Discrete Structures

Worksheet 10.2

Spring 2021
March 12

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1. **Warm up:** For each of the following words, find the number of unique strings that can be formed from its letters.

   (a) `country` (b) `brittle` (c) `popping` (d) `crevice` (e) `mississippi`

2. Answer questions about groups of 3:

   (a) How many ways are there to split a group of 9 people into 3 groups, where each group has at least one member?

   (b) How many ways are there to split a group of 9 people into 3 groups, where each group has exactly 3 members?

   (c) What is the coefficient for $x^3y^3z^3$ in the expansion of $(x+y+z)^9$?

   Note. Polynomial expansions can also be verified by typing them into Wolfram Alpha.

3. Suppose that there are n people in a group, $n \geqslant 2$. Every person has a birthday; assume that it is one of the 365 calendar dates, and nobody has birthday on February 29.

   (a) When $n=2$, what is the probability that both people have the same birthday?

   (b) When $n=3$, what is the probability that at least two people out of the three have the same birthday?

   (c) Find the smallest n so that it is guaranteed that two people in the group have the same birthday.

   Note. In this exercise a *probability* (of coinciding birthdays etc.) is a ratio m/n, where n is the total number of all ways how to assign birthdays to n people, but m is the number of those birthday assignments, where some people have the same birthday.

4. Evaluate coefficients in the polynomial expressions, rational fractions and derivatives:

   (a) Let $f(x)=(x+1)^4$. Express it as $f(x)=a_4x^4+a_3x^3+a_2x^2+a_1x+a_0$.

   (b) Let $f(x)=(x+y)^{20}$. Find the coefficients for $x^{11}y^9$, $x^{10}y^{10}$ and x^9y^{11} in its expansion.

   (c) Let $f(x)=(2x+5y)^{10}$. Find the largest coefficient in its expansion.

   (d) Let $f(x)=(x+y^2)^{12}$. Find the coefficient for x^8y^8.

   (e) Let $f(x)=\left(x+\frac{1}{x}\right)^8$. What powers of x are present in the expansion? Find the constant term in the expansion.

   (f) Find the term for x^6 in the expansion of the squared polynomial:

   $$\left(\binom{6}{0}x^6+\binom{6}{1}x^5+\binom{6}{2}x^4+\binom{6}{3}x^3+\binom{6}{4}x^2+\binom{6}{5}x^1+\binom{6}{6}x^0\right)^2.$$

   (g) Let $f(x)=(x+1)^5$. Find the value of its derivative $f'(1)$. Also express $f'(1)$ using the binomial coefficients $\binom{5}{k}$.

5. **Identities involving binomial coefficients.** Let n be a positive integer. Verify that the following identities are true:

(a)

$$\binom{2n}{0}+\binom{2n}{2}+\ldots+\binom{2n}{2n-2}+\binom{2n}{2n}=\binom{2n}{1}+\binom{2n}{3}+\ldots\binom{2n}{2n-3}+\binom{2n}{2n-1}.$$

(b)

$$\sum_{i=1}^{n} k\cdot\binom{n}{k}=n\cdot 2^{n-1}.$$

(c)

$$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\ldots+\binom{n}{n}^2=\binom{2n}{n}.$$

Note. Some favorite methods to prove identities with binomial coefficients involve using binomial formula (and substituting certain values for x and/or y), or using polynomial derivatives or some algebraic manipulations with polynomials. You can also use "interpretations" – translate the identity into a combinatorial problem.

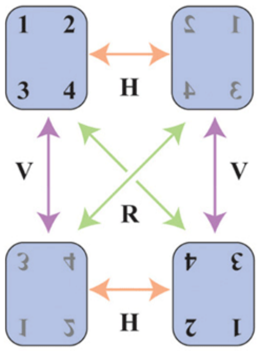

Figure 1: 4 ways to insert a mattress in a bed frame.

6. **Composing permutations.** Permutations on a set $A_n = \{1, 2, \ldots, n\}$ are just bijective functions from the set A_n to itself; so their compositions can be written, and they may lead to new permutations. In this problem we study two different collections of permutations on the set $A_4 = \{1, 2, 3, 4\}$.

   Let us have a rectangular bed frame of size $1m \times 2m$. It has a mattress that can be rotated in 4 ways before it is inserted into the frame. We introduce permutations on 4 mattress vertices. For example, flipping the mattress over the short edge (orange arrow H in the Figure 1) and flipping the mattress over the long edge (purple arrow V in the same Figure). We write them in tables that tell where every vertex goes.

$$\mathbf{H}=\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}.\quad \mathbf{V}=\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}. \tag{1}$$

   Let us also introduce some permutations that multiply the congruence classes $\{1, 2, 3, 4\}$ by numbers $a \not\equiv 5 \pmod 5$ (all multiplication is modulo 5):

$$M_2=\begin{pmatrix}1&2&3&4\\2&4&1&3\end{pmatrix}.\quad M_3=\begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix}. \tag{2}$$

(a) Compute the compositions of two permutations $(V \circ H)$ and $(H \circ V)$ from the "mattress permutations" (1). Write the result as a 2×4 table that tells where every number $1, 2, 3, 4$ is mapped.

(b) Compute the composition of two permutations $M_3 \circ M_2$ from the "multiplication permutations" (2).

(c) Compute the composition of permutation V with M_2 in two ways: $V \circ M_2$ and also the opposite one: $M_2 \circ V$.

Note. During the composition of permutations (just like composition of functions and of binary relations) the rightmost permutation (function, relation) is applied first; the compositions read from right to left.

7. **Combinations with or without Repetition.**

(a) How many ways to choose 20 candies, if there are 3 varieties of candy. (The order of selecting does not matter; just the count of each variety.)

(b) How many nonnegative integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 17.$$

(c) There is a line of 20 chairs. In how many ways can we seat 3 people in three chairs on that line? (Assume that all chairs are distinguishable, e.g. enumerated from #1 to #20, but the people are not distinguishable.)

(d) How many ways are there to seat 5 people on 20 chairs arranged in a line so that there is *physical isolation*: No two people can sit on adjacent chairs. (Once again, assume that people are not distinguishable: If some of them exchange their seats, it is still the same arrangement.)

8. **Pigeonhole-based Proof.** Consider the set containing the first 100 positive integers: $A = \{1, 2, 3, \ldots, 100\}$. Let $S \subseteq A$ and $|S| = 51$. We want to prove that S contains two numbers numbers which are relatively prime. We build the proof in multiple steps:

(a) How many different subsets S of size 51 exist? Could we just analyze all such subsets S, find two relatively prime numbers in every possible S?

(b) How to partition the set A into 50 "buckets" of equal size so that whenever two elements from the same "bucket" belong to S, they are relatively prime?

(c) Formulate the Pigeonhole principle for the set S and the buckets you defined in **(b)**.