Discrete Structures

Worksheet 3

Spring 2021
January 20/22

Some notation is not agreed upon by everyone, for sets A, B.

- their **difference** is $A - B = A \setminus B = \{x \in A \text{ and } x \notin B\}$
- their **symmetric difference** is $A \oplus B = A \triangle B = \{x \in A \text{ or } x \in B, \text{ but } x \notin A \cap B\}$

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1. **Warm up:** Answer the following questions.

   (a) What are the sizes of the following sets:

   $$A = \{x \ : \ x(x-2)(x-1) = 0\} \qquad B = \mathcal{P}(A) \qquad C = A \times B$$

   (b) Describe the following sets using a single symbol for each:

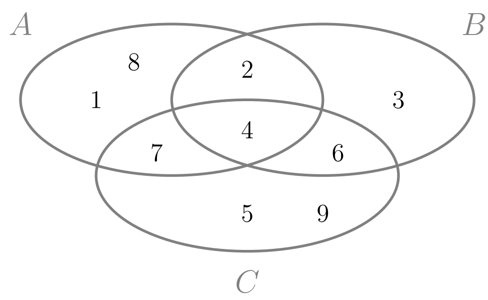
   $$X = \{\tfrac{x+y}{z} \ : \ x, y \in \mathbf{Z},\ z \in \mathbf{N}\} \quad Y = \{\tfrac{a^2-b^2}{a-b} \ : \ a, b \in \mathbf{N},\ a \neq b\} \quad Z = \{10q \ : \ q \in \mathbf{Q}\}$$

   (c) Determine which of the following statements are True and which are False.

   $$A \cup \emptyset = A \qquad \{\emptyset\} = \emptyset \qquad (A \cup B) - C = (A - C) \cup (B - C) \qquad \overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$$

   Note. Notation $\{\frac{x+y}{z} \ : \ x, y \in \mathbf{Z},\ z \in \mathbf{N}\}$ etc. is named "extended set-builder notation"; is introduced in (Rosen, 2.3.1, Definition 4, p.149); also `https://bit.ly/3qAlZOS`.

2. Consder the sets A, B, C of natural numbers, presented as a Venn diagram.



   Write out all the elements contained in the following sets.

   (a) $A \cup B$
   (b) $C - B$
   (c) $C \cap A$
   (d) $(A \cup C) - (B \cap C)$
   (e) $A \cap B \cap C$
   (f) $(A \cap B \cap C) - C$
   (g) $\overline{A \cup B}$
   (h) $\overline{A} \cup \overline{B}$

3. (Adapted from Rosen ex. 2.2.35) Prove the following set identity:

   $$\overline{A \cup B} \cap \overline{B \cup C} \cap \overline{A \cup C} = \overline{A} \cap \overline{B} \cap \overline{C}.$$

   Choose the proof method you prefer. For example:

   - Use set identities (Rosen, p.136)
   - Build a membership table (Rosen, p.138)
   - Shade regions in two Venn diagrams and compare the left-side and the right-side.

4. *(Adapted from Rosen ex. 2.2.56)* Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ for each of the following A_i, where i is a natural number.

   (a) $A_i = \{i, i+1, i+2, \dots\}$
   (b) $A_i = \{0, i\}$
   (c) $A_i = \{-i, i\}$
   (d) $A_i = (0, i)$
   (e) $A_i = [-i, i]$
   (f) $A_i = (i, \infty)$
   (g) $A_i = [i, \infty)$
   (h) $A_i = \{-i, -i+1, \dots, i-1, i\}$

5. Let A, B be sets, and let $f \colon A \to B$ be a function.

   (a) Using logical symbols, express the following statements.
      i. f is injective
      ii. f is surjective
      iii. the range of f is a proper subset of B
      iv. there is an element in B whose preimage contains three distinct elements

   (b) Let $f_1 \colon A_1 \to B_1$ be a function, with $A_1 \subseteq A$, $B_1 \subseteq B$, and $f_1(a) = f(a)$ for every $a \in A_1$.
      i. Prove that if f is injective, then f_1 is injective.
      ii. Prove that if f_1 is surjective and $B_1 = B$, then f is surjective.

   Note. Recall that the set B from the function $f : A \to B$ is called the *codomain* of f, but the set $\{b \in B \mid \exists a \in A\ (f(a) = b)\}$ is called the *range* of f. Function is surjective iff the range is the same as the codomain.

