Compact course notes

Pure Math 955 Winter 2014

The Atiyah–Singer index theorem

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Note: Sections marked with a vertical line on the left side contain background information not presented in class.

0.1 Introduction

The Atiyah–Singer index theorem (1962-8) is the most important mathematical result of the 20th century. It combines geometry, topology, algebra, and analysis. It is the generalization of several theorems, among them the following:

· Gauss–Bonnet: For M^2 compact, oriented in $\mathbf{R}^3$, $\int_{M^2} KdA = 2\pi\chi = 2\pi(2g-2)$, where K is the Gauss curvature and χ is the Euler characteristic of M^2.

· Gauss–Bonnet–Chern: For M^{2n} a compact, oriented Riemann $2n$-manifold, $\int_{M^{2n}} e(M) = (2\pi)\chi$, where e is the Euler form.

· Riemann–Roch: For X a Riemann surface and L a holomorphic vector bundle over X, $h^0(X,L) - h^0(X, N^{-1}\otimes K) = \deg(L) + 1 - g$, where h^0 is the dimension of the space of holomorphic sections, and g is the genus.

· Riemann–Roch–Hirzebruch: For X a compact oriented manifold, E a holomorphic vector bundle, the holomorphic Euler characteristic of E is $\chi(X,E) =$ (alternating sum of dimensions of sheaf cohomology groups) $= \int_X c(E)Td(X)$, where c is the Chern class and Td is the Todd character.

· Hirzebruch signature theorem: Let M^{4n} be a compact, oriented manifold. Then there exists an intersection form $H^{2n}(M,\mathbf{R})$, a symmetric bilinear form of the signature (p,q). The theorem says that $\text{sign}(M) = p - q = \int_{M^{4n}} L(M)$, where L is the L-genus.

· Lefschetz fixed-point theorem.

All of these results are of the form (integral of curvature stuff) = (topological invariants). The Atiyah–Singer theorem states that this also equals the index of an elliptic operator, which is

$$\text{ind}(p) = \dim(\ker(p)) - \dim(\text{coker}(p)) = \int_M (\text{char. classes of } E,F)\cdot(\text{homotopy class of symbol of } p),$$

where, given two vector bundles $E \xrightarrow{\pi} M$ and $F \xrightarrow{\tau} M$, the map $p : E \to F$ relates the two total spaces.

Remark 0.1.1. Consider a "baby" version of the Atiyah–Singer theorem. Suppose V^n, W^m are finite-dimensional real vector spaces, and $p : V \to W$ is a linear map. Then $\ker(p)$ is a subspace of V and measures the failure of p to be injective. Similarly, $\text{coker}(p) = W/\text{Im}(p)$ is a quotiont space of W that measures the failure of p to be surjective. Then

$$\text{ind}(p) = \dim(\ker(p)) - \dim(\text{coker}(p)) = \dim(\ker(p)) - (\dim(W) - \dim(\text{Im}(p))) = \dim(V) - \dim(W).$$

So, with reference to solutions for the above, ker is an obstruction to solutions existing, and coker is an obstruction to solutions being unique.

Remark 0.1.2. In this course, we will make the following assumptions:
· All manifolds are Hausdorff and 2nd-countable (so we have partitions of unity)
· All manifolds are smooth and with a fixed smooth structure
· All maps are smooth (i.e. C^∞), unless otherwise stated

Definition 0.1.3. A *smooth structure* on a manifold M^n is a collection of atlases with smooth transition functions, where an atlas is a collection of charts that cover all of M^n, such that the union of any two of these atlases is also an atlas with smooth transition functions.

1 Vector bundles

1.1 k-vector bundles

Definition 1.1.1. Let $r \geqslant 0$ be an integer. A *smooth* $\mathbf{K}$*-vector bundle* of rank r over a smooth manifold M^n, termed the *base space*, is given by (E, M, π), where

· E is a smooth manifold, termed the *total space*

· $\pi : E \to M$ is a smooth surjective submersion (i.e. $(\pi_*)_e : T_eE \to T_{\pi(e)}M$ is surjective for all $e \in E$)

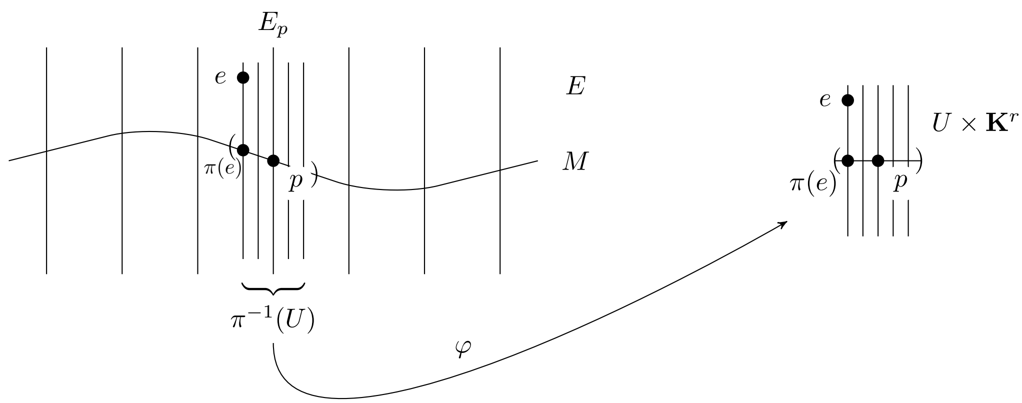
· for all $p \in M$, we call $E_p = \pi^{-1}(p)$ the *fiber of E over p*. Each E_p has the structure of an r-dimensional vector space over $\mathbf{K}$. Notice that $E = \bigsqcup_{p\in M} E_p$, so E is a disjoint union of k-vector spaces.

· for all $p \in M$, there exists a (not unique) open neighborhood $U \ni p$ and a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times \mathbf{K}^r$ such that

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times \mathbf{K}^r \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi_1} & \\ & U & \end{array}$$

commutes, i.e. $\pi_1 \circ \varphi = \pi$, so $\varphi(e) = (\pi(e), f(e))$ for $f : \pi^{-1}(U) \to \mathbf{K}^r$ smooth and such that $\varphi|_{E_p} \to \{p\} \times \mathbf{K}^r$ is a linear isomorphism of $\mathbf{K}$-vector spaces.

What does the above mean? It means that E is a family of r-dimensional vector spaces over $\mathbf{K}$, parametrized by M, such that, locally (near any $p \in M$), this family is a “trivial” cartesian product. That is, we have the following action:



The second-last coondition in the definition above says that φ is fiber-preserving. The last condition says that fibers are mapped to corresponding fibers linearly isomorphically. Further, the pair (U, φ) is called a *local trivialization* of the bundle (E, M, π). note we can always shrink U so that U is the domain of a coordinate chart for M but we don't need to.

It follows from the definition of the vector bundle that $\dim(E) = n + r$, if $\mathbf{K} = \mathbf{R}$, or $\dim(E) = n + 2r$, if $\mathbf{K} = \mathbf{C}$.

Example 1.1.2. Consider the following examples of vector bundles:

· Let M be an n-manifold. The tangent bundle of M is (TM, M, π), which is a rank n real vector bundle on M. The induced charts $T_\varphi : \varphi^{-1}(U) \to \varphi(U) \times \mathbf{R}^n$, where (U, φ) is a chart for M satisfying conditions for a local trivialization.

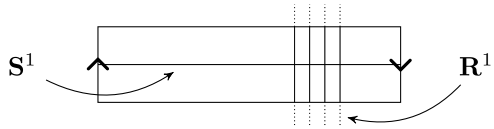
· T^*M is also a rank n real vector bundle over M, the cotangent bundle.

· For $k, \ell \geqslant 0$, $T^k_\ell(M) =$ bundle of type (k, ℓ)-tensors on M. The fiber over p is $(\bigotimes^k T^*_pM) \otimes (\bigotimes^\ell T_pM)$. This is a real vector bundle over M of rank $k + \ell$.

· For $0 \leqslant k \leqslant n$, $\bigwedge^k(T^*M) =$ bundle of k-forms on M. This is a vector bundle of rank $\binom{n}{k}$.

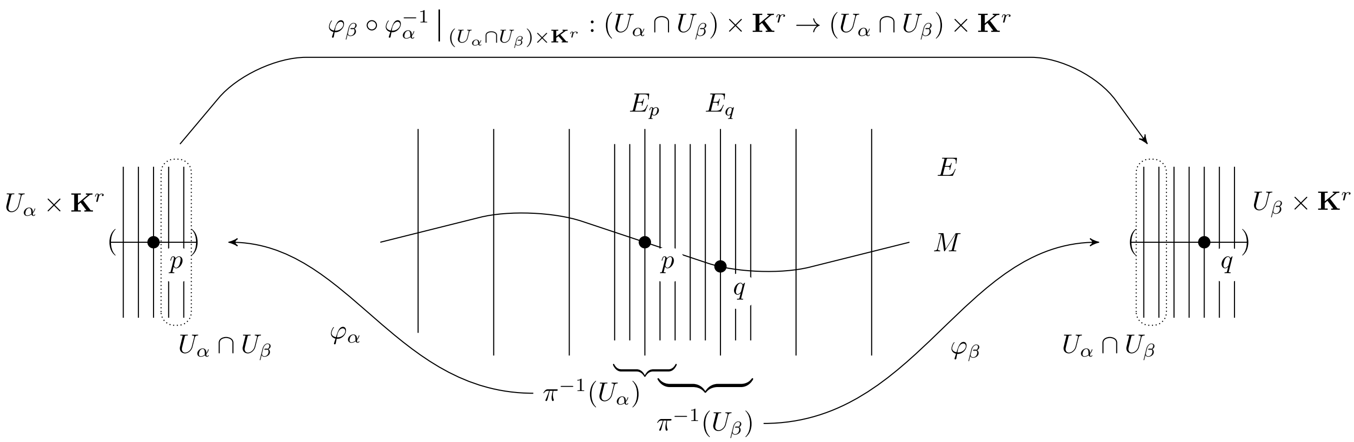
Remark 1.1.3. The above are “intrinsic” vector bundles, which are defined given only the base M. There exist also extrinsic bundles, the most important of which will be the spinor bundle. Another example is the

Möbius bundle, which is a rank-1 non-trivial bundle over $\mathbf{S}^1$.



Definition 1.1.4. A rank-1 vector bundle is called a *line bundle*. The rank r bundle $\pi : M \times \mathbf{K}^r \to M$ by $\pi(p, v) = v$ is called the *trivial rank r bundle* over M. This is called a trivial bundle because there exists a "global trivialization," i.e. a trivialization with domain all of t.

We now need to define an appropriate notion of equivalence (isomorphism) of vector bundles. Before we define this, we will define "gluing cocycles" and "transition maps." To construct the context, first let (E, M, π) be a vector bundle over $\mathbf{K}$ and $(U_\alpha, \varphi_\alpha)$ a cover of M by local trivializations of E, so $M = \bigcup_{\alpha \in A} U_\alpha$.



The map $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a diffeomorphism of $(U_\alpha \cap U_\beta) \times \mathbf{K}^r$ onto itself such that $(\varphi_\beta \circ \varphi_\alpha^{-1})(p, v) = (p, g_{\beta\alpha}(p)v)$ for $p \in U_\alpha \cap U_\beta$ and $g_{\beta\alpha}(p)$ an isomorphism of $\mathbf{K}^r$ onto itself. That is, we have $g_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(r, \mathbf{K})$, which is the gluing cocyle. This allows us to formulate the following definition.

Definition 1.1.5. Given a cover $(U_\alpha, \varphi_\alpha)$ of M by local trivializations of E, the maps $g_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(r, \mathbf{K})$, for all α, β with $U_\alpha \cap U_\beta \neq \emptyset$ are called the *transition functions* for this cover by trivializations.

Consider the following properties of the transition functions, for all α, β, γ such that $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$:
· $g_{\alpha\alpha} : U_\alpha \to GL(r, \mathbf{K})$ is the constant map $g_{\alpha\alpha}(p) = \mathrm{id}_{r\times r}(p)$
· $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$, i.e. $g_{\alpha\beta}(p) = (g_{\beta\alpha}(p))^{-1}$
· $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = \mathrm{id}_{r\times r}$, i.e. $g_{\alpha\beta}(p)g_{\beta\gamma}(p)g_{\gamma\alpha}(p) = \mathrm{id}_{r\times r}(p)$

Now consider $\bigsqcup_{\alpha\in\mathcal{A}} U_\alpha \times \mathbf{K}^r$. Put an equivalence relation on this set by

$$U_\alpha \times \mathbf{K}^r \ni (p_\alpha, v_\alpha) \sim (p_\beta, v_\beta) \in U_\beta \times \mathbf{K}^r \qquad \text{iff} \qquad p_\alpha = p_\beta \text{ and } v_\beta = g_{\beta\alpha}(p)v_\alpha.$$

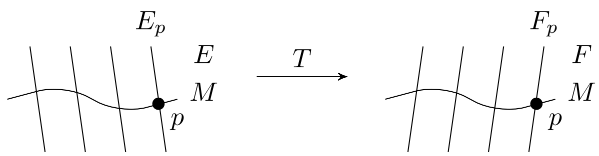
Then the above properties of the transition functions define reflexivity, symmetry, and transitivity, respectively, of the relation $\sim$. Now set $E = \left(\bigsqcup_{\alpha\in\mathcal{A}} U_\alpha \times \mathbf{K}^r\right) \Big/ \sim$.

✠ Exercise 1.1.6.
· Show that E is a smooth manifold. Note it is made up of smooth manifolds glued together by diffeomorphisms.
· Define $\pi : E \to M$ by $\pi : ([(p_\alpha, v_\alpha)]) = p_\alpha$. Show it is well-defined and show π is a smooth surjective submersion.
· Show the natural map $\psi_\alpha : U_\alpha \times \mathbf{K}^r \to \pi^{-1}(U_\alpha)$, given by $\psi_\alpha(p_\alpha, v_\alpha) = [(p_\alpha, v_\alpha)] \in E$ is a diffeomorphism and $\varphi_\alpha = \psi_\alpha^{-1}$ is a local trivialization of E (i.e. E is a rank r $\mathbf{K}$-vector bundle over M).

Definition 1.1.7. Let (E, M, π_E) and (F, M, π_F) be vector bundles of rank k and ℓ, respectively, over the same M. A *vector bundle isomorphism* from (E, M, π_E) to (F, M, π_F) is a smooth map $T : E \to F$ such that

$$\begin{array}{ccc} E & \xrightarrow{\quad T \quad} & F \\ & {\scriptstyle \pi_E} \searrow \quad \swarrow {\scriptstyle \pi_F} & \\ & M & \end{array}$$

commutes (i.e. $\pi_F(T(p)) = \pi_E(p)$), and $T|_{E_p} : E_p \to F_p$ is linear over $\mathbf{K}$. That is, such that the diagram below on the right commutes.



$$\begin{array}{ccc} E_p & \xrightarrow[\text{lin. isom.}]{\varphi_\alpha} & \{p\} \times \mathbf{K}^r \\ {\scriptstyle T}\downarrow & & \downarrow {\scriptstyle \text{linear}} \\ F_p & \xrightarrow[\psi_\beta]{} & \{p\} \times \mathbf{K}^r \end{array}$$

This is all in trems of covers of E, F by local trivializations $(U_\alpha, \varphi_\alpha)$ and (V_β, ψ_β), respectively.

Definition 1.1.8. Let $\underline{\mathcal{H}om}(E, F)$ denote the set of all bundle morphisms from E to F.

Proposition 1.1.9. $\underline{\mathcal{H}om}(E, F)$ is a $\mathbf{K}$-vector space.

