Discrete Structures

Worksheet 8

Spring 2021
February 26

Let A be a set. Recall that a **binary relation** on A is either

- a function R that takes in two elements of A and returns "true" or "false", or
- a subset of $A \times A$.

The expression "$R(a, b)$" or "$a \sim b$" is said "a is related to b". A binary relation R on A is

- **reflexive** if $R(a, a)$
- **symmetric** if $R(a, b)$ iff $R(b, a)$
- **anti-symmetric** if $R(a, b)$ and $R(b, a)$ implies $a = b$
- **transitive** if $R(a, b)$ and $R(b, c)$ implies $R(a, c)$

A relation $\sim$ on A is an **equivalence relation** if it is reflexive, symmetric, and transitive. For such relations, we write $[a] = \{b \in A \ : \ a \sim b\}$ for the **equivalence class** of a.

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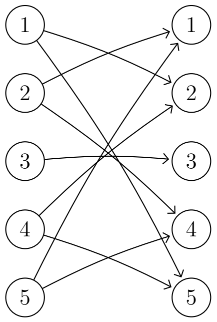
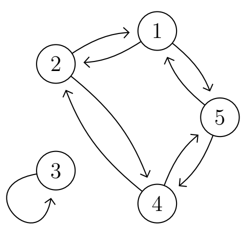
1. **Warm up:** Fill in "yes" or "no" identifying properties of relations on $\mathbf{Z}$.

relation	reflexive	symmetric	anti-symmetric	transitive
$a \geqslant b$				
$a > b$				
$\lvert a\rvert = \lvert b\rvert$				
$a = b$				
$a \equiv b \pmod n$				
$a = 2 + b$				
$a \leqslant 2 - b$				

2. Let $A = \{1, 2, 3, 4, 5\}$, and consider the relation $\sim$ represented in four equivalent ways:



graphical (sets separated)



graphical (sets identified)

	1	2	3	4	5
1	F	T	F	F	T
2	T	F	F	T	F
3	F	F	T	F	F
4	F	T	F	F	T
5	T	F	F	T	F

table (or matrix)

$$R \colon A \times A \to \{\text{true, false}\}, \quad (a, b) \mapsto \begin{cases} \text{true} & \text{if } 3 \mid a + b \\ \text{false} & \text{if } 3 \nmid a + b \end{cases}$$

function

(a) Is $\sim$ an equivalence relation?

(b) For each $a \in A$, let $\overline{a} = \{b \in A \ : \ a \sim a_1 \sim a_2 \sim \cdots \sim a_n \sim b \text{ for some } a_i \in A\}$. Compute $\overline{a}$ for all $a \in A$.

(c) Let $S_1 = \{(1,2),(2,3),(3,4),(4,5),(5,1)\}$ be the relation on A with $S_1(a,b)$ whenever $a+1 \equiv b \pmod 5$.

i. Define a relation S_2 on A such that $S_2 \oplus S_1 = R$.

ii. Define a relation S_3 on A such that $S_3 \circ S_1 = R$.

3. Let $A = \{1,2,3,4\}$. The following of subset of $A \times A$ is defined by the relation $a \geqslant b$.

	1	2	3	4
1	T	T	T	T
2	F	T	T	T
3	F	F	T	T
4	F	F	F	T

For each of the following subsets of $A \times A$, come up with relations that define them.

(a)

	1	2	3	4
1	T	F	T	F
2	F	T	F	T
3	T	F	T	F
4	F	T	F	T

(b)

	1	2	3	4
1	T	T	T	T
2	F	F	F	F
3	T	T	T	T
4	T	T	T	T

(c)

	1	2	3	4
1	T	F	F	T
2	T	F	F	T
3	F	T	T	F
4	F	F	F	F

4. Recall that the number of relations on A is $2^{|A \times A|}$.

(a) How many reflexive relations are there on A?

(b) Which of the relations from Question 3 are equivalence relations?

(c) Let $A = \{1,2,3,4\}$, and consider the relation $a \sim b$ whenever $ab \not\equiv 0 \pmod 3$.

i. Is $\sim$ an equivalence relation?

ii. On the grid $A \times A$, shade in the region that represents $\sim$.

