Discrete Structures

Worksheet 12.1

Spring 2021
March 24

1. **Warm up:** Answer the following True / False questions.
   (a) The expression $a_n = 4n + 5(n-1)$ is a recurrence relation.
   (b) A constant sequence of numbers can be described as a recurrence relation.
   (c) The recurrence relation $a_n = a_{n-1} + a_{n-2}$ has infinitely many solutions, depending on what a_0 and a_1 are.

2. The password system `SillyPass` allows passwords that have at least one lowercase letter and at least one uppercase letter. In an alphabet of 26 letters, find the recurrence relation for allowed passwords of n letters.

3. Let $n, m \in \mathbf{N}$ and consider a lattice with points (i, j) for $0 \leqslant i, j \leqslant n$. You start at $(0,0)$, and from (i, j) you are allowed to "move" on this lattice only to $(i+1, j)$ or to $(i, j+1)$. Your goal is to get to (n, m).
   (a) Draw all the possible ways to get from $(0,0)$ to (n, m) for:
   i. $n = 1, m = 1$ ii. $n = 2, m = 1$ iii. $n = 3, m = 1$ iv. $n = 2, m = 2$
   (b) Let $C(n, m)$ be the number of ways to get from $(0,0)$ to (n, m), so your answers to part (a) give $C(1,1)$, $C(2,1)$, $C(3,1)$, $C(2,2)$, respectively. Express $C(3,3)$ using these four expressions.

4. Consider the scenario from Question 3, and add a probability to each "move." That is, at each (i, j), the probability of going to $(i+1, j)$ is 0.4 and the probability of going to $(i, j+1)$ is 0.6. If only one of the two is possible, it has probability 1.
   (a) Draw a lattice starting at $(0,0)$ and ending at $(3,2)$. What is $C(3,2)$?
   (b) For each edge, label the probability of moving from the left (or bottom) to the right (or top).
   (c) What is the probability that a path from $(0,0)$ to $(3,2)$ will involve three consecutive moves to the right?
   (d) Find the path from $(0,0)$ to $(3,2)$ with the highest and with the lowest probability.

5. For each of the following relations, identify which are linear, recurrent, homogeneous.

	linear?	recurrent?	homogeneous?
$a_n = a_{n-2}^2 + 3a_{n-1} - 9a_{n-3}$			
$b_n = 5b_{n-1} - 2b_{n-2}$			
$c_n = 7n + 25$			
$d_n = d_{n-2}/2 + 5(d_{n-3})^{-1}$			
$e_n = e_{n-1}^3 + 2$			
$f_n = 6 + 2f_{n-1}$			
$g_n = (n-1)g_{n-1} + (n-2)g_{n-2}$			

6. Consider the relation $a_n = a_{n-1} + 2a_{n-2}$, with $a_0 = 5$ and $a_1 = 4$.
   (a) What is its characteristic equation?
   (b) What are its characteristic roots?
   (c) Give the general solution to this recurrence relation.
7. Using $\{a_n\}$ from Question 6, let $b_n = a_n - \frac{9}{8}a_{n-2}$, with $b_0 = a_0$ and $b_1 = a_1$.
   (a) Express this relation without using any a_n terms.
   (b) What is its characteristic equation?
   (c) What are its characteristic roots?
   (d) Give the general solution to this recurrence relation.
8. Consider the equation $r(r-2)^2(r+3)^3(r-4) = 0$.
   (a) Expand out the left side of the equation. You may use a calculator.
   (b) Give an example of a linear homgeneous recurrence relation that has this equation as its characteristic equation.
   (c) For your example from part (b), find the general form of its solution.
9. Consider the relation $a_n = 3a_{n-1} - 2a_{n-2} + (n-2)3^n$, with $a_1 = 1$ and $a_2 = 1$.
   (a) What is the associated homogeneous recurrence relation and what are the roots of its characteristic equation?
   (b) Find a particular solution to this recurrence relation.