Discrete Structures

Worksheet 2

Spring 2021
January 13/15

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1. **Warm up:** Answer the following questions.
   (a) For $x \in \mathbf{Z}$, use the existential quantifier to express "x is even" with logical symbols.
   (b) Given 4 points in the plane, how many different lines exist that go through at least two of them?
   (c) True or False: for all $n \in \mathbf{N}$, the number $3n^2 + 3n + 23$ is prime.

2. Let $P(x, y, z)$ be the statement "dividing x by y leaves a remainder z". Write the following quantifications as sentences in English.
   (a) $\forall\, x \in \mathbf{N}\ P(x, 1, 0)$
   (b) $\exists\, z \in \mathbf{R}\ (\forall\, x \in \mathbf{R}\ \neg P(x, y, z))$
   (c) $\forall\, x \in \mathbf{Z}\ ((P(1, x, z)\ \wedge\ (1 \leqslant x))\ \rightarrow\ ((z = 1)\ \vee\ (z = 0)))$
   (d) $\forall\, x_1 \in \mathbf{R}\ (\forall\, x_2 \in \mathbf{R}_{\neq 0}\ (((\exists\, a \in \mathbf{R}) \wedge (\exists\, b \in \mathbf{R}))\ (P(x_1, x_2, z) \wedge (a < z) \wedge (z < b))))$

3. *(Adapted from Rosen, ex. 1.6.19)* All variables in this question are real numbers. Determine whether each of these arguments is valid. If an argument is correct, what rule of inference is being used? If it is not, what logical error occurs?
   (a) If $n > 1$, then $n^2 > 1$. Suppose that $n^2 > 1$. Then $n > 1$.
   (b) If $n > 3$, then $n^2 > 9$. Suppose that $n^2 \leqslant 9$. Then $n \leqslant 3$.
   (c) If $n > 2$, then $n^2 > 4$. Suppose that $n \leqslant 2$. Then $n^2 \leqslant 4$.
   (d) If $n > 0$, then $n + 1 > 0$. Suppose that $n > 0$. Then $n + 3 > 0$.

4. Write the negation of each of the following statements. (If the statement is an implication, also write its contrapositive.) Use informal English.
   (a) You are rich or you are happy.
   (b) There is a person older than 120.
   (c) The square root of $2x + 1$ is an integer if $3x - 1$ is a perfect square.
   (d) If a function f has a root at $x = a$, then for some $\epsilon > 0$ either $f(a - \epsilon) > 0$ and $f(a + \epsilon) < 0$, or $f(a - \epsilon) < 0$ and $f(a + \epsilon) > 0$.

5. Write the negation of each of the following statements.
   (a) If I am older than 21 then I am older than 18.
   (b) Everyone living is Chicago was born in Chicago.
   (c) If $a, b \in \mathbf{N}$, then either $\frac{a}{b} < 1$ or $\frac{a}{b} = 1$ or $\frac{a}{b} = 1$.
   (d) If n is even, then $3n + 6$ is even.

6. Let x and y be distinct positive real numbers. Show that $\frac{x}{y} + \frac{y}{x} > 2$.

7. All variables are integers in this question. Consider the following statements:

- $P(x, y)$: "x is divisible by y", or equivalently, "y divides x", or equivalently, "there exists q with $x = qy$"
- $Q(x, y)$: "x and y have no common factors", or equivalently, "if $P(x, d)$ and $P(y, d)$ are true, then $d = \pm 1$"

(a) Using $P(x, y)$, write a quantification $R(x)$ that asserts x is a prime number.

(b) Write the following statement in English:

$$S(x) \ : \ \forall\, a, b, c \in \mathbf{Z}\ (R(x)\ \wedge\ (ax = bc))\ \rightarrow\ (P(b, x)\ \vee\ P(c, x))$$

(c) Use $S(x)$ to prove the following statement:

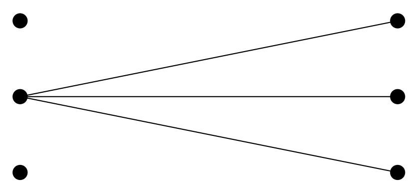
$$T(x, y) \ : \ (R(x)\ \wedge\ P(y^2, x))\ \rightarrow\ P(y, x)$$

(d) Write the following statement in English:

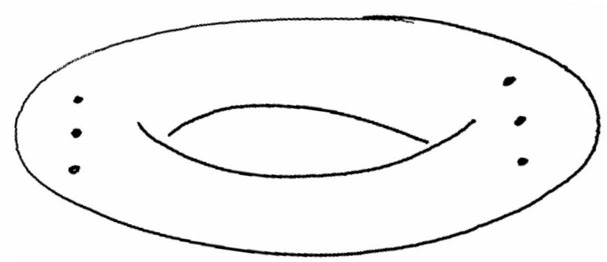
$$U(x, y) \ : \ Q(x, y)\ \wedge\ (\sqrt{2} = \tfrac{x}{y})$$

(e) Prove that $U(x, y)$ is a contradiction for all integers x, y.

