Discrete Structures

Worksheet 2

Spring 2022
January 7

This worksheet will use the following definition:

- a **rational number** may be written as $\frac{a}{b}$, where $a, b \in \mathbf{Z}$, $q \neq 0$, and there no common factors between p and q

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1. **Warm up:** Answer the following True / False questions.

   (a) The negation of (P and Q) is ($\neg P$ and $\neg Q$).

   (b) A statement that is a tautology is always false.

   (c) If a statement is unsatisfiable, then it is a contradiction.

2. This question is about **proof by contradiction**. The following is a good example:

   Claim: The number $3n^2 + 3n + 23$ is prime for every natural number n.
   Counterexample: For $n = 22$, we have $3n^2 + 3n + 23 = 1541 = 23 \cdot 67$. Since 1541 is not prime, the claim is false.

   Disprove the following claims by counterexample.

   (a) Let a, b, c be fixed integers. The number $an^2 + bn + c$ is prime for every natural number n.

   (b) The set $\mathbf{N}$ of natural numbers has a largest element.

   (c) There exists a smallest positive rational number.

3. This question is about **direct proofs**. The following is a good example:

   Claim: Given any two rational numbers, half their sum is a rational number.
   Proof: Let $\frac{a}{b}$, $\frac{c}{d}$ be two rational numbers, so $a, b, c, d \in \mathbf{Z}$ and $b, d \neq 0$. Half their sum is $\frac{\frac{a}{b}+\frac{c}{d}}{2}$, which, as written, is not a ratio of two integers, so it is not clear we have a ratonal number. Observe that

   $$\frac{\frac{a}{b}+\frac{c}{d}}{2} = \frac{\frac{a}{b}+\frac{c}{d}}{2} \cdot \frac{bd}{bd} = \frac{(\frac{a}{b}+\frac{c}{d})bd}{2bd} = \frac{\frac{abd}{b}+\frac{bcd}{d}}{2bd} = \frac{ad+bc}{2bd}.$$

   Since $a, b, c, d \in \mathbf{Z}$, it follows that $ad + bc \in \mathbf{Z}$. Since $b, d \neq 0$, it follows that $2bd \neq 0$. Hence this number is of the necessary form, and is an integer.

   Prove the following claims directly.

   (a) Given any two different rational numbers, half their sum is less than one and greater than the other number.

   (b) Let x, y be two positive numbers. If $x > y$, then $\sqrt{x} \geqslant \sqrt{y}$.

4. This question is about **proof by cases**. The following is a good example:

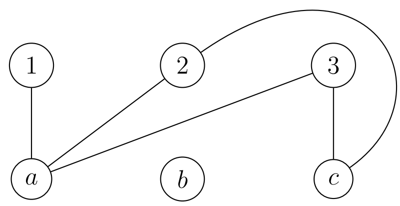
Claim: Every group of n people, for $n \geqslant 6$, can be split up into groups of size 3 or 4.
Proof: For every natural number n, division by 3 leaves either a remainder of 0, 1, or 3.

- *if the remainder is 0:* The natural number $n = 3k$ is a multiple of 3, and so can be split up into k groups of size 3. This allows groups of size $n = 3, 6, 9, 12, \ldots$.
- *if the remainder is 1:* The natural number $n = 3k + 1$ can be split up into $k - 1$ groups of size 3 and one group of size 4. Indeed, $3(k-1)+4 = 3k-3+4 = 3k+1 = n$. This allows groups of size $n = 4, 7, 10, 13, \ldots$.
- *if the remainder is 2:* The natural number $n = 3k + 2$ can be split up into $k - 2$ groups of size 3 and two groups of size 4. Indeed, $3(k-2)+4\cdot 2 = 3k-6+8 = 3k+2$. This allows groups of size $n = 8, 11, 14, 17, \ldots$.

Every number $n \geqslant 3$ is allowed, except for $n = 5$. Hence all groups of size $n > 5$, or $n \geqslant 6$ are allowed.

Prove the following claims by cases.

(a) In the picture below it is not possible to connect c to 1, and b to every one of $1, 2, 3$ at the same time, without having lines cross each other.



(b) The number $n^2 + n + 6$ is even for all integers n.