6. (a) Prove that $f \colon \mathbf{R} \to \mathbf{R}$ given by $f(x) = x$ is injective.
   (b) Prove that $g \colon \mathbf{R} \to \mathbf{R}^2$ given by $g(x) = (x, 0)$ is injective.
   (c) Prove that $k \colon \mathbf{R}^2 \to \mathbf{R}$ given by $k(x, y) = x$ is surjective.

7. Consider the following functions.

$$
\begin{aligned}
f \colon \mathbf{R} &\to \mathbf{R} \\
x &\mapsto x^3
\end{aligned}
\qquad
\begin{aligned}
g \colon \mathbf{R} &\to \mathbf{R} \\
x &\mapsto 5x - \tfrac{4}{3}
\end{aligned}
\qquad
\begin{aligned}
h \colon \mathbf{N} &\to \mathbf{Z} \\
n &\mapsto \begin{cases} \frac{n}{2} & n \text{ is even} \\ \frac{-n-1}{2} & n \text{ is odd} \end{cases}
\end{aligned}
$$

   (a) Find inverses of each of them.
   (b) Prove by construction that the all the functions f, g, h are surjections.

8. Compute the range of the following functions.

   (a) $f \colon \mathbf{R} \to \mathbf{R}$ given by $f(x) = \lfloor 2x + 5 \rfloor$
   (b) $g \colon [0, \infty) \to \mathbf{R}$ given by $g(x) = |x + 3| - 1$
   (c) $h \colon (-\infty, 1] \to \mathbf{R}$ given by $h(x) = e^x \sin^2(x)/2$
   (d) $k \colon \mathbf{R} \to \mathbf{R}$ given by $k(x) = \arctan(x)$

9. Recall **Russel's paradox**: let X be the set of all sets, and let $S = \{Y \in X : Y \notin Y\}$. Then the claim $S \in S$ is equivalent to the claim $S \notin S$. Use Russel's paradox to prove that $0 = 1$.

10. Do some experiments in the Coq environment.

    (a) Tautologies from SUNY Buffalo CSE 191 File (the file in Week3 in ORTUS).

    (b) Two Nonconstructive Proofs of the Same Lemma (Week3 in ORTUS).

    (c) Proofs from Rosen2019 textbook (1.7.5, 1.7.6) (Week3 in ORTUS).

11. Optionally, you can do some set/list operations in Python to solve numeric examples.

    (a) **Check universal quantifier using "all".**
    "The square of a positive integer $n \in [1; 1000]$ *never gives remainder* 3 *when divided by* 7 *(but it does sometimes give remainder* 2 *when divided by* 7*)"*:

    $$\forall n \in \mathbf{Z}^+ \; \forall k \in \mathbf{Z} \left(1 \leq n \leq 1000 \;\rightarrow\; n^2 \neq 7k+3\right).$$

    Run from Python command-line:

    ```
    all(map(lambda x: x**2 % 7 != 3, range(1,1001)))
    ```

    You should get output `True`.

    (b) **Use "map".**
    "The last digits of the numbers in this set $\{7x \mid x \in \mathbf{Z} \wedge x \in [a, a+10)\}$ *are all different."* Run from Python command-line:

    ```
    a = 2021
    list(range(a,a+10))
    list(map(lambda x: 7*x % 10, range(a,a+10)))
    len(set(list(map(lambda x: 7*x % 10, range(a,a+10)))))
    ```

    You should get output `[7, 4, 1, 8, 5, 2, 9, 6, 3, 0]` and `10`.

    (c) **Use Cartesian product and "filter".**
    "The equation $u^2+v^2 = 113$ *has an integer solution* (u, v)*, but the equation* $u^2+v^2 = 127$ *does not."* (See Fermat's Christmas Theorem, `https://youtu.be/DjI1NICfjOk"` – any prime number in the form $4k+1$ can be represented as a sum of two squares in exactly one way.).

    $$\exists u, v \in \mathbf{Z}^+ \left(u^2 + v^2 = 113\right).$$

    Run from Python command-line:

    ```
    from itertools import product
    x = list(product(range(1,12),range(1,12)))
    list(filter(lambda x: x[0]**2 + x[1]**2 == 113, x))
    ```

    You should get this output: `[(7, 8), (8, 7)]`.