8. (a) Prove by exhaustion that it is not possible to connect all the dots on the right to all the dots on the left without at least two lines crossing each other. The lines do not need to be straight.



(b) Prove by construction that is is possible to connect all the dots on the right to all the dots on the left without (not necessarily straight) lines crossing each other, if they lie on the surface of a torus.



9. Denote by $\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$ the set of all positive integers. Let $M(x, y, z)$ be the predicate "the numbers x, y, z satisfy the equality $x \cdot y = z$ for x, y, z in $\mathbf{Z}^+$. Express the following predicates and propositions (using only the given predicate, Boolean connectors and predicates expressed in earlier items).

   (a) $U(x, y, z)$: the numbers x, y, z satisfy $x/y = z$
   (b) $One(x)$: the number x equals 1
   (c) $I(x, y)$: the numbers x and y are the same
   (d) $Divides(x, y)$: the number x divides y
   (e) $Square(x)$: the number x is a perfect square
   (f) $GCD(x, y, z)$: the number z is the greatest common divisor of x and y
   (g) $Prime(x)$: the number x is a prime number
   (h) $P(x)$: if the number x has exactly three positive divisors, then it is a perfect square

10. Test if the numbers are rational or irrational. Justify your answer.

   (a) $\sqrt{5}$
   (b) $\sqrt{6}$
   (c) $\sqrt{2} + \sqrt{3}$
   (d) $\sqrt[3]{7}$
   (e) $\log_8 16$
   (f) $\log_2 3$

11. Prove or disprove each of the following statements.

   (a) If x and y are both irrational, then $x + y$ is also irrational.
   (b) If $x + y$ and xy are both rational, then x and y are also rational.
   (c) If x^2 is irrational then x^3 is also irrational.

12. *(Adapted from IMO 2001./2002.)* Two players interchangeably write natural numbers on the board. They can write numbers from 1 to 8; it is not allowed to write divisors of the numbers already written. The player who cannot make a move at his/her turn, loses. Show that the player who makes the first move can win.

13. Try out the following two proofs of the same tautology in Coq (see `https://bit.ly/35RA15B`):

$$A \to (A \to B) \to (B \to C) \to C.$$

```
Theorem backward_mind: (forall A B C : Prop, A -> (A->B) -> (B->C) -> C).
Proof.
  intros A B C.
  intros hyp_A_true.
  intros hyp_A_implies_B.
  intros hyp_B_implies_C.
  apply hyp_B_implies_C.
  apply hyp_A_implies_B.
  exact hyp_A_true.
Qed.
```

```
Theorem forward_mind: (forall A B C : Prop, A -> (A->B) -> (B->C) -> C).
Proof.
  intros A B C.
  intros hyp_A_true.
  intros hyp_A_implies_B.
  intros hyp_B_implies_C.
  pose (hyp_A_implies_B hyp_A_true) as hyp_B_true.
  pose (hyp_B_implies_C hyp_B_true) as hyp_C_true.
  exact hyp_C_true.
Qed.
```

14. Non-constructive logic: How $\neg\neg a \to a$ (also known as NNPP) implies Peirce's Law (see `https://bit.ly/3oEkXkf`). Try out this in Coq:

```
Require Import Classical_Prop.
Lemma Peirce_Law: forall a b: Prop, ((a -> b) -> a) -> a.
Proof.
  intros a b.
  intros H1.
  apply NNPP.
  unfold not.
  intros aFalse.
  apply aFalse.
  apply H1.
  intros aTrue.
  contradiction (aFalse aTrue).
Qed.
```

If you wish to learn more about non-constructive logic, see the exercises in `https://bit.ly/39nWIR2`. They suggest that you prove a "five-star exercise" showing that one non-constructive axiom leads to all the other non-constructive axioms (and you can choose, which non-constructive axiom you start with). Here is the (incomplete) list of axioms in non-constructive propositional logic:

$$\begin{cases} ((P \to Q) \to P) \to P. & \text{(1) Peirce's Law} \\ \neg\neg P \to P. & \text{(2) Double negation elimination} \\ \neg(\neg P \land \neg Q) \to P \lor Q. & \text{(3) A non-constructive variant of De Morgan's Law} \\ (P \to Q) \to (\neg P \lor Q). & \text{(4) Convert implication into disjunction} \\ P \lor \neg P & \text{(5) Law of excluded middle} \\ (P \to Q) \leftrightarrow (\neg Q \to \neg P) & \text{(6) Counterpositive of an implication} \end{cases}$$

We just showed that (2) $\to$ (1) – assuming double negation elimination, one can prove Peirce's Law.
One can also prove with Coq that they circularly follow each other: (1) $\to$ (2) $\to$ (3) $\to$ (4) $\to$ (5) $\to$ (6) $\to$ (1). Also see the Example 12, (Rosen2019, p.102) – the game "Chomp" about the poisoned cookie (building a non-constructive strategy for a game).