<u>*Proof:*</u> Let $T_1, T_2 \in \underline{\mathcal{H}om}(E, F)$, $\lambda \in k$. Define $\lambda T_1 + T_2 : E \to F$ by $(\lambda T_1 + T_2)(e) = \lambda T_1(e) + T_2(e)$, for $e \in E_{\pi(e)}$. Then $(\lambda T_1 + T_2)(E_p) \subset F_p$ for all $p \in M$. This map is clearly linear. It remains to check that it is smooth. ■

Definition 1.1.10. Let E, F be two $\mathbf{K}$-vector bundles over M. Then E, F are termed *isomorphic* if there exists $T \in \underline{\mathcal{H}om}(E, F)$ and $T \in \underline{\mathcal{H}om}(F, E)$ such that $T \circ S = \mathrm{id}_F$ and $S \circ T = \mathrm{id}_E$. When $F = E$, we write $\underline{\mathcal{H}om}(E, E) = \underline{\mathcal{E}nd}(E)$. If $T \in \underline{\mathcal{E}nd}(E)$, then T is called a *bundle endomorphism.* We also have the set $\underline{\mathcal{A}ut}(E)$, which is the space of bundle automorphisms of E. This set is sometimes denoted $\mathcal{G}_E$, called the *gauge transformations* of E.

Definition 1.1.11. For (E, M, π) a $\mathbf{K}^r$-vector bundle, we say that E is *trivial* if there exists a bundle isomorphism $T : E \to M \times \mathbf{K}^r$. It is clear that E is trivial iff it admits a global trivialization.

Now we see why they are called "local trivializations." A local trivialization $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbf{K}^r$ is a bundle isomorphism between $E|_{U_\alpha} = \pi^{-1}(U_\alpha)$ and $U_\alpha \times \mathbf{K}^r$, the trivial $\mathbf{K}^r$-bundle over $U - \alpha$.

Remark 1.1.12. Suppose $T : E \to F$ is a vector bundle morphism. Let $(U_\alpha, \varphi_\alpha)$, (U_α, ψ_α) be open covers of M, trivializing E, F, respectively. We can do this by intersecting the two open covers. Let $g_{\alpha\beta}$, $h_{\alpha\beta}$ be the transition functions of E, F, respectively, with respect to the trivializations. Then a new map is induced by the commutative diagram below.

$$\begin{array}{ccc} E|_{U_\alpha} & \xrightarrow{\varphi_\alpha} & U_\alpha \times \mathbf{K}^r \\ {\scriptstyle T}\downarrow & & \downarrow \\ F|_{U_\alpha} & \xrightarrow[\psi_\alpha]{} & U_\alpha \times \mathbf{K}^\ell \end{array} \qquad \begin{array}{rccc} \psi_\alpha \circ T \circ \varphi_\alpha^{-1} : & U_\alpha \times \mathbf{K}^r & \to & U_\alpha \times \mathbf{K}^\ell \\ & (p_\alpha, v_\alpha) & \mapsto & (p_\alpha, T_\alpha(p_\alpha) v_\alpha) \end{array}$$

And $T_\alpha : U_\alpha \to M_{\ell\times r}(\mathbf{K})$ is smooth. So locally (over an open set that trivializes both bundles), a vector bundle morphism is a smooth varying family of $\ell \times r$ matrices. Suppose that $U_\alpha \cap U_\beta = W \neq \emptyset$. Then the diagram below commutes.

$$
\begin{array}{ccccccccc}
(p,v) & & W\times\mathbf{K}^r & \xleftarrow{\varphi_\alpha} & E|_W & \xrightarrow{\varphi_\beta} & W\times\mathbf{K}^r & & (p,v) \\
\downarrow & & \vdots & & \downarrow T & & \vdots & & \downarrow \\
(p,T_\alpha(p)v) & & W\times\mathbf{K}^\ell & \xleftarrow[\psi_\alpha]{} & F|_W & \xrightarrow[\psi_\beta]{} & W\times\mathbf{K}^\ell & & (p,T_\beta(p)v)
\end{array}
$$

$$
\begin{aligned}
T_\beta &= h_{\alpha\beta} T_\alpha g_{\alpha\beta} \\
T_\beta(p) &= \underbrace{\underbrace{h_{\beta\alpha}(p)}_{\ell\times\ell}\,\underbrace{T_\alpha(p)}_{\ell\times r}\,\underbrace{g_{\alpha\beta}(p)}_{r\times r}}_{\ell\times r}
\end{aligned}
$$

Next we consider a very important example.

Definition 1.1.13. Define $\mathbf{KP}^n$ to be the set of all 1-dimensional subspaces of $\mathbf{K}^{n+1}$. This is a smooth manifold of dimension n (if $\mathbf{K} = \mathbf{R}$, and $2n$ if $\mathbf{K} = \mathbf{C}$). It is explicitly given by

$$\mathbf{KP}^n = \left(\mathbf{K}^{n+1} \setminus \{0\}\right) \Big/ \left(v \sim w \iff v = \lambda w,\ \lambda \in \mathbf{K},\ \lambda \neq 0\right)$$

From this we are going to build a $\mathbf{K}^1$-vector bundle over $\mathbf{KP}^n = M$. There exists an open cover $U_0, U_1, \ldots, U_n$ of $k\mathbf{P}^n$ given by $U_i = \{[(x^0, \ldots, x^n)], x^i \neq 0\}$, which is a well-defined open set in $\mathbf{KP}^n$. Define E as a subset of the trivial $\mathbf{K}^{n+1}$-bundle over $\mathbf{KP}^n$, so $E \subset \mathbf{KP}^n \times \mathbf{K}^{n+1}$, by

$$E = \{(p, v) \ : \ v \in p\} = \{([(x^0, \ldots, x^n)], \lambda(x^0, \ldots, x^n)) \ : \ \lambda \in \mathbf{K}\}.$$

Each $p \in \mathbf{KP}^n$ is a line ℓ through 0 in $\mathbf{K}^{n+1}$. We attach this line ℓ to p. Next define a projection map $\pi : E \to \mathbf{KP}^n$ by $\pi = \pi_1|_E$ where $\pi_1 : \mathbf{KP}^n \times \mathbf{K}^{n+1} \to \mathbf{KP}^n$ and $\pi(p, v) = p$.

✠ Exercise 1.1.14. Show that $(E, \mathbf{KP}^n, \pi)$ is a $\mathbf{K}^1$-bundle over $\mathbf{KP}^n$. The idea is to construct for $\alpha \in \{0, 1, \ldots, n\}$ a map

$$
\begin{array}{rrcl}
\varphi_\alpha : & \pi^{-1}(U_\alpha) & \to & U_\alpha \times \mathbf{K}^1 \\
 & ([(x^0, \ldots, x^n)], (v^0, \ldots, v^n) = \lambda(x^0, \ldots, x^n)) & \mapsto & ([(x^0, \ldots, x^n)], v^\alpha)
\end{array}
$$

and show that it is a diffeomorphism with the required properties. Further find what the transition functions $g_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(1, \mathbf{K})$ are. Note that this bundle is called the *tautological* $\mathbf{K}^1$*-bundle* over $\mathbf{KP}^n$.

1.2 Methods for constructing new vector bundles

What can we do to vector spaces to get new ones? Let V, W be k-vector spaces. Then

$$
\begin{aligned}
V &\longrightarrow V^* = \mathrm{Hom}_k(V, k), \text{ the dual space} \\
V &\longrightarrow \textstyle\bigwedge^\ell(V), \ \ell\text{th exterior power of } V \\
V, W &\longrightarrow V \oplus W, \text{ the direct sum} \\
V, W &\longrightarrow V \otimes W, \text{ the direct product}
\end{aligned}
$$

Let V_1, V_2, W_1, W_2 be $\mathbf{K}$-vector spaces with $L_i : V_i \to W_i$ bilinear. Then we have maps

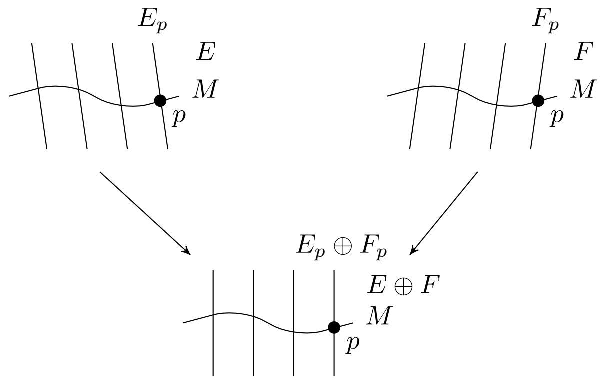
$$
\begin{aligned}
L_1 \oplus L_2 : & V_1 \oplus V_2 \to W_1 \oplus W_2 \\
L_1 \otimes L_2 : & V_1 \otimes V_2 \to W_1 \otimes W_2 \\
\textstyle\bigwedge^\ell(L_i) : & \textstyle\bigwedge^\ell(V_i) \to \bigwedge^\ell(W_i)
\end{aligned}
$$

We can also do these constructions to vector bundles. Let $E \to M$ and $F \to M$ be two $\mathbf{K}$-vector bundles over M. We would like to define E^*, $\bigwedge^\ell(E)$, $E \oplus F$, $E \otimes F$ as vector bundles over M. We begin by letting $\mathcal{U} = \{U_\alpha \ : \ \alpha \in \mathcal{A}\}$ be an open cover of M such that $(U_\alpha, \varphi_\alpha)$ is a trivialization of E and $g_{\alpha\beta}$ are the transition functions, and (U_α, ψ_α) is a trivialization of F with transition functions $h_{\alpha\beta}$.

Define E^* to be the vector bundle over M with transition functions $(\mathcal{U}, (g^{-1})^t)$, i.e. $U_\alpha \times \mathbf{K}^r$ is identified with $U_\beta \times \mathbf{K}^r$ by the inverse transpose of the identification for E. Similarly,

$$\begin{aligned} E \oplus F &= (\mathcal{U}, g_{\alpha\beta} \oplus h_{\alpha\beta}) \\ E \otimes F &= (\mathcal{U}, g_{\alpha\beta} \otimes h_{\alpha\beta}) \\ \textstyle\bigwedge^\ell(E) &= \left(\mathcal{U}, \textstyle\bigwedge^\ell(g_{\alpha\beta})\right) \end{aligned}$$

with $E \oplus F$ described by



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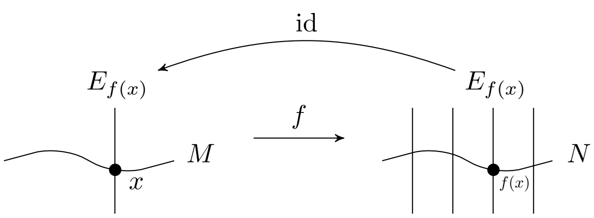
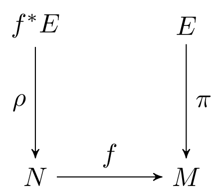
Definition 1.2.1. Let $\pi : E \to M$ be a vector bundle of rank r and $f : N \to M$ smooth. We want to define a bundle f^*E over N, called the *pullback bundle* of E by f, as follows.



$$\begin{aligned} f^*E &= \{(x, e) \in N \times E \ : \ f(x) = \pi(e)\} \\ &= \{(x, e) \ : \ e \in E_{f(x)}\} \end{aligned}$$

Define $\rho : f^*E \to N$ by $\rho(x, e) = x$ and $\rho = \pi_1|_{f^*E}$.

✠ Exercise 1.2.2. Show that this gives a rank r $\mathbf{K}$-vector bundle over N. In terms of transition functions, if $(U_\alpha, \varphi_\alpha)$ is a trivialization of E with transition functions for f^*E with respect to an open cover $\{f^{-1}(U - \alpha) \ : \ \alpha \in \mathcal{A}\}$, they are $f^*g_{\alpha\beta} = g_{\alpha\beta} \circ f : f^{-1}(U_\alpha) \to GL(r, \mathbf{K})$. The bundle makes the below diagram commute.



Example 1.2.3. Let $M = \{p\}$, a single point. Let $E = M \times \mathbf{K}^r$, a trivial k^r bundle over a point. Let N be a manifold with $f : N \to \{p\}$ the constant map. Then

$$f^*E = \{(x, e) \in N \times E \ : \ f(x) = \pi(e) = p\},$$

so $N \times \{p\} \times \mathbf{K}^r \cong \{(x, (p, v)) \ : \ v \in \mathbf{K}^r, x \in N\} \cong N \times \mathbf{K}^r$.

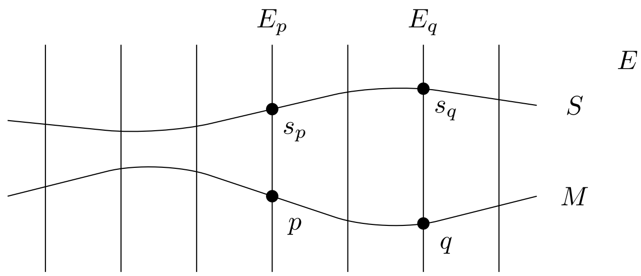
Remark 1.2.4. Recall E^* from above. If E has a trivial cover $\{(U_\alpha, \varphi_\alpha) \ : \ \alpha \in \mathcal{A}\}$, then we may construct the transition functions and gluing cocyles from those of E as below.

in E:	in E^*:
$\varphi_\alpha\|_{E_p} : E_p \xrightarrow{\cong} \{p\} \times \mathbf{K}^r$	$\left(\varphi_\alpha\|_{E_p}\right)^* : \{p\} \times \mathbf{K}^r \xrightarrow{\cong} E_p^*$
$g_{\beta\alpha}$	$\tilde{g}_{\beta\alpha} = (\varphi_\beta^*)^{-1} \circ ((\varphi_\alpha^*)^{-1})^{-1} = ((\varphi_\beta \circ \varphi_\alpha^{-1})^*)^{-1} = ((g_{\beta\alpha})^*)^{-1}$

Above, $(g_{\beta\alpha})^*$ denotes the transpose of $g_{\beta\alpha}$.

1.3 Sections of a vector bundle

Definition 1.3.1. Let $\pi : E \to M$ be a $\mathbf{K}^r$ vector bundle. A (*smooth*) *section* of E is a smooth map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$, i.e. $\pi(s(p)) = p$, so $s(p) \in E_p$ for all $p \in M$.



Define $\Gamma(E)$ to be the space of sections of E, which is an infinite-dimensional $\mathbf{K}$-vector space. That is, for $s_1, s_2 \in \Gamma(E)$, $(s_1 + s_2)_p = (s_1)_p + (s_2)_p$, and for $\lambda \in \mathbf{K}$, $(\lambda s)_p = \lambda s_p$. In fact, $\Gamma(E)$ is a C^∞-module, i.e. if $f \in C^\infty(M)$ and $s \in \Gamma(E)$, then $(fs)_p = f(p)s_p$. Note that any vector bundle always has at least one section, the zero section $0 : M \to E$ (where $0(p) = 0_p \in E_p$).

Remark 1.3.2. Let $(U_\alpha, \varphi_\alpha)$ be a local trivialization of E and $s \in \Gamma(E)$, and consider the maps

$$U_\alpha \xrightarrow{s|_{U_\alpha}} E|_{U_\alpha} \xrightarrow{\varphi_\alpha} U_\alpha \times \mathbf{K}^r$$
$$E|_{U_\alpha} = \pi^{-1}(U_\alpha)$$

which induces a smooth map $\varphi_\alpha \circ s|_{U_\alpha} : U_\alpha \to U_\alpha \times \mathbf{K}^r$ given by $p \mapsto (p, s_\alpha(p))$, where $s_\alpha : U_\alpha \to \mathbf{K}^r$ is smooth. Hence locally, s is an r-tuple of smooth $\mathbf{K}$-valued functions. Now suppose that (U_β, φ_β) is another trivialization of $U_\alpha \cap U_\beta \neq \emptyset$. Then

$$s_\beta = \pi_2(\varphi_\beta \circ s) = \pi_2(\varphi_\beta \circ \varphi_\alpha^{-1} \circ \varphi_\alpha \circ s) = g_{\beta\alpha} \circ s_\alpha.$$

So $s_\beta = g_{\beta\alpha} s_\alpha$ if $U_\beta \cap U_\alpha \neq \emptyset$. That is, $s_\beta(p) = g_{\beta\alpha}(p) s_\alpha(p)$ for all $p \in U_\alpha \cap U_\beta$.

