Discrete Structures

Worksheet 6

Spring 2022
January 21

Be aware of several different ways to write the same thing:

- the set of **natural numbers** is $\mathbf{N} = \{1, 2, 3, \ldots, \}$, but some people use $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$
- the symbol separating elements from statements in **set builder notation** is either $|$ or $:$

$$\{x \in \mathbf{R} \mid 2x^4 = 10\} \quad = \quad \{x \in \mathbf{R} \;:\; 2x^4 = 10\} \quad = \quad \{\sqrt[4]{5}, -\sqrt[4]{5}\}$$

- the **complement** of a set A is either $\overline{A}$ or A^C
- the **symmetric difference** of sets A, B is either $A \oplus B$ or $A \triangle B$
- **generalized** unions and intersections may be written with the indices in different spots

$$\bigcup_{i=1}^{50} \left[-\frac{i}{2}, \frac{1}{2}\right] \quad = \quad \textstyle\bigcup_{i=1}^{50} \left[-\frac{i}{2}, \frac{1}{2}\right] \quad = \quad [-25, 25]$$

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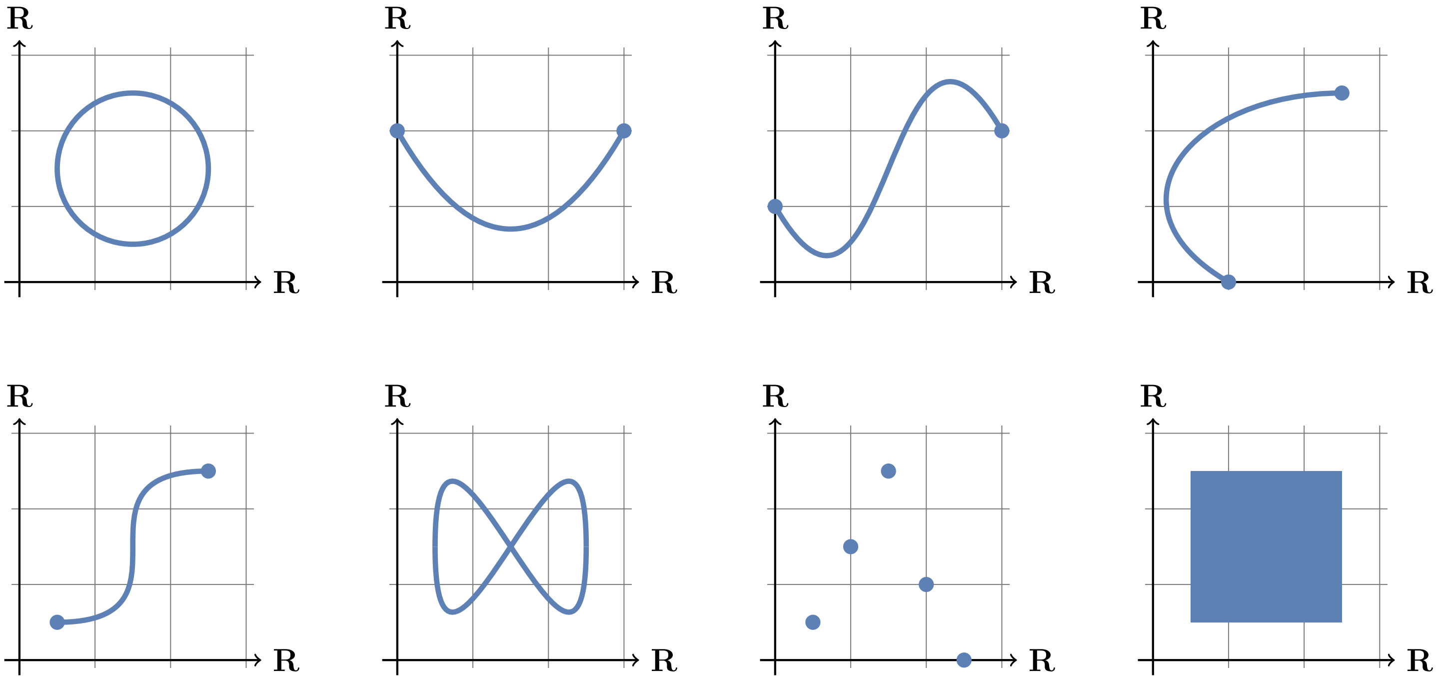
1. **Warm up:** Recall that a **graph** if a function $f\colon A \to B$ is the set of points $\{(a, f(a)) \;:\; a \in A\} \subseteq A \times B$. Consider the following subsets of $\mathbf{R} \times \mathbf{R}$:



For each of the drawings, indicate which can be considered as a graph. For those that can, give the **domain** A and the **codomain** B.

2. Let $A = \{x \;:\; x(x-2)(x-1) = 0\}$ and $B = \{y \;:\; x^2 - 2x + 1 = 0\}$. Find the size of:

$$A \qquad A \cup B \qquad A \cup \{B\} \qquad A \times B \qquad A \times \{B\} \qquad \mathcal{P}(A) \qquad \mathcal{P}(\{A\})$$

3. *(Adapted from Rosen ex. 2.2.56)* Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ for each of the following A_i, where i is a natural number.

(a) $A_i = \{i, i+1, i+2, \dots\}$

(b) $A_i = \{0, i\}$

(c) $A_i = \{-i, i\}$

(d) $A_i = (0, i)$

(e) $A_i = [-i, i]$

(f) $A_i = (i, \infty)$

(g) $A_i = [i, \infty)$

(h) $A_i = \{-i, -i+1, \dots, i-1, i\}$

4. (a) Prove that $f\colon \mathbf{R} \to \mathbf{R}$ given by $f(x) = 2x$ is injective.

   (b) Prove that $g\colon \mathbf{R} \to \mathbf{R}^2$ given by $g(x) = (\frac{x}{2}, 0)$ is injective.

   (c) Prove that $k\colon \mathbf{R}^2 \to \mathbf{R}$ given by $k(x, y) = -6x$ is surjective.

5. Let A, B be sets, and let $f\colon A \to B$ be a function. Using logical symbols, express the following statements.

   (a) f is injective

   (b) f is surjective

   (c) the range of f is a proper subset of B

   (d) there is an element in B whose preimage contains three distinct elements

   Note. Recall the set $\{b \in B \mid \exists a \in A\ (f(a) = b)\}$ is the **range** of f. The function $f\colon A \to B$ is surjective iff the range is the same as the codomain.

6. Find inverses of each of the following functions. Check that $f(f^{-1}(y)) = y$ and $f^{-1}(f(x)) = x$ for each function!

$$f\colon \mathbf{R} \to \mathbf{R},\quad x \mapsto x^3 \qquad g\colon \mathbf{R} \to \mathbf{R},\quad x \mapsto 5x - \frac{4}{3} \qquad h\colon \mathbf{N} \to \mathbf{Z},\quad n \mapsto \begin{cases} \frac{n}{2} & n \text{ is even} \\ \frac{-n-1}{2} & n \text{ is odd} \end{cases}$$

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7. Complete the following tasks for next lab (Tuesday). They will be presented at the beginning of the lab.

   (a) Let $A' \subseteq A, B' \subseteq B$ be sets, and $g\colon A' \to B'$ a function. Suppose that $f(a) = g(a)$ for every $a \in A'$.

      i. Prove that if f is injective, then g is injective.

      ii. Prove that if g is surjective and $B' = B$, then f is surjective.

   (b) Prove that each of the three functions in Task 6. are surjections.

   (c) *(Adapted from Cummings ex. 8.28)* Let A, B, C, D be sets with $C, D \subseteq A$. Let $f\colon A \to B$ be a function, and consider the claim

$$f(C \cap D) = f(C) \cap f(D).$$

   Give examples of A, B, C, D, f for which

      i. the claim is true,

      ii. the claim is false.

   (d) Let $f\colon \mathbf{R} \to \mathbf{R}$ be a continuous, differentiable, increasing function. Explain why f is injective.