Discrete Structures

Worksheet 5

Spring 2022
January 18

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1. **Warm up:** Determine which of the following statements are True and which are False.

$$A \cup \emptyset = A \qquad \{\emptyset\} = \emptyset \qquad (A \cup B) - C = (A - C) \cup (B - C) \qquad \overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$$

2. Consider the following game: There is a pile of n chips. A move consists of removing any proper factor of n chips from the pile. The player to leave a pile with one chip wins. For which n does a winning strategy exist? What is it?

   For the purposes of this question, a proper factor of n is any factor, including 1, that is strictly less than n.

3. *(Adapted from Rosen ex. 2.2.35)* Prove the following set identity:

$$\overline{A \cup B} \cap \overline{B \cup C} \cap \overline{A \cup C} = \overline{A} \cap \overline{B} \cap \overline{C}.$$

   Choose the proof method you prefer. For example:

   - Use set identities (Rosen, p.136)
   - Build a membership table (Rosen, p.138)
   - Shade regions in two Venn diagrams and compare the left-side and the right-side.

4. *(Adapted from Cummings Proposition 3.5)* Consider the following sets:

$$A = \{n \in \mathbf{Z} \ : \ 12 \mid n\}, \qquad B = \{n \in \mathbf{Z} \ : \ 3 \mid n\}, \qquad C = \{n \in \mathbf{Z} \ : \ 4 \mid n\}.$$

   Prove that $A = B \cap C$.

5. *(Adapted from Cummings Proposition 3.15)* Let A, B be sets. Recall that the **power set** of A is the set $\mathcal{P}(A)$, which is the set of all subsets of A. Prove the following statement:

$$\text{If } \mathcal{P}(A) \subseteq \mathcal{P}(B), \text{ then } A \subseteq B.$$

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6. Complete the following tasks for next lab (Friday). They will be presented at the beginning of the lab.

   (a) Let $\{a_n\}_{n=1}^{\infty}$ be the sequence of numbers defined by $a_1 = \frac{5}{2}$ and $a_{n+1} = \frac{1}{2}(a_n^2 + 6)$.

   i. What are the first six elements in the sequence?

   ii. Prove that the sequence is decreasing. That is, show that $a_{n+1} < a_n$ for every $n \in \mathbf{N}$.

   (b) *(Adapted from Cummings Exercise 3.21)* Consider the following sets:

$$A = \{n \in \mathbf{Z} \ : \ 2 \mid n\}, \qquad B = \{n \in \mathbf{Z} \ : \ 9 \mid n\}, \qquad C = \{n \in \mathbf{Z} \ : \ 6 \mid n\}.$$

   Prove that $A \cap B \subseteq C$.

   (c) *(Adapted from Cummings Exercise 3.34)* Prove the following statement, for any sets A, B:

$$A \subseteq B \text{ if and only if } A \cap B = A.$$

   As before, since this is an "if and only if" $\leftrightarrow$ statement, you should have two parts: one of the forwards direction $\rightarrow$, and one of the backwards direction $\leftarrow$.