Example 1.3.3. Consider the following examples of sections.
· A section of TM is a vector field
· A section $\bigwedge^k(T^*M)$ is a k-form, with $\Gamma(\bigwedge^k(T^*M)) = \Omega^k(M)$. More specifically, $\bigwedge^0(T^*M) = M \times \mathbf{R}$, the trivial real line bundle over M. Further, $\Omega^0(M) = \Gamma(\bigwedge^0(T^*M)) = \Gamma(M \times \mathbf{R}) \cong c^\infty(M)$, where $s : M \to M \times \mathbf{R}$ will be given by $s(p) = (p, f(p))$, $f \in C^\infty(M)$.

Proposition 1.3.4. Let E be a rank r $\mathbf{K}$-vector bundle. Then E is trivial iff it admits r sections that are everywhere linearly independent.

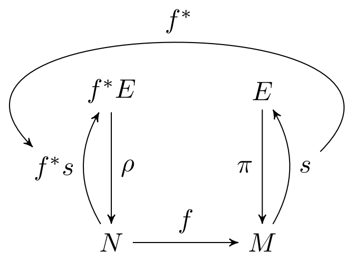
Proof: First suppose $E = M \times \mathbf{K}^r$. Define $s_i : M \to M \times \mathbf{K}^r = E$ by $s_i(p) = (p, e_i) \in E_p$, where $\{e_1, \ldots, e_r\}$ is the standard basis of $\mathbf{K}^r$. For the other direction, suppose that there exist $s_1, \ldots, s_r$ everywhere linearly independent. Define $\varphi : M \times \mathbf{K}^r \to E$ by

$$\varphi\left(\sum_{i=1}^{r} t^i e_i\right) = \sum_{i=1}^{l} t^i s_i(p) \in E_p,$$

for $t^i \in \mathbf{K}$. It remains to check that φ is a bundle homomorphism. ■

✠ Exercise 1.3.5. Note that $\Gamma(E^* \otimes F) \cong \underline{\mathcal{H}om}(E, F)$ because $W^* \otimes W \cong \mathrm{Hom}_{\mathbf{K}}(V, W)$ canonically.
· Show that $E^* \otimes E$ has a nowhere-zero section.
· If E is a line bundle, show that $E^* \otimes E$ is trivial.

Definition 1.3.6. Let $\pi : E \to M$ be a vector bundle and $f : N \to M$ a smooth map. Then there exists a canonical map $f^* : \Gamma(E) \to \Gamma(f^*E)$ called the *pullback of sections*, defined by $s \in \Gamma(E)$.



$$s \leftrightarrow s_\alpha \ : \ U_\alpha \to \mathbf{K}^r$$
$$f^*s \leftrightarrow (f^*s)_\alpha \ : \ f^{-1}(U_\alpha) \to \mathbf{K}^r$$

Above, $f^*s : N \to f^*E$ and $(f^*s)_p = s_{f(p)} \in E_{f(p)} = (f^*E)_p$ for all $p \in N,\ f(p) \in M$. Note that sections are always immersions.

Remark 1.3.7. Suppose that $f : N \to M$ is smooth. Then for $\omega \in \Omega^k(M) = \Gamma(\bigwedge^k(T^*M))$, there exists a pullback form $f^*\omega \in \Omega^k(N)$. This is not quite the same as the pullback of sections. As a section, $f^*\omega \in \Gamma(f^*(\bigwedge^k(T^*M)))$, which is not the same as, but related to, $\Omega^k(N)$.

Remark 1.3.8. Let E be a vector bundle over M. The space $\Omega^k(E) = \Gamma(\bigwedge^k(T^*M) \otimes E)$ contains E-valued k-forms locally (in local coordinates for M) with $\omega = \omega_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. Sections of E over some open set $U \subset M$ are maps $U \to E|_U = \pi^{-1}(U)$. Further, $\Omega^k(M)$" = "$\Omega^k(M \times \mathbf{R}) = \Gamma(\bigwedge^k(T^*M) \otimes (M \times \mathbf{R}))$. Finally, if $U \subset M$ is open, then $E|_U = \pi^{-1}(U)$ is a vector bundle over U. We write this as

$$\Gamma(E|_U) = \Gamma_U(E) = \Gamma(U, E)$$

and call it the global sections of E.

1.4 Metrics and other structures on vector bundles

Definition 1.4.1. Let $\pi : E \to M$ be a $\mathbf{K}^r$-bundle. Then E is termed $\mathbf{K}$-*orientable* if $\bigwedge^r E$ is trivial. Note $\bigwedge^r E$ is a rank 1 vector bundle over M, called the *determinant line bundle* $\det(E)$ of E.

Equivalently, E is $\mathbf{K}$-orientable iff there exists a nowhere-zero section of $\det(E) = \bigwedge^r(E)$. Now suppose that $\mathbf{K} = \mathbf{R}$ and E is $\mathbf{K}$-orientable. Let μ, ν be nowhere-zero sections of $\det(E)$. Then there exists $f \in C^\infty(M)$ such that f is nowhere-zero with $\mu = f\nu$.

An $\mathbf{R}$-orientation of E is a choice of equivalence class. The # of orientations equals $2^{\#}$ of connected components of M.

Example 1.4.2. A smooth manifold M^n is $\mathbf{R}$-orientable iff TM is an $\mathbf{R}$-orientable vector bundle. Note that as a manifold, TM is always orientable, i.e. $T(TM)$ is always an $\mathbf{R}$-orientable vector bundle.

Definition 1.4.3. Let $\pi : E \to M$ be a real vector bundle. A *Riemannian fiber metric* on E is a section $h \in \Gamma(E^* \otimes E^*)$ such that for all $s_1, s_2 \in \Gamma(E)$, $h(s_1, s_2) = h(s_2, s_1)$ and $h(s_1, s_1) \geqslant 0$ with equality iff $s_1 = 0$. This is a smoothly varying family of positive definite symmetric bilinear form on the fibers.

Note that a Riemannian metric on M is a Riemannian fiber metric on the vector bundle TM. However in general, a Riemannian fiber metric on E (which is a metric on TE)is different from a Riemannian metric on a smooth manifold E.

Remark 1.4.4. Any real vector bundle admits lots of Riemannian fiber metrics. The proof is identical to that of the claim that any Riemannian manifold admits lots of Riemannian metrics, with partitions of unity. However, in the complex case, we first need to define conjugate bundles.

Definition 1.4.5. Let $\pi : E \to M$ be a $\mathbf{C}^r$-vector bundle. The *conjugate bundle* $\pi : \overline{E} \to M$ is a $\mathbf{C}^r$-vector bundle over M defined by changing the scalar multiplication on each fiber as follows. The total space $\overline{E} = E$ as a set. The map $\pi : \overline{E} \to M$ is also the same as $\pi : E \to M$. For $\overline{E}_p$ the fiber of $\overline{E}$ over p (as a set, $\overline{E}_p = E_p$) and for $\lambda \in \mathbf{C}$ and $v \in \overline{E}_p$, define scalar multiplication by

$$\underbrace{\lambda v}_{\in \overline{E}_p} = \underbrace{\overline{\lambda} v}_{\in E_p}.$$

Proposition 1.4.6. Let $(\mathcal{U}, g_{\cdot\cdot})$ be a gluing cocycle for E. Then $(\mathcal{U}, g^*_{\cdot\cdot})$ is a gluing cocycle for $\overline{E}$, where * represents complex conjugation and $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbf{C})$.

<u>*Proof:*</u> Let (U_α, U_β) be a local trivialization of E. Then $\varphi_\alpha(e) = (\pi(e), f_\alpha(e))$ for $f_\alpha : E|_{U_\alpha} \to \mathbf{C}^r$. If $p \in M$, then $f_\alpha|E_p : E_p \to \mathbf{C}^r$, which is smooth. Define $c \circ \varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbf{C}^r$ by $(c \circ \varphi_\alpha)(e) = (\pi(e), \overline{f_\alpha(e)})$. Then

$$\begin{aligned} \varphi_\alpha(\lambda v) &= (p, f_\alpha(\lambda v)) = (p, \lambda f_\alpha(v)) \\ \text{implies} \quad (c \circ \varphi_\alpha)(\lambda v) &= (p, \overline{f_\alpha(\lambda v)}) = (p, \overline{\lambda f_\alpha(v)}) = (p, \overline{\lambda}\ \overline{f_\alpha(v)}) = (p, \overline{\lambda}, (c \circ f_\alpha)(v)). \end{aligned}$$

So the maps $c \circ \varphi_\alpha$ are local trivializations for $\overline{E}$. The transition functions $\widetilde{g}_{\beta\alpha}$ for $\overline{E}$ with respect to this cover $\mathcal{U}$ are

$$\begin{aligned} (p, \widetilde{g}_{\beta\alpha}(v)) &= (c \circ \varphi_\beta) \circ (c \circ \varphi_\alpha^{-1})(p, v) \\ &= (p, (c \circ f_\beta) \circ (c \circ f_\alpha)^{-1}(v)) \\ &= (p, (c \circ f_\beta \circ f_\alpha^{-1} \circ c)(v)). \end{aligned}$$

Hence $\widetilde{g}_{\beta\alpha} v = \overline{g_{\beta\alpha}\overline{v}} = \overline{g}_{\beta\alpha} v$, so $\widetilde{g}_{\beta\alpha} = \overline{g}_{\beta\alpha}$. Although this finishes the proof, all the expressions above should be evaluated at p. ■

Definition 1.4.7. Let $\pi : E \to M$ be a <u>$\mathbf{C}^r$-bundle</u>. A *Hermitian fiber metric* on E is a section $h \in \Gamma(E^* \otimes \overline{E}^*)$ such that for $s_1, s_2 \in \Gamma(E)$, $h(s_1, s_2) = \overline{h(s_2, s_1)}$ and $h(s_1, s_1) \geqslant 0$ with equality iff $s_1 = 0$. Note that for f a $\mathbf{C}$-valued smooth function on M,

$$\begin{aligned} h(f s_1, s_2) &= f h(s_1, s_2) \\ h(s_1, f s_2) &= \overline{f} h(s_1, s_2). \end{aligned}$$

Remark 1.4.8. Any $\mathbf{C}^r$-vector bundle admits lots of Hermitian fiber metrics. The proof is still the same as previously mentioned. Also, complex conjugation $c : E \to \overline{E}$ is a bundle isomorphism of E onto E as real vector bundles but not as complex vector bundles.

Proposition 1.4.9. Let E be a $\mathbf{C}^r$-vector bundle over M. Then E can also be regarded as an $\mathbf{R}^{2r}$-vector bundle over M, hence E is always $\mathbf{R}$-orientable.

<u>*Proof:*</u> Let E be determined by the gluing cocyle $(\mathcal{U}, g_{\cdot\cdot})$. There exists a canonical group homomorphism

$$\begin{array}{rccc} j: & GL(r, \mathbf{C}) & \to & GL(2r, \mathbf{R}) \\ & A + iB & \mapsto & \left(\begin{smallmatrix} A & -B \\ B & A \end{smallmatrix}\right) \end{array}.$$

Define a cocycle $\widetilde{g}_{\alpha\beta} = j \circ g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(2r, \mathbf{R})$ that satisfies the cocycle conditions $\widetilde{g}_{\alpha\alpha} = \widetilde{g}_{\alpha\beta}\widetilde{g}_{\beta\alpha} = \widetilde{g}_{\alpha\beta}\widetilde{g}_{\beta\gamma}\widetilde{g}_{\gamma\alpha} = 1$. Define $w_{\alpha\beta} = \det(\widetilde{g}_{\alpha\beta})$, which are the transition functions of $\det_{\mathbf{R}}(E_{\mathbf{R}})$, which is a real line bundle. We also have that $\det(\widetilde{g}_{\alpha\beta}) = |\det(g_{\alpha\beta})|^2 > 0$ for all α, β.

Next, let $f_{\alpha\beta} = \log(w_{\alpha\beta})$ for $w_{\alpha\beta} = e^{f_{\alpha\beta}}$ and $f_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbf{R}$ we have $w_{\gamma\alpha} = w_{\alpha\beta}w_{\beta\gamma}$, implying that $f_{\gamma\alpha} = f_{\gamma\beta} + f_{\beta\alpha}$. Let $\{\rho_\alpha : \alpha \in \mathcal{A}\}$ be a partition of unity subordinate to $\mathcal{U}$. Define, for all $\alpha \in \mathcal{A}$, maps

$$\begin{array}{rccl} f_\alpha : & U_\alpha & \to & \mathbf{R} \\ & p & \mapsto & \sum_{\substack{\gamma\in\mathcal{A} \\ U_\alpha\cap U_\gamma\neq\emptyset}} \rho_\gamma f_{\gamma\alpha} \end{array},$$

which is a smooth and well-defined map. This gives us that

$$f_{\gamma\alpha} - f_{\gamma\beta} = f_{\gamma\alpha} + f_{\beta\gamma} = f_{\beta\alpha}$$

$$\text{and} \quad f_\alpha - f_\beta = \sum_{\gamma\in\mathcal{A}} \rho_\gamma \left(f_{\gamma\alpha} - f_{\gamma\beta}\right) = \underbrace{\left(\sum_{\gamma\in\mathcal{A}} \rho_\gamma\right)}_{=\,1} f_{\beta\alpha} = f_{\beta\alpha}.$$

So from the cocycle conditions, we have constructed $f_\alpha : U_\alpha \to \mathbf{R}$ smooth such that

$$f_\alpha - f_\beta = f_{\beta\alpha} \quad \Longrightarrow \quad -f_\beta = -f_\alpha + f_{\beta\alpha} \quad \Longrightarrow \quad e^{-f_\beta} = g_{\beta\alpha}e^{-f_\alpha}.$$

The above says that $s_\beta = g_{\beta\alpha}s_\alpha$. So the section s of $\det_{\mathbf{R}}(E_{\mathbf{R}})$ is given locally by $s_\alpha = e^{-f_\alpha}$, and it is well-defined and nowhere-zero. So s is a global nowhere vanishing section of $\det_{\mathbf{R}}(E_{\mathbf{R}})$, so $E_{\mathbf{R}}$ is $\mathbf{R}$-orientable, for $E_{\mathbf{R}}$ the underlying $\mathbf{R}$-vector bundle. ■

2 Characteristic classes

2.1 Connections

Definition 2.1.1. Let $\pi : E \to M$ be a $\mathbf{K}^r$-vector bundle. A *connection* ∇ on E is a $\mathbf{K}$-linear map $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ such that $\nabla(fs) = df \otimes s + f\nabla s$ for all $f \in C^\infty(M)$ and $s \in \Gamma(E)$ (this is the Leibniz rule).

If $X \in \Gamma(TM)$ (i.e. X is a vector field), define $\nabla_X s = (\nabla s)(X)$, which is contraction of the T^*M factor with X at every point. That is, for $(s_i)_p \in E_p$ and $(\alpha_i)_p \in T_p^*M$,

$$(\nabla s)_p = \sum_{i=1}^n (\alpha_i)_p \otimes (s_i)_p, \qquad \text{so} \qquad ((\nabla s)(X))_p = \sum_{i=1}^n (\alpha_i)_p(X_p)(s_i)_p.$$

The Leibniz rule then becomes

$$\nabla_X(fs) = (df \otimes s)(X) = f\nabla_X s = (Xf)s + f\nabla_X s.$$

We then call $\nabla_X s$ the *covariant derivative* of the section s in the direction of the vector field X. Notice since $\nabla s : \Gamma(TM) \to \Gamma(E)$, it follows that $\nabla s \in \underline{\mathcal{H}om}(TM, E)$.

Remark 2.1.2. How do we get new connections from existing ones?

First, consider $T : E \to F$ a vector bundle isomorphism for E, F over M. If ∇ in a connection on E, then $T\nabla T^{-1}$ is a connection on F (it remains to check that the Leibniz rule still holds).

