Discrete Structures

Worksheet 14

Spring 2021
April 16

1. **Warm up:** Answer the following questions about the 3-ary (ternary) trees:
   (a) Every full ternary tree of height 3 has at least x vertices and at most y vertices.
   (b) A full ternary tree with 11 internal vertices has z vertices.
   (c) There are u full (ordered) ternary trees with 6 vertices.
   (d) There are v full (ordered) ternary trees with 7 vertices.

2. Answer the following True/False questions:
   (a) Every tree is bipartite.
   (b) There is a tree with degrees $3, 2, 2, 2, 1, 1, 1, 1, 1$.
   (c) There is a tree with degrees $3, 3, 2, 2, 1, 1, 1, 1$.
   (d) If two trees have the same number of vertices and the same degrees, then the two trees are isomorphic.
   (e) If T is a tree with n vertices, the largest degree that any vertex can have is $n - 1$.
   (f) If T is a binary tree with 15 vertices, then there is a simple path in T of length 6.
   (g) In a binary tree with 16 vertices, there must be a path of length 4.
   (h) If T is a rooted binary tree of height 5, then T has at most 25 leaves.

3. Consider the binary tree in Figure 1.

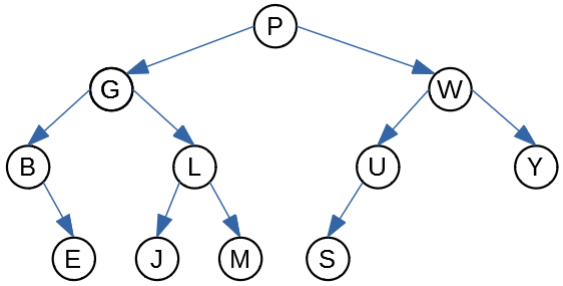

Figure 1: A Labeled Binary Tree

   (a) List the vertice labels, if they are visited in preorder, inorder and postorder sequence.
   (b) Draw a tree that is the mirror image of Figure 1. List the vertices of this tree in preorder, inorder and postorder sequence.
   (c) Is any of the sequences obtained in **(a)** and **(b)** a reverse image of another?

4. Prove or disprove the following statement: If the degrees of vertices are positive integers $d_1, \ldots, d_n$ and their total is $2(n - 1)$, then there exists a tree with n vertices and these degrees.

5. Construct trees T_1 and T_2 with the following preorder traversal of the 11 vertices (labeled a to k):

$$a, b, c, d, e, f, g, h, i, j, k.$$

(a) Tree T_1 is a regular ordered tree. In the tree T_1 vertex a has 1 child, vertex b has 3 children, vertex d has 2 children, vertex g has 4 children (and the remaining vertices are leaves).

(b) Tree T_2 is a binary tree; children are ordered; left and right children are distinguishable (internal vertices may have just the left child, or just the right child, or both left and right children). In the tree T_2 vertex a has two children, vertex b has two children, vertex d has one child (right only), vertex f has two children, vertex g has two children, vertex j has one child (left only).

6. Construct trees T_1 and T_2 (containing 8 vertices each), if they have the following preorder and postorder traversal orders:

(a) Tree T_1 has these traversals:

$$\begin{cases} \text{Preorder traversal:} & a, b, c, d, e, f, g, h; \\ \text{Postorder traversal:} & c, d, b, g, h, f, e, a \end{cases}$$

(b) Tree T_2 has these traversals:

$$\begin{cases} \text{Preorder traversal:} & a, b, c, d, e, f, g, h; \\ \text{Postorder traversal:} & c, d, e, b, g, h, f, a. \end{cases}$$

7. Figure 2 shows a directed graph.

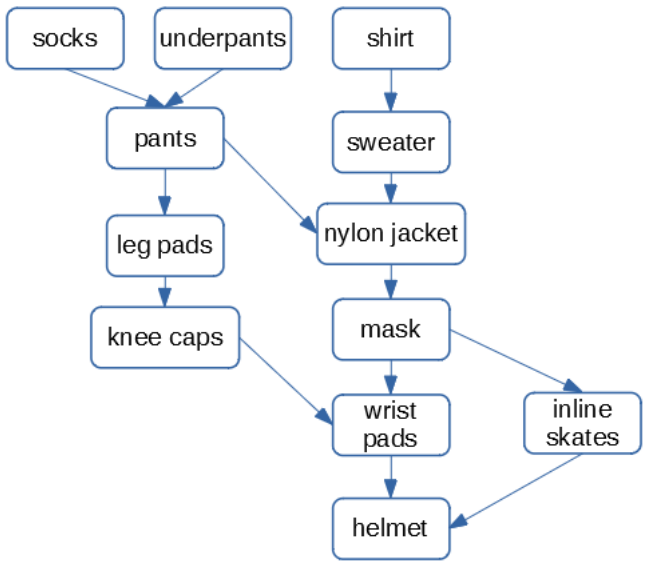

Figure 2: Directed graph for topological sorting

(a) Do a DFS traversal on this tree, show the discovery and finishing times for ever vertex. (Start from the alphabetically first garment as a root. If the DFS vertex discovery runs out of vertices at some point, select the next alphabetically smallest name as a new root.)

(b) Label all edges into 4 categories: as tree edges, back edges, forward edges and cross edges. (Tree edges belong to the DFS tree; back edges point from some vertex to its ancestor; forward edges point from an ancestor to its descendant - but were not chosen for the tree; cross edges – all the remaining ones).

(c) Perform topological sort in this graph. (Arrange the items in decreasing order by their finishing times; verify that this is a feasible order how to put on these garments)

8. Some questions on Trees and Catalan numbers:

   (a) Count how many full binary trees are there with 3 inner nodes (all the other nodes are leaves).

   (b) Count how many arithmetic expressions one can write using 4 variables A, B, C, D, exactly three binary minus operations, and no more than 4 unary minus operations (denote them by tilde symbol to avoid confusion with binary minus).

   (c) Describe a method, how to determine, if an expression containing letters, binary minus ($-$) and unary minus ($\sim$) is a syntactically correct postfix notation.