hasse diagram of $(S, \leqslant)$.

(c) Consider the matrix below. A partition of the entries into three rectangles is given, in two rearranged ways.



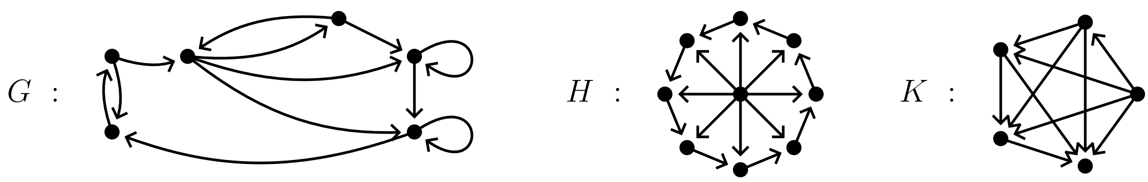
The rearrangements are not *different*: to be different, at least one of the rectangles has to be of a different size.

i. How many different partitions into three rectangles of this matrix do there exist?

ii. For what natural numbers n do there exist partitions of the matrix into n rectangles? Construct an example for each n.

(d) This question is about Hasse diagrams.

i. For each of the following digraphs, find its Hasse diagram.



ii. For each of the following Hasse diagrams, add as many edges as you can so that the transitive closure of the diagram remains unchanged.