Next, consider $\nabla, \widetilde{\nabla}$ two connections on E, with $A = \widetilde{\nabla} - \nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$, which is $\mathbf{K}$-linear. Then

$$A(fs) = \widetilde{\nabla}(fs) - \nabla(fs) = df \otimes s + f\widetilde{\nabla}s - (df \otimes s + f\nabla s) = fA(s).$$

Hence $A \in \Gamma(E^* \otimes T^*M \otimes E) \cong \Gamma(T^*M \otimes E^* \otimes E) \cong \Gamma(T^*M \otimes \mathrm{End}(E)) = \Omega^1(\mathrm{End}(E))$. So the difference between any two connections on E is an $\mathrm{End}(E)$-valued 1-form. Conversely, if ∇ is a connection on E and $A \in \Omega^1(\mathrm{End}(E))$, then $\widetilde{\nabla} = \nabla + A$ is a connection on E. This shows that the space of cennections $\mathcal{A}_E$ on E is an affine space modeled on the vector space $\Omega^1(\mathrm{End}(E))$.

Example 2.1.3. Consider connections on the following spaces.

$\cdot$ $E = M \times \mathbf{R}$. Then $\Gamma(E) \cong C^\infty(M)$, with

$$\begin{array}{ccc} d : C^\infty(M) & \longrightarrow & \Omega^1(M), \\ \wr\Vert & & \wr\Vert \\ \Gamma(E) & & \Gamma(T^*M \otimes (M \otimes \mathbf{R})) \end{array}$$

with $d(f) = df$ and $d(fs) = (df)s + fds$ for $f, s \in C^\infty(M)$. Hence d is a connection on $M \times \mathbf{R}$, and is a *trivial connection*.

$\cdot$ $E = M \times \mathbf{K}^r$, the trivial $\mathbf{K}^r$-bundle. A section of E in $s = \begin{bmatrix} s_1 \\ \vdots \\ s^r \end{bmatrix}$ for $s : M \to \mathbf{K}^r$ an r-tuple of $\mathbf{K}$-valued smooth functions. Define $\nabla^0 \begin{bmatrix} s_1 \\ \vdots \\ s^r \end{bmatrix} = \begin{bmatrix} ds_1 \\ \vdots \\ ds^r \end{bmatrix}$ to be a connection on $M \times \mathbf{K}^r$. This is the general form of the *trivial connection*. Note that in general, on a non-trivial bundle there does not exist any analog of the trivial connection. Now let ∇ be any other connection on $M \times \mathbf{K}^r$. Then $\nabla = \nabla^0 + A$ for some $A \in \Omega^1(\mathrm{End}(E))$. In local coordinates $(x^1, \ldots, x^n)$ for M and $\{e_1, \ldots, e_r\}$ the standard basis of $\mathbf{K}^r$, we have that $A = A^i_j e^j \otimes e_i$, where A^i_j is a 1-form on M, so $A^i_j = A^i_{jk} dx^k$ with $1 \leqslant i, j \leqslant r$ and $1 \leqslant k \leqslant n$. These are locally defined smooth functions. It follows that

$$\begin{bmatrix} ds^1 \\ \vdots \\ ds^r \end{bmatrix} + \begin{bmatrix} A^1_j s^j \\ \vdots \\ A^r_j s^j \end{bmatrix} = \begin{bmatrix} \text{(1-form on } M) \\ \vdots \\ \text{(1-form on } M) \end{bmatrix},$$

for $\nabla s = (\nabla^0 + A)s$. Above we begin the use of the Einstein notation, where a repeated index in the superscript and subscript of a term indicates a sum over that index.

Remark 2.1.4. The above may be generalized. Let $\pi : E \to M$ be a $\mathbf{K}^r$-bundle. Let $U \subset M$ be open such that $E|_U$ is trivial (i.e. bundle-isomorphic to $V \times \mathbf{K}^r$), so there exists $\{e_1, \ldots, e_r\}$ a global frame for $E|_U$. That is, $e_1, \ldots, e_r \in \Gamma(E|_U)$ that are linearly independent at every point. Note that ∇ may be restricted to a connection ∇ on $E|_U$. We will do so, and denote both by the same symbol.

So ∇e_i is a section of $(T^*M \otimes E)|_U$. Hence there exists $A^i_j \in (T^*M)|_U$ and $A^i_j \in \Omega^1(U)$ such that $\nabla e_i = A^j_i e_j$. Let $s \in \Gamma(E|_U) = \Gamma(U, E)$, so $s = s^i e_i$ for unique $s^i \in C^\infty(U)$, which are $\mathbf{K}$-valued. Then

$$\nabla s = \nabla(s^i e_i) = (ds^i) \otimes e_i + s^1 \nabla e_i = ds^j \otimes e_j + s^i A^j_i \otimes e_j = (ds^j + a^j_i s^i) \otimes e_j.$$

Hence locally, every connection is completely determined by these A^j_is, which are the connection matrices with respect to the local frame.

Remark 2.1.5. How does the above compare to Christoffel symbols for a connection on $E = TM$? For ∇ a connection on TM, we have $A^j_i = A^j_{ik} e^k$, and from above $\nabla e_i = A^j_i e_j$. The Christoffel symbols originally are $\nabla_{e_k} e_i = A^j_{ik} e_j$, which follows by switching the indices.

We return to the previous remark. Let $\{\tilde{e}_1, \ldots, \tilde{e}_r\}$ be another local frame for E over U. Then $\nabla \tilde{e}_i = \tilde{A}^j_i \tilde{e}_j$, where $\tilde{e}_i = e_j g^j_i$, for g^j_i the change of basis matrix, i.e. $g^j_i : U \to GL(r, \mathbf{K})$ is smooth. We now see that

$$\begin{aligned} \nabla \tilde{e}_i &= \nabla(e_j g^j_i) \\ &= \tilde{A}^j_i e_k g^k_j \end{aligned} \qquad \text{and} \qquad \begin{aligned} \nabla(e_j g^j_i) &= \nabla(g^j_i e_j) \\ &= dg^j_i \otimes e_j + g^j_i \nabla e_j \\ &= dg^k_i \otimes e_k + g^j_i A^k_j \otimes e_k \\ &= \tilde{A}^j_i g^k_j \otimes e_k. \end{aligned}$$

Hence $\tilde{A}_i^j g_j^k = dg_i^k + g_i^j A_j^k$. Now multiply both sides by $(g^{-1})_k^\ell$ and sum over k to get that

$$\tilde{A}_i^\ell = (g^{-1})_k^\ell A_j^k g_i^j + (g^{-1})_k^\ell dg_i^k \quad \text{implying} \quad \tilde{A} = g^{-1}Ag + g^{-1}dg$$

is the relation between connection matrices $A, \tilde{A}$ for a connection ∇ with respect to two local frames $\{e_1, \dots, e_r\}$ and $\{\tilde{e}_1, \dots, \tilde{e}_r\}$, related by g. So if $(\mathcal{U}, g_{\cdot\cdot})$ is a gluing cocyle for E and A_α are the connection matrices of ∇ with respect to the local trivializations $(\varphi_\alpha, U_\alpha)$, we have the map given by

$$\begin{array}{rcl} A_\alpha & : & U_\alpha \to GL(r, \mathbf{K}) \\ A_\alpha & = & g_{\beta\alpha}^{-1} A_\beta g_{\beta\alpha} + g_{\beta\alpha}^{-1} dg_{\beta\alpha} \end{array}.$$

Proposition 2.1.6. Let $\pi : E \to M$ be a $\mathbf{K}^r$-bundle. Then there exist lots of connections on E.

Proof: Let $(\mathcal{U}, g_{\cdot\cdot})$ be a gluing cocyle. The map $\psi_\alpha = \varphi_\alpha^{-1} : U_\alpha \times \mathbf{K}^r \to E|_{U_\alpha}$ is a bundle isomorphism. Let ∇_α be the trivial connection on $U_\alpha \times \mathbf{K}^r$. Define $\widehat{\nabla}_\alpha = \psi_\alpha \circ \nabla_\alpha \circ \psi_\alpha^{-1}$, which is a connection on $E|_{U_\alpha}$. Let $\{\rho_\alpha \ : \ \alpha \in \mathcal{A}\}$ be a partition of unity subordinate to $\mathcal{U}$. If $s \in \Gamma(E)$, then $\rho_\alpha(s) \in \Gamma(E|_{U_\alpha})$. Hence $\widehat{\nabla}_\alpha(\rho_\alpha(s)) \in \Gamma((T^*M \otimes E)|_{U_\alpha})$. Define the map

$$\begin{array}{rrcl} \nabla : & \Gamma(E) & \to & \Gamma(T^*M \otimes E) \\ & s & \mapsto & \sum_{\alpha,\beta \in \mathcal{A}} \rho_\beta \left(\widehat{\nabla}_\alpha(\rho_\alpha(s)) \right) \end{array}.$$

This map is $\mathbf{K}$-linear. We need to show that the Leibniz rule holds for $f \in C^\infty(M)$. This follows as

$$\begin{aligned} \nabla(fs) &= \sum_{\alpha,\beta \in \mathcal{A}} \rho_\beta \left(\widehat{\nabla}_\alpha(\rho_\alpha(fs)) \right) \\ &= \sum_{\alpha,\beta \in \mathcal{A}} \rho_\beta \left(\widehat{\nabla}_\alpha f|_{U_\alpha}(\rho_\alpha(s)) \right) \\ &= \sum_{\alpha,\beta \in \mathcal{A}} \rho_\beta \left((df)|_{U_\alpha} \otimes (\rho_\alpha(s)) + f\widehat{\nabla}_\alpha(\rho_\alpha(s)) \right) \\ &= \left(\sum_{\alpha \in \mathcal{A}} \rho_\alpha \right) \left(\sum_{\beta \in \mathcal{A}} \rho_\beta \right) (df \otimes s + f\nabla s) \\ &= df \otimes s + f\nabla s. \end{aligned}$$

So ∇ is indeed a connection, and we are done. ■

Given vector bundles with connections, we get naturally induced connections on new vector bundles constructed from them.

Proposition 2.1.7. Let (E_1, ∇^1) and (E_2, ∇^2) be vector bundles over M with connections. Then:

i. There exists a connection ∇ on $E_1 \oplus E_2$ defined by $\nabla(s_1 \oplus s_2) = \nabla(s_1) \oplus \nabla(s_2)$, i.e. $\nabla_X(s_1 \oplus s_2) = (\nabla_X s_1) \oplus (\nabla_X s_2)$.

ii. There exists a connection ∇ on $E_1 \otimes E_2$ defined by $\nabla(s_1 \otimes s_2) = (\nabla s_1) \otimes s_2 + s_1 \otimes (\nabla s_2)$, i.e. $\nabla_X(s_1 \otimes s_2) = (\nabla_X s_1) \otimes s_2 + s_1 \otimes (\nabla_X s_2)$. Extend this to all sections of $E_1 \otimes E_2$ by $\mathbf{K}$-linearity.

Let (E, ∇) be a vector bundle with a connection and let $f : N \to M$ be smooth. Then:

ii. There exists a connection ∇ on E^* defined, for all $\alpha \in \Gamma(E^*)$ and $s \in \Gamma(E)$, by $d(\alpha(s)) = (\nabla_\alpha)s + \alpha(\nabla s)$, i.e. $X(\alpha(s)) = (\nabla_X \alpha)(s) = \alpha(\nabla_X s)$.

iv. There exists a connection ∇ an $\bigwedge^k(E)$ given by $\nabla_X(s_1 \wedge \cdots \wedge s_k) = \sum_{j=1}^k s_1 \wedge \cdots \wedge \nabla s_j \wedge \cdots \wedge s_k$.

v. There exists a connection $f^*\nabla$ on the pullback bundle f^*E over N such that $f^*\nabla : \Gamma(f^*E) \to \Gamma(T^*M \otimes f^*E)$ is $\mathbf{K}$-linear.

Proof: We will only prove **v.** here. We construct $f^*\nabla$ by describing it in terms of a gluing cocycle for f^*E. Let $(\mathcal{U}, g_{..})$ be a gluing cocycle for $\pi : E \to M$. The connection ∇ on E is described locally by $A_\alpha : U_\alpha \to GL(\mathbf{K}^r) \otimes \Omega^1(U_\alpha)$ such that $A_\beta = g_{\beta\alpha}A_\alpha g_{\beta\alpha}^{-1} - (dg_{\beta\alpha})g_{\beta\alpha}^{-1}$. Recall that $\{f^{-1}(U_\alpha) \ : \ \alpha \in A\}$ is an open cover of N and $f^*g_{\beta\alpha} = g_{\beta\alpha} \circ f : f^{-1}(U_\alpha) \cap f^{-1}(U_\beta) \to GL(\mathbf{K}^r)$ are transition functions for f^*E. Next let

$$(f^*A)_\alpha = f^*A_\alpha = A_\alpha \circ f : f^{-1}(U_\alpha) \to GL(\mathbf{K}^r) \otimes \Omega^1(f^{-1}(U_\alpha)),$$

and pull it back by f to get

$$\begin{aligned} A_\beta \circ f &= (g_\beta \circ f)(A_\alpha \circ f)(g_{\beta\alpha}^{-1} \circ f) - (dg_{\beta\alpha} \circ f)(g_{\beta\alpha}^{-1} \circ f) \\ \text{and} \qquad (f^*A)_\beta &= (f^*g)_{\beta\alpha}(f^*A)_\alpha(f^*g)_{\beta\alpha}^{-1} - d(f^*g)_{\beta\alpha}(f^*g)_{\beta\alpha}^{-1}. \end{aligned}$$

Hence the maps $(f^*A)_\alpha$ define a connection on E. ■

Remark 2.1.8. Recall that if ∇ is a connection on E and $\{e_1, \ldots, e_r\}$ is a local frame for E (over $U \subset M$), then over U with $s \in \Gamma(U|_U)$ given by $s = s^i e_i$ for $s^i \in C^\infty(M)$, we have that

$$\nabla s = \underbrace{\left(ds^i + A^i_j s_i\right)}_{\in\ \Omega^1(E)} e_i \quad \text{for} \quad A^i_j \in \Omega^1(U) \quad \text{and} \quad A \in GL(\mathbf{K}^r) \otimes \Omega^1(U).$$

Let $\{e^1, \ldots, e^r\}$ be the dual coframe for E^*, so $e^i \in \Gamma(E^*|_U)$ with $e_i e^j = \delta^j_i$. What does the matrix A^* look like in terms of A? That is , if $\alpha \in \Gamma(E^*|_U)$, $\alpha = \alpha_i e^i$ for $\alpha_i \in C^\infty(M)$, then $\nabla\alpha = (d\alpha_i + (A^*)^j_i \alpha_j)e^i$.

Proposition 2.1.9. In the context of the above remark, $(A^*)^i_j = -A^j_i$.

