Discrete Structures

Worksheet 4

Spring 2021
January 27/29

---

1. **Warm up:** Answer the following questions.

   (a) Find the first five terms of the sequences $\{a_n\}$ and $\{b_n\}$, where
   $$a_n = \tfrac{1}{2}(\cos(n\pi) + 1) \qquad , \qquad b_n = a_1 + a_2 + \cdots + a_n.$$

   (b) What does the sequence $\{c_n\}$ converge to, for $c_n = \frac{3n^2-7n+4}{2n^2+n-1}$?

   (c) True / False: The sequence $\{\frac{1}{n}\}$ converges.

   (d) True / False: The sequence $\{1 + \frac{1}{2} + \cdots + \frac{1}{n}\}$ converges.

2. Consider the sequence $\{a_n\} = \{0, 3, 6, 9, 12, \dots\}$.

   (a) Define $\{a_n\}$ using a recurrence relation in which a_n is

   i. defined in terms of a_{n-1}, ii. defined in terms of a_{n-1} and a_{n-2}.

   (b) Compute the sum $\displaystyle\sum_{k=1}^{n}(a_k)^2$.

   (c) Consider the sequence $\{b_n\}$ given by $b_0 = 5$ and $b_n = a_n + 2b_{n-1} + 4$. Find a solution to this recurrence relation.

3. Let $\{a_n\}$ be the sequence for which every term is a string in the alphabet of 26 lower-case letters. Specifically,

$a_1 = \texttt{a}$, $a_2 = \texttt{b}$, ... , $a_{26} = \texttt{z}$, $a_{27} = \texttt{aa}$, $a_{28} = \texttt{ba}$, $a_{29} = \texttt{ca}$, ... $a_{52} = \texttt{za}$, $a_{53} = \texttt{ab}$, $a_{54} = \texttt{bb}$, ...

   (a) Find indices n_1, n_2, n_3, n_4 such that:

   i. $a_{n_1} = \texttt{ad}$ ii. $a_{n_2} = \texttt{ada}$ iii. $a_{n_3} = \texttt{cab}$ iv. $a_{n_4} = \texttt{barb}$

   (b) Let f be the function that takes in a letter ℓ and that outputs the index n such that $a_n = \ell$, that is, $a_{f(\ell)} = \ell$. What is the domain and range of f?

   (c) Let $s = s_1 s_2 \cdots s_k$ be a string of k letters. Using f, find n_s such that $a_{n_s} = s$.

   Hint. You may need to use order relation in the alphabet. For example, $\displaystyle\sum_{s_i < \texttt{d}} 26 = 26+26+26 = 78$ denotes summation for s_i alphabetically preceding `d`; that is $s_i \in \{\texttt{a}, \texttt{b}, \texttt{c}\}$.

   Note. It is sometimes useful to order strings grouping them by the last letter, then by second-to-last letter, and so on. It is called *reflected lexicographic order*; it allows linguists to locate words with similar endings – `https://bit.ly/3of6vy8`.

4. Consider the sum $\displaystyle\sum_{k=10}^{50}(1-k)(2k^2+5)(4k^3-1)$. In Python:

   (a) Compute the sum using `sum(map(...))` on the list `range(51)`.

   (b) Define the summands in the sum in a list `L` of length 51, and compute the sum using `reduce(...)` from the package `functools` on `L`.

5. Evaluate the following expressions.

(a) $\sum_{i=0}^{7} \sum_{j=0}^{10} ij^2$ (b) $\prod_{i=0}^{7} \sum_{j=0}^{10} ij^2$ (c) $\sum_{i=0}^{5} \prod_{j=0}^{6} \sum_{k=0}^{7} (i+j+k)$

6. Answer the following questions regarding infinite geometric progressions and decimal notation:

(a) Express this as a rational number: $\frac{1}{1000} + \frac{1}{1000^2} + \frac{1}{1000^3} + \ldots$.

(b) Find the rational number p/q having this decimal notation: $p/q = 0.(027) = 0.027027027\ldots$ (infinite fraction having a 3-digit period "027").

(c) Compute the infinite decimal representation of $(64 \cdot 37)^{-1}$. How many digits precede the period of this eventually periodic decimal fraction? How long is the period?

7. Use functions and injectivity and surjectivity to answer parts (a), (b) in your own words.

(a) What does it mean for a set to be *countable*?

(b) What does it mean for two sets to have the same *cardinality*?

(c) Prove that the intervals $(0, 1)$ and (a, b) have the same cardinality, for any $a < b \in \mathbf{R}$.

(d) Prove that $|\mathbf{N}| \leqslant |(a, b)|$ by defining an injection $\mathbf{N} \to (a, b)$, for any $a < b \in \mathbf{R}$.