5. The **arity** of a relation is the number of arguments it takes as input. For example, binary relations are 2-ary relations. Let $A = \{1,2,3,4,5\}$ and let $R \colon A \times A \times A \to \{\text{true, false}\}$ be the 3-ary relation given by $R(a,b,c) = \text{true}$ whenever $a+b \equiv c \pmod 5$.

(a) Is the binary relation $S(a,b) = R(a,a,b)$ an equivalence relation?

(b) Is the binary relation $T(a,b) = R(a,5,b)$ an equivalence relation?

(c) Define functions $P_1 \colon A \times A \to A$ and $P_2 \colon A \times A \to \{\text{true, false}\}$ so that $R(a,b,c) = P_2(P_1(a,b),c)$.

(d) Make a suggestion on how to represent R graphically.

6. Figure 1 shows different ways how to introduce partitions in a 5-element set (see `https://bit.ly/3bFBJdr`).

   (a) How many equivalence relations are there in a set of 2, 3 or 4 elements?

   (b) Two partitions $P_1 = (C_1, C_2, \ldots, C_k)$ and $P_2 = (D_1, D_2, \ldots, D_\ell)$ of the same set S are in relation R, if $k = \ell$ (the number of parts is equal), and also they represent the same way how to express the number $n = |S|$ as a sum of positive integers. (For example, in Figure 1 all 15 partitions colored brown have 3 parts each and they separate the set into pieces $5 = 2 + 2 + 1$.)
   Is the binary relation R between the partitions of S itself an equivalence relation?

   (c) In how many ways can one express number 6 as a sum of (one or more) positive integers? (The order in the sum does not matter; $1+1+2+2$ is same as $1+2+1+2$).

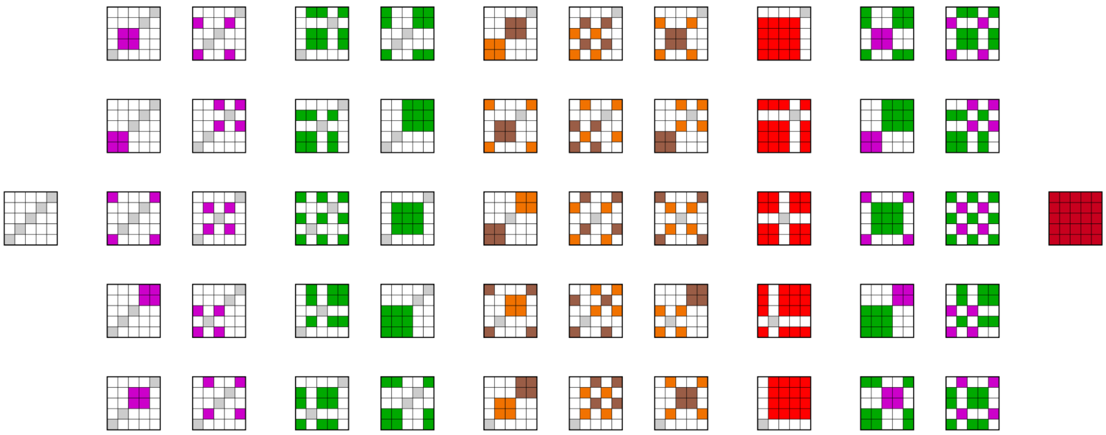


Figure 1: Equivalence relations on a 5-element set.

7. Let f, g be functions $\mathbf{Z}^+ \to \mathbf{R}$ from positive integers to reals. Define a relation R so that $(f, g) \in R$ iff f is in $\Theta(g)$.

   (a) Is relation R reflexive, symmetric or transitive?

   (b) Is R an equivalence relation?

   (c) Are functions $f(n) = \sum_{i=1}^{n} i^2$ and $g(n) = n^3 \cdot (1 + 0.99 \cdot \sin n)$ such that $(f, g) \in R$? How about $(g, f) \in R$?

   Note. The relation symbol R should not be confused with the set of all real numbers $\mathbf{R}$ in this exercise.

8. Take a look at the GCD-related examples (various teorems about the greatest common divisor) and proofs by induction in the Coq-section under `https://bit.ly/3qYAVH8`. (Under H7.Q4.)