Proof: Start with $e^a e_b = \delta^a_b$ and take d of both sides. This gives

$$0 = (\nabla e^a)a_b + e^a(\nabla e_b) \quad \iff \quad 0 = (A^*)^a_k e^k(e_b) + e^a(A^k_a e_k) \quad \iff \quad 0 = (A^*)^b_a + A^a_b.$$

■

Example 2.1.10. This example is very important. Let (E, ∇) be a vector bundle with a connection. We get an induced connection ∇ on $\mathrm{End}(E) \to M$. Define it as follows. For $B \in \Gamma(\mathrm{End}(E))$ and $s \in \Gamma(E)$, define $\nabla_X B \in \Gamma(\mathrm{End}(E))$ by

$$(\nabla_X B)(s) = \nabla_X(Bs) - B(\nabla_X s).$$

This is $\mathbf{K}$-linear because B and ∇_X on E are $\mathbf{K}$-linear. So we need to checek the Leibniz rule. Let $f \in C^\infty(M)$ and compute

$$\begin{aligned} (\nabla_X(fB))(s) &= \nabla_X((fB)(s)) - (fB)(\nabla_X s) \\ &= \nabla_x(f(B(s))) - f(B(\nabla_X s)) \\ &= (Xf)B(s) + f\nabla_x(B(s)) - fB(\nabla_X s) \\ &= ((Xf)B)(s) + (f(\nabla_X B))(s) \\ &= ((Xf)B + f(\nabla_X B))(s). \end{aligned}$$

Let's now look at what this looks like in a local trivialization $(U_\alpha, \varphi_\alpha)$ for E with $s = s^i e_i$ and $E|_{U_\alpha} \cong U_\alpha \times \mathbf{K}^r$ trivial. Then $\nabla s = (ds^i + A^i_j s^j)e_i$, with $B(s) = (B^k_\ell s^\ell)e_k$ for some $B^k_\ell \in C^\infty(U_\alpha)$. This gives

$$\begin{aligned} \nabla(B(s)) &= (d(B^k_\ell s^\ell) + A^k_\ell(B^\ell_j s^j))e_k \\ \text{and} \quad B(\nabla(s)) &= (B^k_\ell(ds^\ell + A^\ell_j s^j))e_k, \\ \text{implying} \quad (\nabla B)(s) &= \nabla(B(s)) - B(\nabla s) \\ &= ((dB^k_\ell)s^\ell + B^k_\ell ds^\ell + A^\ell_k B^\ell_j s^j - B^k_\ell ds^\ell - B^k_\ell A^\ell_j s^j)e_k \\ &= (dB^k_\ell + [A, B]^k_\ell)s^\ell. \end{aligned}$$

Hence $\nabla B = dB + [A, B]$ is a local trivialization. We present another proof of this fact. Recall that $\mathrm{End}(E) \cong E^* \otimes E$. Let $\{e_1, \ldots, e_r\}$ be a local trivialization for E and $\{e^1, \ldots, e^r\}$ a local trivialization for E^*. Then $B \in \Gamma(\mathrm{End}(E))$ is locally given by $B = B^i_j e^j \otimes e_i \in \Gamma((E^* \otimes E)|_U)$. Explicitly. if $s = s^k e_k$, then

$$B(s) = (B^i_j e^j \otimes e_i)(s^k e_k) = B^i_j s^k e^j(e_k) e_i = B^i_j s^j e_i = B(s),$$

giving that

$$\begin{aligned}\nabla B &= (dB^i_j) e^j \otimes e_i + B^i_j \nabla(e^j \otimes e_i) \\ &= (dB^i_j) e^j \otimes e_i + B_j((\nabla e^j) \otimes e_i + e^j \otimes (\nabla e_i)) \\ &= (dB^i_j) e^j \otimes e_i + B^i_j(-A^j_\ell e^\ell \otimes e_i) + (B^i_j \otimes A^\ell_i e_\ell) \\ &= \end{aligned}$$

We have again shown that $\nabla B = dB + [A, B]$. Note that by the second line we had three different connections, but for ease of notation all were given the same symbol.

Definition 2.1.11. Let h be a fiber metric on E. We say that a connection ∇ on E is compatible with h if for all $s_1, s_2 \in \Gamma(E)$,

$$d(h(s_1, s_2)) = h(\nabla s_1, s_2) + h(s_1, \nabla s_2) \qquad \text{i.e.} \qquad X(h(s_1, s_2)) = h(\nabla_X s_1, s_2) + h(s_1, \nabla_X s_2).$$

Equivalently, ∇ is h-compatible iff $\nabla h = 0$.

There is quite a lot more to say about connections, in terms of parallel sections, parallel transport, holonomy, etc. We move on to more pertinent matters.

2.2 Curvature

Definition 2.2.1. Let ∇ be a connection on E. Let $X, Y \in \Gamma(TM)$ be vector fields. Then we can take $\nabla_X, \nabla_Y, \nabla_{[X,Y]} : \Gamma(E) \to \Gamma(E)$, which are $\mathbf{K}$-linear, but not $C^\infty(M)$-linear, because of the Leibniz rule. Define the *curvature* $F^\nabla(X, Y) : \Gamma(E) \to \Gamma(E)$ on ∇ by

$$F^\nabla(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

✠ Exercise 2.2.2. Show that for all $f \in C^\infty(M)$ and $s \in \Gamma(E)$,

$$F^\nabla(X, Y)(fs) = F^\nabla(fX, Y)s = F^\nabla(X, fY)s = f(F^\nabla(X, Y)s).$$

Remark 2.2.3. Consider a special case of the curvature, when $E = TM$ and ∇ is the Levi-Civita connection of a Riemann fiber metric on M. Then $F^\nabla(X, fY)Z = R(X, Y)Z$, the Riemann curvature tensor. It is clear that $F^\nabla(Y, X) = -F^\nabla(X, Y)$ and $F^\nabla(X, Y)_p = F^\nabla(X_p, Y_p)$. Further, the map

$$\begin{array}{rcl}\Gamma(TM) \times \Gamma(TM) & \to & \underline{\mathcal{E}nd}(E) = \Gamma(\mathrm{End}(E)) \\ (X, Y) & \mapsto & \overline{F^\nabla}(X, Y)\end{array}$$

is skew-symmetric and bilienar over $C^\infty(M)$. So $F^\nabla(\cdot, \cdot) \in \Omega^2(\mathrm{End}(E))$, i.e. the curvature is an $\mathrm{End}(E)$-valued 2-form.

Remark 2.2.4. In local coordinates $(x^1, \ldots, x^n)$ on M with domain U, $F^\nabla = \frac{1}{2} F_{ij} dx^i dx^j$. What are the F_{ij}? Start by supposing that $\{e_1, \ldots, e_r\}$ is a local frame over U. Define

$$\nabla_i s = \nabla_{\frac{\partial}{\partial x^i}} s = \partial_i s + A_i s = (\partial_i s^a + A^a_{bi} s^b) e_a,$$

where $A^a_b = A^a_{bi}dx^i$. Then

$$\begin{aligned}
F_{ij}s &= F^\nabla\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)s \\
&= \nabla_i(\nabla_j s) - \nabla_j(\nabla_i s) - \nabla_{[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}]}s \\
&= \nabla_i(\partial_j s A_j s) - \nabla_j(\partial_i s + A_i s) - \partial_j(\partial_i s + A_i s) - A_j(\partial_i s + A_i s) \\
&= \frac{\partial^2}{\partial x^i \partial x^j}s + (\partial_i A_j)s + A_j\partial_i s + A_i\partial_j s + A_i A_j s - \frac{\partial^2}{\partial x^j \partial x^i}s - (\partial_j A_i)s - A_i\partial_j s - A_j\partial_i s - A_j A_i s \\
&= (\partial_i A_j - \partial_j A_i + A_i A_j - A_j A_i)s,
\end{aligned}$$

where we used the short form $A_j s = (A^a_{bj}s^b)e_a$. This gives us the expression

$$F_{ij} = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} + [A_i, A_j].$$

The commutator represents the non-linear part of the curvature. This is the local coordinate formula for curvature (a matrix-valued 2-form), in terms of the locally defined matrix A representing the connection ∇.

Example 2.2.5. A special case occurs with the line bundle, i.e. when $r = 1$. Since 1×1 matrices commute, and $\mathrm{End}(E) \cong E^* \otimes E \cong M \times \mathbf{K}^1$ is trival, the curvature $F^\nabla \in \Omega^2(M \times \mathbf{K}^1) = \Gamma(\bigwedge^2(T^*M) \otimes (M \times \mathbf{K}^1)) = \Omega^2_{\mathbf{K}}(M)$ is an ordinary $\mathbf{K}$-valued 2-form. Further, $F^\nabla = dA$ locally, i.e. $F_{ij} = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}$, so $F = d(A_i dx^i)$. Hence F is closed because it is locally exact (it is usually not globally exact).

Remark 2.2.6. Let's find a shorthand for the above expression of F_{ij}. We start with

$$F^\nabla = \frac{1}{2}(\partial_i A_j - \partial_j A_i)dx^i \wedge dx^j + \frac{1}{2}(A_i A_j - A_j A_i)dx^i \wedge dx^j = dA + A \wedge A,$$

as $dA = dA_i \wedge dx^i = \frac{\partial A_i}{\partial x^j} \wedge dx^j \wedge dx^i$. This is the "local short form" for curvature. We must be careful, because A does not make sense globally, only locally. This is a matrix-valued 2-form on U.

In a local trivialization $(U_\alpha, \varphi_\alpha)$ of E, $\mathrm{End}(E)$ is also trivial. So the curvature is $F^\nabla_\alpha = F^a_b e^b \otimes e_a$, where $\{e_1, \ldots, e_r\}$ is a local frame with an appropriate coframe.So above we showed that $F_\alpha = dA_\alpha + A_\alpha \wedge A_\alpha$, noting that from now on we drop the ∇ and say $F^\nabla = F$ without confusion. Further, we know if (U_β, φ_β) is another local trivialization on E, then

$$\begin{aligned}
& A_\beta = g_{\beta\alpha}A_\alpha g^{-1}_{\beta\alpha} - (dg_{\beta\alpha})^{-1}g_{\beta\alpha} \\
\text{so} \quad & F_\beta = dA_\alpha + A_\beta \wedge A_\beta \\
&\quad = d(gA_\alpha g^{-1}) - (dg)g^{-1} + (gA_\alpha g^{-1} - (dg)g^{-1}) \wedge (gA_\alpha g^{-1} - (dg)g^{-1}) \\
&\quad = (dg)A_\alpha g^{-1} + g(dA_\alpha)g^{-1} - gA_\alpha(dg^{-1}) - (d^2g)g^{-1} + (dg) \wedge (dg^{-1}) + (gA_\alpha g^{-1}) \wedge (gA_\alpha g^{-1}) \\
&\qquad - (dg)g^{-1} \wedge gA_\alpha g^{-1} - gA_\alpha g^{-1} \wedge (dg)g^{-1} + (dg)g^{-1} \wedge (dg)g^{-1} \\
&\quad = g(dA_\alpha + A_\alpha \wedge A_\alpha)g^{-1},
\end{aligned}$$

where we employed the shorthand $g = g_{\beta\alpha}$ and $(dg)g^{-1} + g(dg^{-1}) = 0$ since $gg^{-1} = \mathrm{id}$. Hence $F_\beta = gF_\alpha g^{-1}$. This is what we expected, since for all $p \in M$, $F_p \in \bigwedge^2(T^*_p M) \otimes \mathrm{End}(E_p)$. In particular, $(F_p)_{ij} \in \mathrm{End}(E_p)$.

Now let us look at another interpretation of the curvature, which we need to formalize the Chern–Weil construction of characteristic classes.

Definition 2.2.7. Let ∇ be a connection on E. Define a k-linear operator d^∇ by

$$\begin{array}{rccc} d^\nabla : & \Omega^k(E) & \to & \Omega^{k+1}(E) \\ & \omega \otimes s & \mapsto & d\omega \otimes s + (-1)^k \omega \wedge \nabla s \end{array} \quad \text{for} \quad \begin{array}{ll} \omega \in \Omega^k(M) & d\omega \in \Omega^{k+1}(M) \\ s \in \Gamma(E) & \nabla s \in \Gamma(T^*M \otimes E) \end{array}.$$

To see that this is well-defined, observe that $(f\omega)\otimes s = \omega\otimes(fs)$ and so

$$\begin{aligned} d^\nabla((f\omega)\otimes s) &= d(f\omega)\otimes s + (-1)^k f\omega\wedge\nabla s \\ &= (df)\wedge\omega\otimes s + fd\omega\otimes s + (-1)^k f\omega\wedge\nabla s \\ \text{and}\quad d^\nabla(\omega\otimes(fs)) &= d\omega\otimes(fs) + (-1)^k\omega\wedge\nabla(fs) \\ &= f(d\omega)\otimes s + (-1)^k\omega\wedge(df\otimes s + f\nabla s), \end{aligned}$$

which are the same thing. Hence d^∇ is well-defined.

Example 2.2.8. Consider the special case when $E = M\times\mathbf{K}^1$, the trivial line bundle. Then $\Omega^k(E) = \Gamma(\bigwedge^k(T^*M)\otimes\mathbf{K}^1) = \Omega^k_{\mathbf{K}}(M)$, then set of $\mathbf{K}$-valued k-forms on M. Let $\nabla - d$, the trivial connection on E. Then

$$d^\nabla(\underbrace{\omega\otimes f}_{=\ f\omega}) = d\omega\otimes f + (-1)^k\omega\wedge df = fd\omega + df\wedge\omega = d(\omega f),$$

so $d^\nabla = d$ in this case. So really d^∇ is a generalization of d to non-trivial bundles E and non-trivial connections ∇.

Remark 2.2.9. The space $\Omega^\cdot(E)$ is not an algebra. However, $\Omega^k(\mathrm{End}(E))$ is an algebra. So define, for $T, S\in\mathrm{End}(E)$,

$$(\omega\otimes T)\wedge(\eta\otimes S) = (\omega\wedge\eta)\otimes TS.$$

ω	in $\Omega^k(M)$
η	in $\Omega^\ell(M)$
T, S	in $\Gamma(\mathrm{End}(E))$
TS	in $\Gamma(\mathrm{End}(E))$
$\omega\wedge\eta$	in $\Omega^{k+\ell}(M)$

Further, if $\omega\otimes T\in\Omega^k(\mathrm{End}(E))$ and $\eta\otimes s\in\Omega^\ell(E)$, define $(\omega\otimes T)\wedge(\eta\otimes s) = (\omega\wedge\eta)\otimes T(s)\in\Omega^{k+\ell}(E)$. Now we have maps

$$\begin{array}{rcl} \Omega^k(\mathrm{End}(E))\times\Omega^\ell(\mathrm{End}(E)) & \to & \Omega^{k+\ell}(E) \\ \Omega^k(\mathrm{End}(E))\times\Omega^\ell(E) & \to & \Omega k+\ell(E) \end{array},$$

where the first one is not super-commutative. We claim that for both of these products the Leibniz rule holds. To check this, let $\omega\in\Omega^k(M)$, $\eta\in\Omega^\ell(M)$, $S,T\in\Gamma(\mathrm{End}(E))$ and d^∇ be defined on $\mathrm{End}(E)$-valued forms induced by a connection ∇ on $\mathrm{End}(E)$. We then have

$$\begin{aligned} d^\nabla((\omega\otimes T)\wedge(\eta\otimes s)) - d^\nabla((\omega\wedge\eta)\otimes TS) &= d(\omega\wedge\eta)\otimes TS + (-1)^k\omega\wedge\eta\wedge\nabla(TS) \\ &= (d\omega\wedge\eta + (-1)^k\omega\wedge d\eta)\otimes TS \\ &\qquad + (-1)^{k+\ell}\omega\wedge\eta\wedge((\nabla T)S + T(\nabla S)). \end{aligned}$$

Next observe that

$$\begin{aligned} d^\nabla(\omega\otimes T) &= d\omega\otimes T + (-1)^k\omega\wedge dT \\ \text{and}\quad d^\nabla(\eta\otimes S) &= d\eta\otimes S + (-1)^\ell\eta\wedge dS, \end{aligned}$$

implying

$$\begin{aligned} (d^\nabla(\omega\otimes T))\wedge(\eta\otimes S) &= (d\omega\wedge\eta)\otimes TS + (-1)^{k+\ell}\omega\wedge\eta\wedge(\nabla T)S \\ \text{and}\quad (-1)^k(\omega\otimes T)\wedge d^\nabla(\eta\otimes S) &= (-1)^k(\omega\wedge d\eta)TS + (-1)^{\ell+k}\omega\wedge\eta\wedge T(\nabla S). \end{aligned}$$