8. All pairs of positive integers are enumerated as shown in Figure 1.

	j=1	j=2	j=3	j=4	j=5				
i=1	1	2	4	7	11	16	22		...
i=2	3	5	8	12	17	23			...
i=3	6	9	13	18	24				...
i=4	10	14	19	25					...
i=5	15	20	26						...
	21	27							...

Figure 1: A bijective function $F : \mathbf{Z}^+ \times \mathbf{Z}^+ \to \mathbf{Z}^+$.

(a) Which row i and which column j is the value 2021 in this table?

(b) Which number is written in the cell $(i, j) = (1000, 1000)$.

(c) Evaluate the formula $F(i, j) = \frac{(i+j-2)(i+j-1)}{2} + i$ for $i = j = 1000$. (You should get the same value as in the previous item.)

Note. Compare this enumeration with (Rosen, p.183), where the rational numbers are visited sweeping diagonally back and forth (in both directions). There are multiple "natural" ways to define a bijection $F : \mathbf{Z}^+ \times \mathbf{Z}^+ \to \mathbf{Z}^+$.

9. Let A, B, C be sets with A, B countable and $|C| = |\mathbf{R}|$.

(a) Prove that $A \cup B$ is countable.

(b) Prove that $|A \cup C| > |\mathbf{N}|$.

(c) Prove that $A \times B$ is countable.

(d) Prove that $|A \times B| = |\mathbf{N}|$. You may assume that $|\mathbf{N} \times \mathbf{N}| = |\mathbf{N}|$.

(e) Prove that $|\underbrace{\mathbf{N} \times \cdots \times \mathbf{N}}_{n \text{ times}}| = |\mathbf{N}|$.

10. Let X, Y, Z be sets, and suppose that there exist:

   - an injection $f\colon X \to Y$,
   - an injection $g\colon Y \to Z$,
   - a surjection $h\colon Z \to X$.

   Using these assumptions, answer the following questions.

   (a) Prove that X, Y, and Z all have the same cardinality.

   (b) Construct an injection $X \cup Y \to Z \times \{0, 1\}$.

   (c) Let $X = \mathbf{N}$. Find examples of sets Y, Z and functions f, g, h that satisfy the given assumptions, and so that none of X, Y, Z are the same.

11. (a) Find two 2×2 matrices A, B such that $AB = BA$.

   (b) Let $A = \left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right]$. Compute the matrix power A^n for any $n \in \mathbf{N}$.

   (c) The *Hadamard product* of two $m \times n$ matrices A, B is $(A \circ B)_{ij} = A_{ij} \cdot B_{ij}$. Prove that this is a commutative operation.

12. For this question, a matrix in Python is a list of lists, for example $\left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right]$ = `[[1,2],[0,1]]`.

   (a) Define a function `add(`A`,`B`)` that takes in two matrices A, B of the same size and returns their sum.

   (b) Define a function `mult(`A`,`B`)` that takes in two matrices A, B of the appropriate size and returns their product.

   (c) Define a function `pow(`A`,`n`)` that takes in a square matrix A and returns its nth power. Use `mult` from part (b).

13. Let f be a function mapping positive integers $\mathbf{Z}^+$ to positive integers. In other words, $f(1), f(2), f(3), \ldots$ is an infinite sequence, its terms are elements from $\mathbf{Z}^+$. Translate the following predicate expressions into plain English:

   (a) $\forall c \in \mathbf{Z}^+\ \forall a \in \mathbf{Z}^+\ \exists b \in \mathbf{Z}^+\ (c > a\ \to f(c + b) = f(c))$.

   (b) $\exists a \in \mathbf{Z}^+\ \exists b \in \mathbf{Z}^+\ \forall c \in \mathbf{Z}^+\ (c \geq a\ \to\ f(c) = b)$.

   (c) $\exists a \in \mathbf{Z}^+\ \forall c \in \mathbf{Z}^+\ \exists b \in \mathbf{Z}^+\ (c \geq a\ \to\ f(c) = b)$.

   (d) $\forall c \in \mathbf{Z}^+\ \forall a \in \mathbf{Z}^+\ \exists b \in \mathbf{Z}^+\ (c > a\ \wedge\ f(c) = b)$.

14. Download the file `worksheet04-traffic.v` from ORTUS (under Week4). It contains 3 statements about the plane and rail traffic between cities A, B, C (the situation is similar to Homework 2 Question 2). Please complete the three proofs in this file. (Homework 4 contains three more statements to prove.)

   Hint. If Coq IDE crashes when you double-click the file, try opening Coq IDE application without any file and copy-paste the text from `worksheet04-traffic.v` into the editor. (For MS Windows users there is also a way to run Coq from Visual Studio Code.)