**Discrete Structures**

# Worksheet 7

**Spring 2022**
**January 25**

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1. **Warm up:** Answer the following questions.

   (a) Find the first five terms of the sequences $\{a_n\}$ and $\{b_n\}$, where

   $$a_n = \frac{1}{2}(\cos(n\pi) + 1) \qquad , \qquad b_n = a_1 + a_2 + \cdots + a_n.$$

   (b) What does the sequence $\{c_n\}$ converge to, for $c_n = \frac{3n^2-7n+4}{2n^2+n-1}$?

   (c) True / False: The sequence $\{\frac{1}{n}\}$ converges.

   (d) True / False: The sequence $\{1 + \frac{1}{2} + \cdots + \frac{1}{n}\}$ converges.

2. This question is about sequences and logic.

   (a) What does it mean to be a sequence of real numbers? Can you describe a sequence using predicate logic?

   (b) Using predicate logic, write the statement "The sequence $\{a_n\}$ converges to 0."

3. Evaluate the following expressions.

   (a) $\displaystyle\sum_{i=0}^{7}\sum_{j=0}^{10} ij^2$ (b) $\displaystyle\prod_{i=0}^{7}\sum_{j=0}^{10} ij^2$ (c) $\displaystyle\sum_{i=0}^{5}\prod_{j=0}^{6}\sum_{k=0}^{7}(i+j+k)$

4. This question is about infinite geometric progressions and decimal notation.

   (a) Express this as a rational number: $\dfrac{1}{1000} + \dfrac{1}{1000^2} + \dfrac{1}{1000^3} + \ldots.$

   (b) Find the rational number $p/q$ having this decimal notation: $p/q = 0.(027) = 0.027027027\ldots$ (infinite fraction having a 3-digit period "027").

   (c) Compute the infinite decimal representation of $(64 \cdot 37)^{-1}$. How many digits precede the period of this eventually periodic decimal fraction? How long is the period?

5. Use functions and injectivity and surjectivity to answer parts (a), (b) in your own words.

   (a) What does it mean for a set to be *countable*?

   (b) What does it mean for two sets to have the same *cardinality*?

   (c) Prove that the intervals $(0, 1)$ and $(a, b)$ have the same cardinality, for any $a < b \in \mathbf{R}$.

   (d) Prove that $|\mathbf{N}| \leqslant |(a, b)|$ by defining an injection $\mathbf{N} \to (a, b)$, for any $a < b \in \mathbf{R}$.

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**6.** Complete the following tasks for next lab (Friday). They will be presented at the beginning of the lab.

   (a) Show that $|(0, 1)| = |(a, b)|$ by constructing a bijection between the two intervals, for any real numbers $a < b$.7.6.c

   (b) Let $A, B$ be countable sets. Using the definition of countability, prove that $A \cup B$ is countable.

   (c) Let $A$ be a countable set. Using the definition of countability, prove that $A \times \{1, \ldots, n\}$ is countable, for any $n \in \mathbf{N}$.

   (d) Construct a function $[0, 1) \to$ (points on a circle). Make it a bijection.