Hence we get that

$$\begin{aligned} d^\nabla((\omega\otimes T)\wedge(\eta\otimes S)) &= (d^\nabla(\omega\otimes T))\wedge(\eta\otimes S) + (-1)^k(\omega\otimes T)\wedge(d^\nabla(\eta\otimes S)) \\ \text{and}\quad d^\nabla((\omega\otimes T)\wedge(\eta\otimes s)) &= d^\nabla(\omega\otimes T)\wedge(\eta\otimes s) + (-1)^k(\omega\otimes T)\wedge d^\nabla(\eta\otimes s), \end{aligned}$$

where the second statement comes from an analogous proof. Note that above the d^∇ used were different, but it causes no confusion because of the arguments of each. Now let's see what d^∇ looks like in a local trivialization $\{e_1, \dots, e_r\}$. An E-valued k-form is a finite sum of $\omega \otimes s$, for $\omega \in \Omega^k(M)$, $s = s^i e_i \in \Gamma(E)$ and s^i smooth $\mathbf{K}$-valued functions. Then $\omega \otimes s = \omega \otimes (s^i e_i) = (s^i \omega) \otimes e_i$, i.e.

$$\omega \otimes s = \begin{bmatrix} \omega s^1 \\ \vdots \\ \omega s^r \end{bmatrix} \qquad \text{for} \qquad s = \begin{bmatrix} s^1 \\ \vdots \\ s^r \end{bmatrix},$$

and

$$\begin{aligned} d^\nabla(\omega \otimes s) &= d^\nabla(s^i \omega \otimes e_i) \\ &= d(s^i \omega) \otimes e_i + (-1)^k (s^i \omega) \wedge \nabla e_i \\ &= d(s^i \omega) \otimes e_j + (-1)^k s^i \omega \wedge a A_i^j e_j \\ &= (d(s^j \omega) + (A_i^j s^i) \wedge \omega) e_j \\ &= (d + A \wedge \cdot)(\omega \otimes s), \end{aligned}$$

that is,

$$d^\nabla(\omega \otimes s) = d \begin{bmatrix} \omega s^1 \\ \vdots \\ \omega s^r \end{bmatrix} + A \begin{bmatrix} \omega s^1 \\ \vdots \\ \omega s^r \end{bmatrix}.$$

In words, in a fixed local trivialization, $d^\nabla = d + A \wedge \cdot$. Similarly, if in a local trivialization $\{e_1, \dots, e_r\}$ on E, we get a local trivialization of $\mathrm{End}(E)$,

$$\omega \otimes S = \omega \begin{bmatrix} s^{11} & \cdots & s^{1r} \\ \vdots & \ddots & \vdots \\ s^{r1} & \cdots & s^{rr} \end{bmatrix} = \begin{bmatrix} \omega s^{11} & \cdots & \omega s^{1r} \\ \vdots & \ddots & \vdots \\ \omega s^{r1} & \cdots & \omega s^{rr} \end{bmatrix}.$$

✠ Exercise 2.2.10. Check that $d^\nabla(\omega \otimes s) = (d + [A, \cdot])(\omega \otimes s)$. This shows that $d^\nabla = d + [A, \cdot]$.

Lemma 2.2.11. The map $(d^\nabla)^2 : \Omega^k(E) \to \Omega^{k+2}(E)$ is linear over $\Omega^k(M)$. That is, $(d^\nabla)^2(\omega \otimes s) = \omega \wedge (d^\nabla)^2 s$ for all $\omega \in \Omega^k(M)$ and $s \in \Gamma(E)$.

Proof: This follows from the calculation below:

$$\begin{aligned} (d^\nabla)^2(\omega \wedge s) &= d^\nabla(d\omega \otimes s + (-1)^k \omega \wedge \nabla s) \\ &= d^2 \omega \otimes s + (-1)^k d\omega \wedge \nabla s + (-1)^k d\omega \wedge \nabla s + (-1)^{k+k} \omega (d^\nabla)^2 s \\ &= \omega \wedge (d^\nabla)^2 s. \end{aligned}$$

■

A special case occurs when $k = 0$, for which $(d^\nabla)^2(fs) = f(d^\nabla)^2 s$.

Proposition 2.2.12. The equation $(d^\nabla)^2(\omega \otimes s) = \omega \wedge (d^\nabla)^2 s = \omega \wedge F^\nabla \wedge s$ holds. That is, $(d^\nabla)^2 s = F^\nabla \wedge s$, for $F^\nabla \in \Omega^2(\mathrm{End}(E))$ and $s \in \Omega^0(E)$.

Proof: Let $\{e_1, \dots, e_r\}$ be a local frame for E. Let $\{\theta_1, \dots, \theta_r\}$ be a local frame for E. Let $\{\theta_1, \dots, \theta_n\}$ be a local frame for TM. Then $\{\theta^1, \dots, \theta^n\}$ is the dual coframe of T^*M. Then

$$\begin{aligned} (d^\nabla)^2(s) &= d^\nabla(d^\nabla s) \\ &= d^\nabla(\delta s) \\ &= d^\nabla(\theta^k \otimes \nabla_{\theta_k} s) \\ &= d\theta^k \otimes \nabla_{\theta_k} s + (-1)^1 \theta^k \wedge \nabla(\nabla_{\theta_k} s) \\ &= d\theta^k \otimes \nabla_{\theta_k} s - \theta^k \wedge \theta^j \otimes \nabla_{\theta_j}(\nabla_{\theta_k} s). \end{aligned}$$

The above followed as $\nabla_X s = \nabla_{X^i\theta_i} s = X^i \nabla_{\theta_i} s = \theta^i(X) \otimes \nabla_{\theta_i} s = (\theta^i \otimes \nabla_{\theta_i} s)(X)$. We now let $X = X^a\theta_a$ and $Y = Y^b\theta_b$. Then

$$\begin{aligned}
((d^\nabla)^2 s)(X,Y) &= (d\theta^k)(X,Y)\nabla_{\theta_k} s - (\theta^k \wedge \theta^j)(X,Y)\nabla_{\theta_j}\nabla_{\theta_k} s \\
&= (X(\theta^k(Y)) - Y(\theta^k(X)) - \theta^k([X,Y]))\nabla_{\theta_k} s - (\theta^k(X)\theta^j(Y) - \theta^k(Y)\theta^j(X))\nabla_{\theta_j}\nabla_{\theta_k} s \\
&= (X(Y^k) - Y(X^k) - [X,Y]^k)\nabla_{\theta_k} s - (X^kY^j - X^jY^k)\nabla_{\theta_j}(\nabla_{\theta_k} s) \\
&= X(Y^k)\nabla_{\theta_k} s - Y(X^k)\nabla_{\theta_k} s - \nabla_{[X,Y]} s - X^k\nabla_Y(\nabla_{\theta_k} s) + Y^k\nabla_X(\nabla_{\theta_k} s) \\
&= \nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X,Y]} s \\
&= F^\nabla(X,Y)s.
\end{aligned}$$

■

Hence the non-vanishing of the curvature F^∇ of ∇ measures the failure of $d^\nabla : \Omega^\bullet(E) \to \Omega^{\bullet+1}(E)$ to be a complex. As an aside, note that $(d\alpha)(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y])$.

Proposition 2.2.13. Let ∇ be a connection on E. Let $B \in \Omega^1(\mathrm{End}(E))$. Then $\widetilde{\nabla} = \nabla + B$ is a connection on E. Moreover, $F^{\nabla+B} = F^\nabla + d^\nabla(B) + B \wedge B$.

Proof: This can be proven true by showing it is true locally (since no choice is involved). So in a local trivialization, $\nabla = d + A$ for $A \in \Omega^1(\mathrm{End}(E))$. Then

$$\begin{aligned}
F^\nabla &= dA + A \wedge A, \\
\widetilde{\nabla} &= d + A + B, \\
F^{\widetilde{\nabla}} &= d(A+B) + (A+B) \wedge (A+B) \\
&= dA + A \wedge A + dB + A \wedge B + B \wedge A + B \wedge B.
\end{aligned}$$

Now note that

$$\begin{aligned}
d^\nabla(B) &= d^\nabla(dx^i \otimes B_i) \\
&= -dx^i \wedge \nabla B_i \\
&= -dx^i \wedge (dB_i + [A, B_i]) \\
&= dB_i \wedge dx^i - dx^i \wedge (dx^j \otimes A_jB_i - B_i dx^j \otimes A_j) \\
&= dB + A \wedge B + B \wedge A \\
&= d^\nabla(B).
\end{aligned}$$

■

The Bianchi identity introduced below corresponds to the 2nd Bianchi identity from Riemannian geometry.

Proposition 2.2.14. [Bianchi identity]
Let ∇ be a connection on E. Then $d^\nabla(F^\nabla) = 0$.

Proof: (First proof) In a fixed local trivialization, $d^\nabla = d + [A, \cdot]$ on $\Omega^\bullet(\mathrm{End}(E))$, and $F^\nabla = dA + A \wedge A$. Now comupte

$$\begin{aligned}
d^\nabla(F^\nabla) &= d(dA + A \wedge A) + [A, dA + A \wedge A] \\
&= d^2A + dA \wedge A - A \wedge dA + A \wedge dA + A \wedge A \wedge A - A \wedge A \wedge A - dA \wedge A \\
&= 0.
\end{aligned}$$

Proof: (Second proof) Note that $d^\nabla(F^\nabla) \in \Omega^3(\mathrm{End}(E))$. By Leibniz, for $s \in \Omega^\bullet(E)$,

$$\begin{aligned} d^\nabla(F^\nabla) \wedge s &= d^\nabla(F^\nabla \wedge s) - F^\nabla \wedge d^\nabla s \\ &= d^\nabla((d^\nabla)^2 s) - (d^\nabla)^2(d^\nabla s) \\ &= (d^\nabla)^3 s - (d^\nabla)^3 s \\ &= 0. \end{aligned}$$

So $d^\nabla(F^\nabla) \wedge s = 0$ for all $s \in \Gamma(E)$, so $d^\nabla(F^\nabla) = 0$. ■

Proposition 2.2.15. Let (E^1, ∇^1), (E^2, ∇^2) be vector bundles with connections. Recall that $E^1 \oplus E^2$ has an induced connection $\nabla = \nabla^1 \oplus \nabla^2$ and $E^1 \otimes E^2$ also has an induced connection $\nabla = \nabla^1 \otimes \nabla^2$. Then

$$F^\nabla = F^{\nabla^1} \oplus F^{\nabla^2} \qquad \text{and} \qquad F^\nabla = F^{\nabla^1} \otimes \mathrm{id}_{E^2} + \mathrm{id}_{E^1} \otimes F^{\nabla^2},$$

for the appropriate ∇ in each expression.

Proof: Since $\nabla_X(s_1 \oplus s_2) = (\nabla^1_X s_1) \oplus (\nabla^2_X s_2)$, we have that $(d^\nabla)^2(s_1 \oplus s_2) = (d^{\nabla^1})^2 s_1 \oplus (d^{\nabla^2})^2 s_2$ (this remains to be checked). Hence $F^\nabla \in \Omega^2(\mathrm{End}(E^1 \oplus E^2))$, so

$$F^\nabla = \begin{pmatrix} F^{\nabla^1} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & F^{\nabla^2} \end{pmatrix} = \begin{pmatrix} F^{\nabla^1} & 0 \\ 0 & F^{\nabla^2} \end{pmatrix}.$$

For the second identity, proceed as above, with $\nabla_X(s_1 \otimes s_2) = (\nabla^1_X s_1) \otimes s_2 + s_1 \otimes (\nabla^2_X s_2)$. This is left as an exercise. ■

We would like to compute the curvature of the dual connection with respect to the curvature of the original connection.

Proposition 2.2.16. Let (E, ∇) be a vector bundle with a connection. Let (E^*, ∇^*) be the dual bundle with the dual connection, so $F^{\nabla^*} \in \Omega^2(\mathrm{End}(E^*)) = \Omega^2((\mathrm{End}(E))^2)$. Then $F^{\nabla^*} = -(F^\nabla)^*$.

Proof: Let $s \in \Omega^0(E)$ and $\omega \in \Omega^0(E^*)$, so $\omega(s) \in \Omega^0(M) = C^\infty(M)$. Then

$$\begin{aligned} X(\omega(s)) &= (\nabla^*_X \omega)(s) + \omega(\nabla_X s) \\ \text{and} \quad Y(X(\omega(s))) &= (\nabla^*_Y \nabla^*_X \omega)(s) + (\nabla^*_X \omega)(\nabla_Y s) + (\nabla^*_Y \omega)(\nabla_X s) + \omega(\nabla_Y \nabla_X s), \end{aligned}$$

implying

$$\begin{aligned} 0 &= X(Y(\omega(s))) - Y(X(\omega(s))) - [X,Y](\omega(s)) \\ &= (F^{\nabla^*}\omega)(s) + \omega(F^\nabla s) \\ &= (F^{\nabla^*}\omega)(s) + ((F^\nabla)^*)(s). \end{aligned}$$

That is, $F^{\nabla^*} = -(F^\nabla)^*$. ■

Remark 2.2.17. Let (E, ∇) be a bundle with connection, and $f : N \to M$ smooth. Then $(f^*E, f^*\nabla)$ is also a bundle with a connection. It is left as an exercise to show that

$$F^{f^*\nabla} = f^*(F^\nabla).$$

If $F^\nabla = 0$, then ∇ is called a *flat* connection. Not all bundles admit flat connections (trivial ones always do). Further, $F^\nabla = 0$ iff for all $p \in M$, there exists $U \ni p$ such that $E|_U$ admits a global parallel frame.

2.3 Chern–Weil theory of characteristic classes

Characteristic classes of E with $\pi : E \to M$ are cohomology classes of M, so $c(E) \in H^\bullet(M, \mathbf{K})$, in the de Rham cohomology. They "measure" the non-triviality of E. We begin with some algebra.

Definition 2.3.1. Let V be a finite-dimensional vector space over $\mathbf{K}$. Let $P : V^{\times k} \to \mathbf{K}$ be a k-linear symmetric map. Define $\tilde{P} : V \to \mathbf{K}$ by $\tilde{P}(v) = P(v, \ldots, v)$. Notice that $\tilde{P}(\lambda v) = \lambda^k \tilde{P}(v)$, so we say that $\tilde{P}$ is *homogeneous* of degree k. Moreover,

$$P(v_1, \ldots, v_k) = \frac{1}{k!} \frac{\partial^k}{\partial t_1 \cdots \partial t_k} \tilde{P}(t_1 v_1 + \cdots + t_k v_k).$$

We say that P is obtained from $\tilde{P}$ by *polarization*. Proof of the polarization identity is left as an exercise. In fact, if $\tilde{P} : V \to \mathbf{K}$ is homogeneous of degree k, then the identity above defines a k-linear symmetric map.

Definition 2.3.2. Let $V = \mathrm{End}(\mathbf{K}^r) \cong gl(r, \mathbf{K})$. A k-linear symmetric map $P : gl(r, \mathbf{K})^{\times k} \to \mathbf{K}$ is called *invariant* if for all $Q \in GL(r, \mathbf{K}) = \mathrm{Aut}(r, \mathbf{K})$ and all $B_1, \ldots, B_k \in gl(r, \mathbf{K})$, we have

$$P(QB_1Q^{-1}, \ldots, QB_kQ^{-1}) = P(B_1, \ldots, B_k).$$

This is equivalent to $\tilde{P}(QBQ^{-1}) = \tilde{P}(B)$ for all $B \in gl(r, \mathbf{K})$.

Example 2.3.3. Let $B \in gl(r, \mathbf{K})$ and observe that

$$\det(I + tB) = \sum_{k=0}^{r} t^k \underbrace{\sigma_k(B)}_{\in\ \mathbf{K}},$$

where $\sigma_0(B) = 1$, $\sigma_1(B) = \mathrm{Tr}(B), \ldots, \sigma_r(B) = \det(B)$, which are the *elementary symmetric polynomials* of B. Note that each $\sigma_k : gl(r, \mathbf{K}) \to \mathbf{K}$ is homogeneous of degree k and invariant, hence determined by polarization on invariant $\mathbf{K}$-linear symmetric maps.

