Discrete Structures

Worksheet 4

Spring 2022
January 14

This worksheet uses the following definitions:

- The symbols " $a \mid b$ ", for $a, b \in \mathbf{Z}$, mean "a divides b", or in other words, there exists $k \in \mathbf{Z}$ with $ak = b$
- a statement is in **conjunctive normal form** (CNF) if $P = (\cdots \vee \cdots) \wedge (\cdots \vee \cdots)$
- a statement is in **disjunctive normal form** (DNF) if $P = (\cdots \wedge \cdots) \vee (\cdots \wedge \cdots)$

In other words, CNF is "ands of ors" and DNF is "ors of ands".

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1. **Warm up:** Answer the following questions.

   (a) What is the negation of $\exists\, x \in X,\ P(x)$?

   (b) What is the negation of $\forall\, x \in X,\ Q(x)$?

   (c) The set $\mathbf{Z}\times\mathbf{Z}$ contains ordered pairs (a, b), where $a, b \in \mathbf{Z}$. On what subset $X \subseteq \mathbf{Z}\times\mathbf{Z}$ is the statement $\forall\, x \in X,\ \forall\, y \in X, (x^2 + y^2 \leqslant 4)$ true?

2. For each of the following statements, find the truth value when $X = \mathbf{N}$ and when $X = \mathbf{Q}$.

Statement	Truth value for $X = \mathbf{N}$	Truth value for $X = \mathbf{Q}$
$\forall\, x \in X,\ (-x) \in X$		
$\forall\, x \in X, \exists\, y \in X,\ xy = 1$		
$\exists\, x \in X,\ \forall\, y \in X,\ x < y$		
$\exists\, x \in X,\ \forall\, y \in X,\ x \leqslant y$		
$\forall\, x \in X,\ \forall\, y \in X, \exists\, z\ \in X, ((x < y) \to ((x < z) \wedge (z < y)))$		
$\forall\, x \in X,\ \forall\, y \in X, \exists\, z\ \in X, ((x < y) \to ((x < z) \wedge (z \leqslant y)))$		

3. This question is about disjunctive and conjunctive normal form. Let P, Q, R be atomic predicates.

   (a) Express the following statements in conjunctive and disjunctive normal form. Use a truth table to help you out.

   $$P \to Q \qquad\qquad P \to (Q \to R) \qquad\qquad (P \to Q) \to R$$

   (b) In what cases is CNF the same as DNF? Give an example.

   (c) Use part (a) to rephrase the following senetence using "and" and "or", and not using "if...then" or "whenever":

   If I win the lottery, then whenever I see a
   homeless person, I will give them 100 euros.

4. *(Adapted from Cummings, Exercise 5.4)* Each of the following statements can be written in the form $P \wedge Q$, or $P \vee Q$, or $\neg P$. For each statement, decide which form it takes, and give the explicit statement for P and Q.

   (a) $3 \mid 12$ and $6 \mid 12$
   (b) $3 \neq 5$
   (c) $x < y$
   (d) $x \leqslant y$
   (e) 10 is odd while 11 is not
   (f) $x \in \mathbf{R} \setminus \mathbf{Z}$

5. *(Adapted from Cummings, Exercise 6.16)* Each of the following statements are false. For each, find a counterexample.

   (a) If $n \in \mathbf{N}$, then $2n^2 - 4n + 31$ is prime.
   (b) If $a, b \in \mathbf{N}$, then $a + b < ab$.
   (c) If $a, b \in \mathbf{Z}$, $a \mid b$ and $b \mid a$, then $a = b$.
   (d) If $x \in \mathbf{R}$, then $x^2 - x \geqslant 0$.
   (e) If $x \in \mathbf{R}$, then $\frac{x^2+x}{x^2-x} = \frac{x+1}{x-1}$.
   (f) If $x, y \in \mathbf{R}$, then $|x + y| = |x| + |y|$.
   (g) If $x, y \in \mathbf{R}$ and $|x + y| = |x - y|$, then $y = 0$.

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6. Complete the following tasks for next lab (Tuesday). They will be presented at the beginning of the lab.

(a) Prove the following statement:

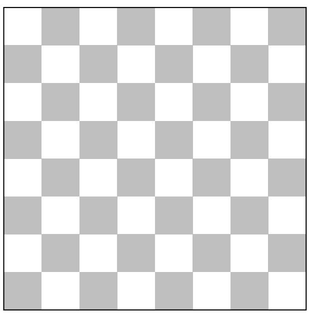
For every $n \in \mathbf{N}$, n is odd if and only if $3n + 5$ is even.

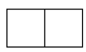
Since this is an "if and only if" $\leftrightarrow$ statement, you should have two parts to your proof: one of the forwards direction $\rightarrow$, and one of the backwards direction $\leftarrow$.

(b) Prove the following statement, for the sequence $\{a_n\}_{n=1}^{\infty}$, where $a_n = \frac{3n+1}{n+2}$:

For every $\epsilon > 0$, there exists $n \in \mathbf{N}$ with $|a_n - 3| < \epsilon$.

(c) This problem is about chessboards and dominoes.

chessboard: 

domino:  or 

i. Prove by construction that the chessboard can be covered with with dominoes.
ii. Prove that if any one square is removed from the chessboard, then it can not be covered by dominoes. *Hint: How many squares does a domino cover? How many squares does a chessboard with one removed have?*