Discrete Structures

Worksheet 8

Spring 2022
January 28

1. **Warm up:** Answer the following questions.
   (a) Let A, B be sets. True / False: if there exists an injection $A \to B$, then $|A| \leqslant |B|$.
   (b) How many distinct 5-character strings do there exist, using the English 26-letter alphabet (all lowercase)?
   (c) Order these sets in terms of their cardinality: $\mathbf{N}, \mathbf{Z}, \mathbf{Q}$

2. Let X, Y, Z be sets, and suppose that there exist:
   - an injection $f\colon X \to Y$,
   - an injection $g\colon Y \to Z$,
   - an injection $h\colon Z \to X$.

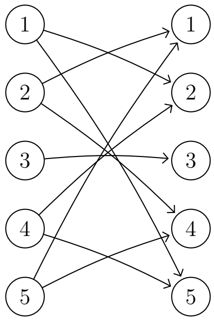
   Using these assumptions, answer the following questions.
   (a) Prove that X, Y, and Z all have the same cardinality.
   (b) Construct an injection $X \cup Y \to Z \times \{0, 1\}$.
   (c) Let $X = \mathbf{N}$. Find examples of sets Y, Z and functions f, g, h that satisfy the given assumptions, and so that none of X, Y, Z are the same.

3. Let f be a function mapping positive integers $\mathbf{Z}^+$ to positive integers. In other words, $f(1), f(2), f(3), \dots$ is an infinite sequence, its terms are elements from $\mathbf{Z}^+$. Translate the following predicate expressions into plain English:
   (a) $\forall c \in \mathbf{Z}^+\ \forall a \in \mathbf{Z}^+\ \exists b \in \mathbf{Z}^+\ (c > a\ \to f(c + b) = f(c))$.
   (b) $\exists a \in \mathbf{Z}^+\ \exists b \in \mathbf{Z}^+\ \forall c \in \mathbf{Z}^+\ (c \geq a\ \to\ f(c) = b)$.
   (c) $\exists a \in \mathbf{Z}^+\ \forall c \in \mathbf{Z}^+\ \exists b \in \mathbf{Z}^+\ (c \geq a\ \to\ f(c) = b)$.
   (d) $\forall c \in \mathbf{Z}^+\ \forall a \in \mathbf{Z}^+\ \exists b \in \mathbf{Z}^+\ (c > a\ \wedge\ f(c) = b)$.

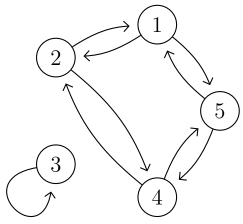
4. Fill in "yes" or "no" identifying properties of relations on $\mathbf{Z}$. The number $n \in \mathbf{Z}$ is fixed.

relation	reflexive	symmetric	anti-symmetric	transitive
$a \geqslant b$				
$a > b$				
$\lvert a\rvert = \lvert b\rvert$				
$a = b$				
$a = 2 + b$				
$a \leqslant 2 - b$				
$a = b + kn$ for some $k \in \mathbf{Z}$				

5. Let $A = \{1, 2, 3, 4, 5\}$, and consider the relation $\sim$ represented in four equivalent ways:



graphical (sets separated)



graphical (sets identified)

	1	2	3	4	5
1	F	T	F	F	T
2	T	F	F	T	F
3	F	F	T	F	F
4	F	T	F	F	T
5	T	F	F	T	F

table (or matrix)

$$R\colon A \times A \to \{\text{true, false}\},\qquad (a,b) \mapsto \begin{cases} \text{true} & \text{if } 3 \mid a+b \\ \text{false} & \text{if } 3 \nmid a+b \end{cases}$$

function

(a) Is $\sim$ an equivalence relation?

(b) For each $a \in A$, let $\overline{a} = \{b \in A \;:\; a \sim a_1 \sim a_2 \sim \cdots \sim a_n \sim b$ for some $a_i \in A\}$. Compute $\overline{a}$ for all $a \in A$.

(c) Let $S_1 = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\}$ be the relation on A with $S_1(a, b)$ whenever $a + 1 \equiv b \pmod 5$.

i. Define a relation S_2 on A such that $S_2 \oplus S_1 = R$.

ii. Define a relation S_3 on A such that $S_3 \circ S_1 = R$.

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6. Complete the following tasks for next lab (Tuesday). They will be presented at the beginning of the lab.

(a) Let $A = \{1, 2, 3, 4\}$ and consider the relation $a \sim b$ whenever the product $a \cdot b$ is not a multiple of 3.

i. Show that $\sim$ is not an equivalence relation.

ii. Draw the relation $\sim$ as a table.

(b) Consider the relation on $\mathbf{N}$ given by $a \sim b$ whenever $b = 2^k a$ for some nonnegative integer $k \in \mathbf{Z}_{\geqslant 0}$.

i. Show the relation $\sim$ is reflexive.

ii. Show the relation $\sim$ is antisymmetric.

iii. Show the relation $\sim$ is transitive.

(c) How many transitive relations are there on the set $\{1, 2\}$?

(d) Let A be the set of all characters in the English alphabet, that is,

$$A = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}, \mathtt{f}, \mathtt{g}, \mathtt{h}, \mathtt{i}, \mathtt{j}, \mathtt{k}, \mathtt{l}, \mathtt{m}, \mathtt{n}, \mathtt{o}, \mathtt{p}, \mathtt{q}, \mathtt{r}, \mathtt{s}, \mathtt{t}, \mathtt{u}, \mathtt{v}, \mathtt{w}, \mathtt{x}, \mathtt{y}, \mathtt{z}\}.$$

Two letters are related to each other if they have the same number of holes. For example, `c` has no holes and `b` has one hole.

i. Explain why this is an equivalence relation.

ii. How many "groups" does this relation split the set A into? In each group, every element should be related to every other element (and no elements in other groups).