Lemma 2.3.4. If P is invariant, then for all $B, B_i \in gl(r, \mathbf{K})$,

$$\sum_{i=1}^{k} P(B_1, \ldots, B_{i-1}, [B, B_i], B_{i+1}, \ldots, B_k) = 0.$$

Proof: Take $Q = e^{tB}$, $Q^{-1} = e^{-tB} \in GL(r, \mathbf{K})$, for which

$$P(QB_1Q^{-1}, \ldots, QB_kQ^{-1}) = P(B_1, \ldots, B_k).$$

Differentiate this expression with respect to t and set $t = 0$, so

$$\frac{d}{dt}(QB_1Q^{-1}) = \frac{d}{dt}(e^{tB}B_je^{-tB}) = Be^{tB}B_je^{-tB} - e^{tB}B_jBe^{-tB}.$$

Then $t = 0$ evaluates the above expression as $BB_j - B_jB = [B, B_j]$. ■

Proposition 2.3.5. Let P be an invariat k-linear symmetric map on $gl(r, \mathbf{K})$. Then for any vector bundle E of rank r and any partition $i_1 + \cdots + i_k = m$ for $0 \leqslant i_j \leqslant m$, there exists a naturally induced map

$$\begin{array}{rrcl} p: & \Omega^{i_1}(\mathrm{End}(E)) \times \cdots \times \Omega^{i_k}(\mathrm{End}(E)) & \to & \Omega^m_{\mathbf{K}}(M) \\ & (\omega_1 \otimes T_1, \ldots, \omega_k \otimes T_k) & \mapsto & \omega_1 \wedge \cdots \wedge \omega_k P(T_1, \ldots, T_k) \end{array},$$

where $\omega_j \in \Omega^{i_j}(M)$ and $T_j \in \Gamma(\mathrm{End}(E))$.

Proof: In a trivialization, $E_p \cong \mathbf{K}^r$, $\mathrm{End}(E_p) \cong gl(r, \mathbf{K})$, so this is well-defined since P is invariant. ∎

Definition 2.3.6. There exists a bracket $[\,\cdot\,,\,\cdot\,]$ such that for $\omega \in \Omega^k(M)$, $\eta \in \Omega^\ell(M)$ and $T, S \in \Gamma(\mathrm{End}(E))$,

$$\begin{array}{rrcl} [\,\cdot\,,\,\cdot\,]: & \Omega^k(\mathrm{End}(E)) \times \Omega^\ell(\mathrm{End}(E)) & \to & \Omega^{k+\ell}(\mathrm{End}(E)) \\ & (\omega \otimes T, \eta \otimes S) & \mapsto & (\omega \wedge \eta) \otimes [T, S] \end{array}.$$

Remark 2.3.7. Note that $[\,\cdot\,,\,\cdot\,]$ is not always symmetric. That is, for $B \in \Omega^1(\mathrm{End}(E))$, with $B = dx^1 \otimes B_i$ locally and B_i matrices, we have that

$$\begin{aligned} [B, B] &= [dx^i \otimes B_i, dx^j \otimes B_j] \\ &= dx^i \wedge dx^j [B_i, B_j] \\ &= dx^i \wedge dx^j (B_i B_j - B_j B_i) \\ &= 2dx^i \wedge dx^j B_i B_j \\ &= 2B \wedge B. \end{aligned}$$

Hence $B \wedge B = \frac{1}{2}[B, B] \neq 0$ in general.

Lemma 2.3.8. [Generalization of infinitesimal invariants]
Let $C_1, \ldots, C_k \in \Omega^{even}(\mathrm{End}(E))$, where the even index might change for each C_i. Let $B \in \Omega^1(\mathrm{End}(E))$ and $P : gl(r, \mathbf{K}) \to \mathbf{K}$ be a k-linear symmetric invariant map. Then

$$\sum_{j=1}^{k} (-1)^{i_1 + \cdots + i_{j-1}} P(C_1, \ldots, C_{j-1}, [B, C_j], C_{j+1}, \ldots, C_k) = 0.$$

Proof: The proof by linearity, assumes wlog that everything can be decomposable. We start with $\omega \in \Omega^1(M)$, $S \in \Gamma(\mathrm{End}(E))$ and $B = \omega \otimes S$. Also, $\omega_j \in \Omega^{even}(M)$, $T_j \in \Gamma(\mathrm{End}(E))$, and $C_j = \omega_j \otimes T_j$. Then

$$[B, C_j] = \omega \wedge \omega_j [S, T_j]$$

and

$$\begin{aligned} P(C_1, \ldots, C_{j-1}, [B, C_j], C_{j+1}, \ldots, C_k) &= \omega_1 \wedge \cdots \wedge \omega_{j-1} \wedge (\omega \wedge \omega_j) \wedge \omega_{j+1} \wedge \cdots \wedge \omega_k \\ &\qquad P(T_1, \ldots, T_{j-1}, [S, T_j], T_{j+1}, \ldots, T_k) \\ &= \omega \wedge (\omega_1 \wedge \cdots \wedge \omega_k) P(T_1, \ldots, T_{j-1}, [S, T_j], T_{j+1}, \ldots, T_k). \end{aligned}$$

Now sum over all j from 1 to k and apply the next lemma. The proof will be finished below.

Lemma 2.3.9. Let $\gamma_j \in \Omega^{i_j}(\mathrm{End}(E))$ for $j = 1, \ldots, k$, so $P(\gamma_1, \ldots, \gamma_k) \in \Omega^{i_1 + \cdots + i_k}(M)$. Then, for any connection ∇ on E,

$$d(P(\gamma_1, \ldots, \gamma_k)) = \sum_{j=1}^{k} (-1)^{i_1 + \cdots + i_{j-1}} P(\gamma_1, \ldots, \gamma_{j-1} d^\nabla \gamma_j, \gamma_{j+1}, \ldots, \gamma_k).$$

Proof: Fix a local trivialization on U. Then $\gamma_j \in \Omega^{i_j}(\mathbf{K}^{r \times r})$, or equivalently γ_j is a matrix of i_j-forms on U. In this trivalization, d^∇ on $\Omega^\bullet(\mathrm{End}(E))$ is $d + [A, \cdot]$, i.e. $d^\nabla \gamma_j = d\gamma_j + [A, \gamma_j]$. Without loss of generality, $\gamma_j = \omega_j \otimes T_j$ for $\omega_j \in \Omega^{i_j}(M)$ and $T_j \in \Gamma(\mathrm{End}(E))$. First compute

$$\begin{aligned} P(\gamma_1, \ldots, \gamma_k) &= P(\omega_1 \otimes T_1, \ldots, \omega_k \otimes T_k) \\ &= \omega_1 \wedge \cdots \wedge \omega_k P(T_1, \ldots, T_k) \\ &= \omega_1 \wedge \cdots \wedge \omega_k P((T_1)^{a_1}_{b_1} e^{b_1} \otimes e_{a_1}, \ldots, (T_k)^{a_k}_{b_k} e^{b_k} \otimes e_{a_k}) \\ &= (\omega_1 (T_1)^{a_1}_{b_1}) \wedge \cdots \wedge (\omega_k (T_k)^{a_k}_{b_k}) \underbrace{P(e^{b_1} \otimes e_{a_1}, \ldots, e^{b_k} \otimes e_{a_k})}_{\text{constant function}}, \end{aligned}$$

for $T_j = (T_j)^{a_j}_{b_j} e^{b_j} \otimes e_{a_j}$. Then

$$\begin{aligned} d(P(\gamma_1,\dots,\gamma_k)) &= \sum_{j=1}^{k} (-1)^{i_1+\cdots+i_{j-1}} (\gamma_1)^{a_1}_{b_1} \wedge \cdots \wedge d((\gamma_j)^{a_j}_{b_j}) \wedge \cdots \wedge (\gamma_k)^{a_k}_{b_k} P(e^{b_1} \otimes e_{a_1}, \dots, e^{b_k} \otimes e_{a_k}) \\ &= \sum_{j=1}^{k} (-1)^{i_1+\cdots i_{j-1}} P(\gamma_1,\dots,\gamma_{j-1}, d\gamma_j, \gamma_{j+1},\dots,\gamma_k), \end{aligned}$$

for $\gamma_j = (\gamma_j)^a_b e^b \otimes e_a$, i.e. $(\gamma_j)^a_b = \omega(T_j)^a_b$. ■

Remark 2.3.10. Consider the bracket $[\cdot,\cdot]$ on $\Omega^\bullet(\mathrm{End}(E))$, acting on $A = dx^i \otimes A_i$ for $A_i \in \Gamma(\mathrm{End}(E))$, and $\omega \in \Omega^k(U)$. Then $\omega \otimes S \in \Omega^k(\mathrm{End}(E))$ was defined by

$$\begin{aligned} [A, \omega \otimes S] &= [dx^i \otimes A_i, \omega \otimes S] \\ &= dx^i \wedge \omega \otimes [A_i, S] \\ &= dx^i \wedge \omega \otimes (A_i S - S A_i) \\ &= (dx^i \otimes A_i) \wedge (\omega \otimes S) - (-1)^k (\omega \otimes S) \wedge (dx^i \otimes A_i) \\ &= A \wedge (\omega \otimes S) - (-1)^k (\omega \otimes S) \wedge A. \end{aligned}$$

On the next assignment, we will see that d^∇ on $\Omega^\bullet(\mathrm{End}(E))$ is given in a trivialization by $d^\nabla(\gamma) = d\gamma + [A,\gamma]$.

We now finish the proof of Lemma 2.3.8.

Proof: In a local trivialization, $d^\nabla \gamma_j = d\gamma_j + [A,\gamma_j]$ from Lemma 2.3.9. Then

$$\begin{aligned} d(P(\gamma_1,\dots,\gamma_k)) = &\sum_{j=1}^{k} (-1)^{i_1+\cdots i_{j-1}} P(\gamma_1,\dots,\gamma_{j-1}, d^\nabla\gamma_j, \gamma_{j+1},\dots,\gamma_k) \\ &- \sum_{j=1}^{k} (-1)^{i_1+\cdots+i_{j-1}} P(\gamma_1,\dots,\gamma_{j-1}, [A,\gamma_j], \gamma_{j+1},\dots,\gamma_k) \\ = &\ 0 \end{aligned}$$

by Lemma 2.3.9. ■

Theorem 2.3.11. [Chern, Weil]
Let ∇ be any connection on E with curvature $F^\nabla \in \Omega^2(\mathrm{End}(E))$.

1. For any k-linear symmetric invariant map $P : gl(r,\mathbf{K})^{\otimes k} \to \mathbf{K}$, the $\mathbf{K}$-valued $2k$-form $\tilde{P}(cF^\nabla) \in \Omega^{2k}_{\mathbf{K}}(M)$ is closed for all $c \in \mathbf{K}$.

2. If ∇^0, ∇^1 are two connections on E, then

$$[\tilde{P}(cF^{\nabla^0})] = [\tilde{P}(cF^{\nabla^1})] \in H^{2k}(M,\mathbf{C}).$$

Proof: **1.** Note that $\tilde{P}(cF^\nabla) = P(cF^\nabla,\dots,cF^\nabla)$ and

$$d(\tilde{P}(cF^\nabla)) = d(P(cF^\nabla,\dots,cF^\nabla)) = 0$$

by Lemma 2.3.8 and the Bianchi identity $d^\nabla(F^\nabla) = 0$.

2. We know $\nabla^1 = \nabla^0 + B$ for some global $B \in \Omega^1(\mathrm{End}(E))$. Define $\nabla^t = \nabla^0 + tB$ for $t \in [0,1]$, so $\nabla^0 = \nabla^0$ and $\nabla^1 = \nabla^1$. Let $P(t) = \tilde{P}(F^{\nabla^t})$. We need to show that $P(1) - P(0)$ is exact, so let

$$F^t = F^{\nabla^t} = F^{\nabla^0 + tB} = F^{\nabla^0} + t d^{\nabla^0} B + \frac{t^2}{2}[B,B].$$

Then

$$\frac{d}{dt}F^t = d^{\nabla^0}B + t[B,B] = (d^{\nabla^0} + [tB_j,\cdot])B.$$

Further, $\nabla^t = \nabla^0 + tB$ implies that (check this) $d^{\nabla^t} = d^{\nabla^0} + [tB,\cdot]$ on $\Omega^\bullet(\mathrm{End}(E))$, so $\frac{d}{dt}F^t = d^{\nabla^t}B$. Next, define

$$\underbrace{(TP)(\nabla^1,\nabla^0)}_{\in\ \Omega^{2k-1}_{\mathbf{K}}(M)} = \mathbf{K}\int_0^1 P(F^t,\ldots,F^t,B)dt.$$

We will show that $d((TP)(\nabla^1,\nabla^0)) = P(1) - P(0)$, which will complete the proof. First note that

$$\begin{aligned} P(1) - P(0) &= \int_0^1 \left(\frac{d}{dt}P(t)\right)dt \\ &= \int_0^1 \left(P\left(\frac{d}{dt}F^t,\ldots,F^t\right) + \cdots + P\left(F^t,\ldots,\frac{d}{dt}F^t\right)\right)dt, \end{aligned}$$

and since P is symmetric on $\Omega^{even}(\mathrm{End}(E))$,

$$= \mathbf{K}\int_0^1 P(F^t,\ldots,F^t,d^{\nabla^t}B)dt.$$

Finally, since

$$d((TP)(\nabla^1,\nabla^0)) = \mathbf{K}\int_0^1 d(P(F^t,\ldots,F^t,B))dt = \mathbf{K}\int_0^1 P(F^t,\ldots,F^t,d^{\nabla^t}B)dt,$$

the result follows from Lemma 2.3.8 and the Bianchi identity. ■

Remark 2.3.12. So far, we have learned that given $\pi : E \to M$ a $\mathbf{K}^r$-bundle, P a k-linear symmetric invariant map, and $c \in \mathbf{K}$, we get $[\tilde{P}(cF^\nabla)] \in H^{2k}(M,\mathbf{C})$, a well-defined cohomology class.

Example 2.3.13. Let $\mathbf{K} = \mathbf{C}$ and $r = \mathrm{rank}(E)$, Then for $B \in \mathbf{K}^{r\times r}$

$$\det(I + tB) = \sum_{k=0}^r t^k\sigma_k(B),$$

so $\sigma_0(B) = 1$, $\sigma_1(B) = \mathrm{Tr}(B),\ldots,\sigma_r(B) = \det(B)$. We then let $\tilde{P}_k = \sigma_k$, which is an invariant homeomorphism of degree k.

2.4 Chern, Todd, and Pontryagin classes

Definition 2.4.1. Define the kth *Chern form* of E with respect to the connection ∇ to be

$$c_k(E,\nabla) = \sigma_k\left(\frac{i}{2\pi}F^\nabla\right) \in \Omega^{2k}_{\mathbf{K}}(M).$$

Further, the kth *Chern class* of E is defined as

$$c_k(E) = [c_k(E,\nabla)] \in H^{2k}(M,\mathbf{C}).$$

Notice that $c_0(E) = 1$ and $c_k(E) = 0$ for $k > \mathrm{rank}(E)$. Moreover, $c_k(E) = 0$ if $2k > \dim(M)$. Finally, define the *total Chern class* of E to be

$$c(E) = \sum_{k=0}^\infty c_k(E) = \left[\det\left(I + \frac{i}{2\pi}F^\nabla\right)\right] \in H^{even}(M,\mathbf{C}).$$

Example 2.4.2. If $E = M \times \mathbf{C}^r$ is trivial, all $c_k(E) = 0$ because we can take $\nabla = d$, the trivial connection, so $F^d = 0$. However, the converse is not true. Next consider the exponent of a matrix, for which

$$\mathrm{Tr}(e^B) = \mathrm{Tr}\left(\sum_{k=0}^{\infty} \frac{B^k}{k!}\right)$$

is invariant. Note that $e^{QBQ^{-1}} = Qe^B Q^{-1}$, so $\mathrm{Tr}(e^{tB}) = \sum_{k=0}^{\infty} t^k \tilde{P}_k(B)$.

