Discrete Structures

Worksheet 7

Spring 2021
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1. **Warm up:** Answer the following questions.
   (a) What is the difference between *induction* and *strong induction*?
   (b) What does it mean for a set to be *well-ordered*?
   (c) What is the *lexicographic* ordering on a pair of ordered sets?
   (d) What is a *triangulation* of a *polygon*?

2. Use induction to prove that, for all $n \in \mathbf{N}$:
   (a) $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
   (b) $\sum_{k=1}^{n} 6k^2 = 2n^3 + 3n^2 + n$
   (c) $\sum_{k=1}^{n} (3k-1)(3k+2) = 3n^3 + 6n^2 + n$

3. Use induction to prove that:
   (a) $2n + 1 < 2^n$ for all $n \in \mathbf{N}_{\geqslant 3}$
   (b) $n! > 2^n$ for all $n \in \mathbf{N}_{\geqslant 4}$
   (c) $n^3 \leqslant 3^n$ for all $n \in \mathbf{N}$

4. Use induction to prove that:
   (a) $2^n + 3^n$ is divisible by 5 for all odd $n \in \mathbf{N}$
   (b) $5^n + 2 \cdot 11^n$ is divisible by 3 for all $n \in \mathbf{Z}_{\geqslant 0}$
   (c) $4^n + 5^n + 6^n$ is divisible by 15 for all odd $n \in \mathbf{N}$

5. Let $a_=1$, $a_2 = 8$, and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geqslant 3$. Use strong induction to prove that $a_n = 3 \cdot 2^{n-1} + 2 \cdot (-1)^n$ for all $n \in \mathbf{N}$.

6. Let $a_0 = 3$, $b_0 = 4$ and $c_0 = 5$. If

$$a_n = a_{n-1} + 2, \qquad b_n = 2a_{n-1} + b_{n-1} + 2, \qquad c_n = 2a_{n-1} + c_{n-1} + 2$$

   for all $n \in \mathbf{N}$, use strong induction to prove that $c_n - b_n$ is constant for all $n \in \mathbf{N}$.

7. Let $A = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. Find and prove the formulas for the entries of $A^n = \left(\begin{smallmatrix} a_n & b_n \\ c_n & d_n \end{smallmatrix}\right)$.

8. Find an inverse of a modulo m for each of these pairs of relatively prime integers.

   (a) $a = 4$, $m = 9$

   (b) $a = 19$, $m = 141$

   (c) $a = 55$, $m = 89$

   (d) $a = 89$, $m = 232$

9. Denote $A_1 = \{1\}$, $A_2 = \{1, 2\}$, etc. In general, $A_k = \{1, 2, \ldots, k\}$.
   By $A \oplus B = (A - B) \cup (B - A)$ we denote the symmetric difference of two sets. Consider this set:

$$S = \bigoplus_{j=1}^{100} A_{2j-1}.$$

   - Find the 10 smallest elements of S.
   - Describe the set S using the set-builder notation.

10. Some questions about expressions with parentheses.

    - Check that there are exactly 5 ways how to parenthesize an expression with three binary operations (here $\circ$ denotes some binary operator):

$$((a \circ b) \circ c) \circ d;\ \ (a \circ (b \circ c)) \circ d;\ \ (a \circ b) \circ (c \circ d);\ \ a \circ ((b \circ c) \circ d);\ \ a \circ (b \circ (c \circ d)).$$

    - Describe the number of ways how to parenthesize a slightly longer expression:

$$a \circ b \circ c \circ d \circ e.$$

      Can you express this with a recurrent formula?
    - Assume that somebody erased all the letters and operands, and only left the parentheses. Describe rules how to check, if the sequence of parentheses looks "feasible". For example, (()(())) is a feasible sequence of matching parentheses, but)((()(is not.

11. Find the first 6 members of this infinite sequence $(C_0, C_1, C_2, \ldots)$:

$$\begin{cases} C_0 = 1, \\ C_{n+1} = \sum_{i=0}^{n} (C_i \cdot C_{n-i}). \end{cases}$$

    Write the first 7 values $C_0, C_1, C_2, C_3, C_4, C_5$. (The members of this sequence are called *Catalan numbers*.)

12. Define the following sequence:

$$\begin{cases} x_0 = 1, \\ x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right), \text{ if } n \geq 0 \end{cases}$$

    - Write a small software program to compute (x_n), find the numeric value of x_{20}.
    - What can be the value of the limit

$$L = \lim_{n \to \infty} x_n$$

      assuming that it exists?

13. Find all solutions, if any, to each of the following systems of congruences.

    (a) $x \equiv 5 \pmod{6}$, $x \equiv 3 \pmod{10}$, $x \equiv 8 \pmod{15}$

    (b) $y \equiv 7 \pmod{9}$, $y \equiv 4 \pmod{12}$, $y \equiv 16 \pmod{21}$

14. In this question all numbers are integers. In Python:

    (a) Write a function `div(a,b)` that outputs a list `[c,d]`, where `c` is the quotient and `d` is the remainder when `a` is divided by `b`.

    (b) Write a function `gcd(m,n)` that outputs the greatest common denominator of `m` and `n`, and that uses `div` in a non-trivial way.