Discrete Structures

Worksheet 3

Spring 2022
January 11

In this worksheet we use the following propositions:

- $E(x, y)$: The integer x ends with the digit y
- $D(a, b)$: The integer a divides the integer b
- $C(s, t)$: The integers s and t have no common factors

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1. **Warm up:** Answer the following questions.

   (a) What is the negation of "I always take my hat off before my socks." ?

   (b) What is the contrapositive of "If my dog barks, then my baby will wake up." ?

   (c) Fill in the empty cells in the truth table below.

P	Q	$\neg P$		$(\neg P) \wedge Q$		$Q \to (\neg P)$
T	T		T		T	
T	F		T		F	
F	T		T		T	
F	F		F		T	

2. Change each of the following compound expressions into English sentences. Try to make the sentences as "naturally-sounding" as possible!

   (a) $\forall\ x \in \mathbf{Z},\ N(x, 9) \to D(3, x)$

   (b) $\forall\ x \in \mathbf{N},\ D(5, x) \to (E(x, 0) \wedge E(x, 5))$

   (c) $E(x, 2) \to (\exists\ z \in \mathbf{Z},\ 2z = x)$

   (d) $\exists\ x \in \mathbf{Z}, (D(9, x) \leftrightarrow E(x, 9))$

   Which of these compound expressions are true?

3. Express the following sentences using logical symbols.

   (a) For every integer ending with 4, there is an integer ending with 3.

   (b) There does not exist an integer ending with 7 and divisible by 7.

   (c) Every positive integer either ends with 8 or is the square of an integer ending with 9.

4. *(Adapted from Rosen, ex. 1.6.19)* All variables in this question are real numbers. Determine whether each of these arguments is valid. If an argument is correct, what rule of inference is being used? If it is not, what logical error occurs?

   (a) If $n > 1$, then $n^2 > 1$. Suppose that $n^2 > 1$. Then $n > 1$.

   (b) If $n > 3$, then $n^2 > 9$. Suppose that $n^2 \leqslant 9$. Then $n \leqslant 3$.

   (c) If $n > 2$, then $n^2 > 4$. Suppose that $n \leqslant 2$. Then $n^2 \leqslant 4$.

   (d) If $n > 0$, then $n + 1 > 0$. Suppose that $n > 0$. Then $n + 3 > 0$.

5. All variables are integers in this question.

   (a) Express $C(s, t)$ using $D(a, b)$.

   (b) Using $D(a, b)$, write a quantification $Pr(x)$ that asserts x is a prime number.

   (c) Write the following statement in English:

$$S(x) \ : \ \forall\, a, b, c \in \mathbf{Z}\ (Pr(x)\ \wedge\ (ax = bc))\ \rightarrow\ (D(x, b)\ \vee\ D(x, c))$$

   Convince yourself that it is true!

   (d) Use $S(x)$ to prove that $(Pr(x)\ \wedge\ D(x, y^2))\ \rightarrow\ D(x, y)$.

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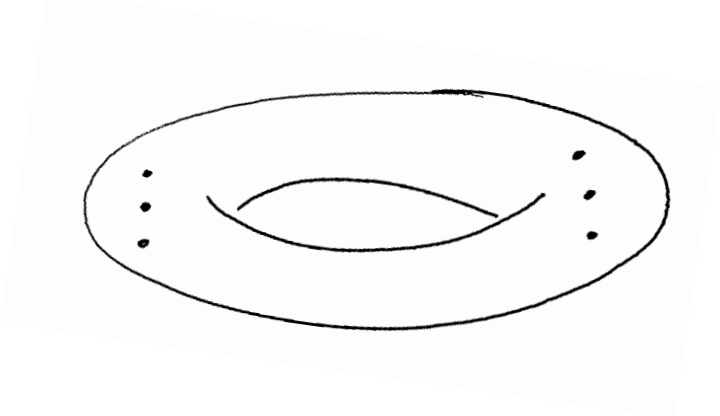
5. Complete the following tasks for next lab (Friday). They will be presented at the beginning of the lab.

   (a) Let x and y be distinct positive real numbers. Prove that $\frac{x}{y} + \frac{y}{x} > 2$.

   (b) Prove by construction that is possible to connect all the dots on the right to all the dots on the left with (not necessarily straight) lines that do not cross each other. The dots are on the surface of a *torus*.



   (c) Let $Pr(x)$ be the proposition "The integer x is prime."

      i. What is the compound expression $\forall\, y \in \mathbf{Z}, \exists\, x \in \mathbf{Z}, ((x > y) \wedge Pr(x))$? Express it as a normal English sentence.

      ii. Express the sentence "For every prime number, there is another prime number smaller than one half of the first number" as a compound expression of logical symbols.

      iii. Prove the statement $\exists!\ x \in \mathbf{Z},\ (Pr(x) \wedge (\exists\ y \in \mathbf{Z}, 2y = x))$ is show it is false.