Definition 2.4.3. Define the kth *Chern character form* of E with respect to ∇ to be

$$ch_k(E, \nabla) = \frac{1}{k!}\mathrm{Tr}\left(\left(\frac{i}{2\pi}F^{\nabla}\right)^k\right) = \tilde{P}_k\left(\frac{i}{2\pi}F^{\nabla}\right) \in \Omega^{2k}_{\mathbf{C}}(M).$$

Further, define the kth *Chern character* of E to be

$$ch_k(E) = [ch_k(E, \nabla)] \in H^{2k}(M, \mathbf{C}).$$

Observe that

$$\begin{aligned} &ch_0(E) = r = \mathrm{rank}(E), \\ &ch_1(E) = \left[\mathrm{Tr}\left(\frac{i}{2\pi}F^{\nabla}\right)\right] = c_1(E), \\ &ch_2(E) \neq c_2(E) \text{ in general, and} \\ &ch_k(E) = 0 \text{ if } 2k > \dim(M), \text{ but } ch_k(F) \text{ may be non-zero for } k > r. \end{aligned}$$

Finally, define the *total Chern character* of E to be

$$ch(E) = \sum_{k=0}^{\infty} ch_k(E) = \left[\mathrm{Tr}\left(e^{\frac{i}{2\pi}F}\right)\right] \in H^{even}(M, \mathbf{C}).$$

Example 2.4.4. Take the expression

$$\underbrace{\frac{\det(tB)}{\det(I - e^{-tB})}}_{\substack{\text{invariant map} \\ \text{on matrices}}} = \sum_{k=0}^{\infty} t^k \underbrace{\tilde{P}_k(B)}_{\substack{\text{homogeneous} \\ \text{of degree } k}}$$

and define the kth *Todd form* of E with respect to ∇, the kth *Todd class* of E, and the *total Todd class* of E by

$$\begin{aligned} td_k(E, \nabla) &= \tilde{P}_k\left(\frac{i}{2\pi}F^{\nabla}\right) \quad \in \Omega^{2k}_{\mathbf{C}}(M), \\ td_k(E) &= [td_k(E, \nabla)] \quad \in H^{2k}(M, \mathbf{C}), \\ td(E) &= \sum_{k=0}^{\infty} td_k(E) \quad \in H^{even}(M, \mathbf{C}). \end{aligned}$$

Lemma 2.4.5. Suppose h is a fiber metric on a $\mathbf{K}^r$-bundle. Let ∇ be a connection on E compatible with h. Then $F^{\nabla} \in \Omega^2(\mathrm{End}_-(E))$.

Here, $\mathrm{End}_-(E)_p = \mathrm{End}_-(E_p)$, the set of endomorphisms of E_p that are infinitesimal isometries with respect to h_p. If $\mathbf{K} = \mathbf{R}$, then $\mathrm{End}_-(E_p) = O(r)$, the set of skew-symmetric matrices. If $\mathbf{K} = \mathbf{C}$, then $\mathrm{End}_-(E_p) = U(r)$, the set of skew-Hermitian matrices.

Proof: In a local trivialization, $\nabla = d + A \wedge \cdot$. Let $\{e_1, \dots, e_r\}$ be a local orthonormal frame with respect to h. Then

$$\begin{aligned} d(h(e_i, e_j')) &= d(\delta_{ij}) = 0 \\ &= h(\nabla e_i, e_j') + h(e_i, \nabla e_j') \\ &= h(A_i^\ell e_\ell, e_j) + h(e_i, A_j^\ell e_\ell) \\ &= A_i^\ell \delta_{\ell j} + \overline{A}_j^\ell \delta_{i\ell} \\ &= A_i^j + \overline{A}_j^i. \end{aligned}$$

In this frame,

$$\begin{aligned} F &= dA + A \wedge A \\ F_i^j &= dA_i^j + A_i^k \wedge A_k^j \\ F_j^i &= dA_j^i + A_j^k \wedge A_k^j \\ &= -d\overline{A}_i^j + \overline{A}_k^j \wedge \overline{A}_i^k \\ &= -\left(\overline{dA_i^j + A_i^k \wedge A_k^j}\right) \\ &= -\overline{F_i^j}, \end{aligned}$$

so F is skew-symmetric (in $\mathbf{R}$) or skew-Hermitian (in $\mathbf{C}$). ■

Corollary 2.4.6. Chern classes, Chern characters, and Todd classes are real, i.e. belong to $H^{even}(M, \mathbf{R})$.

Proof: Let h be any Hermitian metric on E. Let ∇ be any connection compatible with h. In a local trivialization, F^∇ is skew-Hermitian, so $\frac{i}{2\pi} F^\nabla$ is Hermitian. Then the expressions

$$\det\left(I + \frac{i}{2\pi} F\right) \qquad , \qquad \operatorname{Tr}\left(e^{\frac{i}{2\pi}F}\right) \qquad , \qquad \frac{\det\left(\frac{i}{2\pi}F\right)}{\det\left(I - e^{-\frac{i}{2\pi}F}\right)}$$

are all real since $\frac{i}{2\pi}F$ is Hermitian. That is, there exists at each point an invertible Q such that

$$Q\left(\frac{i}{2\pi}F\right)Q^{-1} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_r \end{pmatrix}$$

for $\lambda_i \in \mathbf{R}$. ■

Remark 2.4.7. Let E be a real vector bundle of rank r over M. Consider the expression

$$\det\left(I + \frac{1}{2\pi}F^\nabla\right) = \sum_{k=0}^{r} \underbrace{\sigma_k\left(\frac{1}{2\pi}F^\nabla\right)}_{\in\ \Omega^{2k}_{\mathbf{R}}(M)}.$$

Choose any Riemann fiber metric h on E and any connection ∇ composed with h. Then F is skew-symmetric (with respect to the frame), i.e.

$$F^t = -F \qquad \text{as} \qquad \det\left(I + \frac{t}{2\pi}F\right) = \det\left(\left(I + \frac{t}{2\pi}F\right)^t\right)$$
$$= \det\left(I - \frac{t}{2\pi}F\right)$$
$$= \sum_{k=0}^{r} (-1)^k t^k \sigma_k\left(\frac{1}{2\pi}F\right).$$

This shows that $\sigma(\frac{1}{2\pi}F) = 0$ for k odd, and only for this type of connection. Hence $[\sigma_k(\frac{1}{2\pi}F)] = 0$ for k odd, independent of any choice.

Definition 2.4.8. Define the kth *Pontryagin class* of E to be

$$p_k(E) = \left[\sigma_{2k}\left(\frac{1}{2\pi}F\right)\right] \in H^{4k}(M, \mathbf{R}).$$

Note that $p_k(E) = 0$ if $4k > n = \dim(M)$. Further, define the *total Pontryagin class* of E to be

$$p(E) = \sum_{k=0}^{\infty} p_k(E) = \left[\det\left(I + \frac{1}{2\pi}F^{\nabla}\right)\right] \in H^{4k}(M, \mathbf{R}).$$

Remark 2.4.9. Let E be a $\mathbf{R}^r$-vector bundle. Consider the complexification $E \otimes \mathbf{C}$. This is a $\mathbf{C}^r$-bundle over M whose fiber at p is

$$(E \otimes \mathbf{C})_p = E_p \otimes_{\mathbf{R}} \mathbf{C}.$$

A connection ∇ on E extends to a connection ∇ on $E \otimes \mathbf{C}$ by $\mathbf{C}$-linearity. But what is $c_k(E \otimes \mathbf{C})$? We take $f(t) = \det(I + \frac{t}{2\pi}F)$, so $f(1) = P(E)$ and $f(i) = c(E \otimes \mathbf{C})$. It remains to check the details, but the final result is that

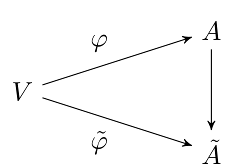
$$p_k(E) = (-1)^k c_{2k}(E \otimes \mathbf{C}) \in H^{4k}(M, \mathbf{R}) \qquad \text{and} \qquad c_k(E \otimes \mathbf{C}) = 0 \quad \text{for } k \text{ odd}.$$

3 Dirac operators on Clifford bundles

The material covered in this section is more general - it applies to any elliptic operator, but we will do just Dirac operators and generalized Laplacians (unless we have more time at the end of the term).

3.1 Clifford algebra

Definition 3.1.1. Let V be an n-dimensional vector space with a symmetric bilinear positive definite form $\langle \cdot\,, \cdot \rangle : V \times V \to \mathbf{R}$. A *Clifford algebra* in V is a real algebra A with unit 1 and a map $\varphi : V \to A$ such that $(\varphi(v))^2 = -\langle v, v\rangle 1$, which is universal with respect to this property. That is, if there exists another map $\tilde{\varphi} : V \to \tilde{A}$ such that $(\tilde{\varphi}(v))^2 = -\langle v, v\rangle 1$, then there exists a unique algebraic homomorphism $A \to \tilde{A}$ such that

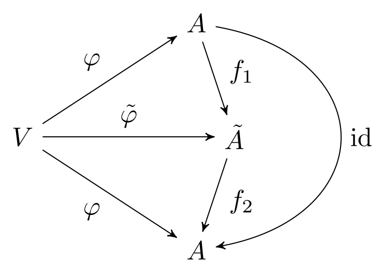


commutes.

Example 3.1.2. Let $\langle\,\cdot\,,\,\cdot\,\rangle = 0$. Then $A = \bigwedge^\bullet V$ is a Clifford algebra for $(V, \langle\,\cdot\,,\,\cdot\,\rangle)$. The map $\varphi : V \to \bigwedge^\bullet V$ is the inclusion, so $(\varphi(v))^2 = v \wedge v = 0 = -\langle v, v\rangle 1$. If $\tilde{\varphi} : V \to \tilde{A}$ is another such map, define $A \to \tilde{A}$ by $v_1 \wedge \cdots \wedge v_k \mapsto \tilde{\varphi}(v_1) \cdots \tilde{\varphi}(v_k)$, which is an algebra homomorphism by construction, and the diagram above commutes. It is left as an exercise to check that this is a unique map.

Proposition 3.1.3. For any $(V, \langle\,\cdot\,,\,\cdot\,\rangle)$, a Clifford algebra exists, and is unique up to isomorphism.

Proof: Let us first check uniqueness. Suppose $A, \tilde{A}$ are two such Clifford algebras. Then the situation may be modeled by the commutative diagram below.



Since the map $A \to A$ on the outside must be id, as it is unique, we have that $f_2 \circ f_1 = \mathrm{id}_A$, so f_1 and f_2 are algebraic isomorphisms.

For existence, let $\{e_1, \ldots, e_n\}$ be any basis for V. Define $A = \mathrm{span}_{\mathbf{R}}\{e_1^{k_1} \cdots e_n^{k_n} \;:\; k_i \in \{0,1\}\}$, so A is a 2^n-dimensional vector space. Define multiplication on A by

$$e_i e_j + e_j e_i = -2\langle e_i, e_j\rangle 1. \tag{1}$$

This rule determines the product of any 2 elements of A. Then takeb $\varphi : V \to A$, given by $e_i \to e_i$. Why is this a Clifford algebra? We want $\varphi(v)\varphi(v) = -\langle v, v\rangle 1$. Let $v = v_1 + v_2$. Then for all $v_1, v_2 \in V$,

$$\begin{aligned} \varphi(v_1+v_2)\varphi(v_1+v_2) &= -\langle v_1+v_2, v_1+v_2\rangle, \\ (\varphi(v_1)+\varphi(v_2))(\varphi(v_1)+\varphi(v_2)) &= -(\langle v_1,v_1\rangle + 2\langle v_1,v_2\rangle + \langle v_2,v_2\rangle)1, \\ \varphi(v_1)\varphi(v_2)+\varphi(v_2)\varphi(v_1) &= -2\langle v_1, v_2\rangle 1. \end{aligned} \tag{2}$$

Hence (1) holds iff (2) holds, by linearity in e_1, e_j. Denote this A by $\mathcal{C}\ell(V, \langle\,\cdot\,,\,\cdot\,\rangle)$, or just by $\langle\,\cdot\,,\,\cdot\,\rangle$ for shorter notation. Note that the map $\varphi : V \to \mathcal{C}\ell(V)$ is injective (as V is a subspace of $C\ell(V)$). ■

Why do we care about $\mathcal{C}\ell(V)$? We will see that Clifford algebras are intimately related to the Laplacian. Let us first look at a special case.

Definition 3.1.4. Let $(V, \langle\,\cdot\,,\,\cdot\,\rangle)$ be as before, $\langle\,\cdot\,,\,\cdot\,\rangle$ positive definite, and $\{e_1, \ldots, e_n\}$ an orthonormal basis. Then $\mathcal{C}\ell(V) \otimes \mathbf{C}$ is the *complexified Clifford algebra*. In this case, we have

$$\begin{aligned} \mathcal{C}\ell(V) \otimes \mathbf{C} &\cong (\mathcal{C}\ell(V \otimes \mathbf{C}), \langle\,\cdot\,,\,\cdot\,\rangle), \\ \varphi(v) \otimes t &\leftrightarrow \varphi(v \otimes t). \end{aligned}$$

Suppose S is a module over $\mathcal{C}\ell(V) \otimes \mathbf{C}$. This means S is a finite-dimensional complex vector space together with a map $(\mathcal{C}\ell(V) \otimes \mathbf{C}) \times S \to S$, where

$$(\alpha+\beta)- = \alpha s + \beta s \qquad , \qquad \alpha(s_1+s_2) = \alpha s_1 + \alpha s_2 \qquad , \qquad \alpha(\beta s) = (\alpha\beta)s.$$

Let $C^\infty(V, S)$ be the space of smooth S-valued functions on V. Then $C^\infty(V, S) = \Gamma(V \times S)$, where $V \times S$ is the trivial $\mathbf{C}$-vector bundle over V with fiber S. Each e_i corresponds to a differential operator $\partial_i = \nabla_{e_i}$, where ∇ is the trivial connection on $V \times S$.

Definition 3.1.5. The *Dirac operator* D on $C^\infty(V, S)$ is a $\mathbf{C}$-linear map $D : C^\infty(V, S) \to C^\infty(V, S)$ given by $Ds = \sum_{i=1}^n e_i(\partial_i s)$, with the multiplication being module multiplication. In other words, $(Ds)_p = \sum_{i=1}^n (\partial_i s)_p$, for $(\partial_i s)_p \in (V \times S)_p = S$.

Proposition 3.1.6. The Dirac operator is independent on the choice of basis.

Proof: Let $\{\tilde{e}_1, \ldots, \tilde{e}_n\}$ be another orthonormal basis, so $\tilde{e}_i = P_i^\ell e_\ell$ for $P \in O(n)$. Then

$$\begin{aligned}
Ds &= \sum_{i=1}^n \tilde{e}_i \nabla_{\tilde{e}_i} s \\
&= \sum_{i=1}^n \left(P_i^\ell e_\ell\right) \left(\nabla_{P_i^k e_k} s\right) \\
&= \sum_{i=1}^n \left(P_i^\ell e_\ell\right) \left(P_i^k \nabla_{e_k} s\right) \\
&= \left(\sum_{i=1}^n P_i^\ell P_i^k\right) e_\ell \left(\nabla_{e_k} s\right) \\
&= \sum_{k=1}^n e_k \nabla_{e_k} s.
\end{aligned}$$

■

Definition 3.1.7. Define the *Laplacian* of s to be

$$\begin{aligned}
D^2 s &= D(Ds) \\
&= \sum_{j=1}^n e_j(\partial_j(Ds)) \\
&= \sum_{j=1}^n e_j \left(\partial_j \sum_{i=1}^n e_i \partial_i s\right) \\
&= \sum_{i,j=1}^n (e_j \cdot e_i)(\partial_j \partial_i s) \\
&= \sum_{i=1}^n (-\partial_i \partial_i s) + \sum_{i \neq j} (e_j \cdot e_i)(\partial_j \partial_i s) \\
&= -\sum_{i=1}^n \partial_i \partial_i s.
\end{aligned}$$

So the Laplacian of s is $-\sum_{i=1}^n \partial^2/\partial(x^i)^2 s$.

Remark 3.1.8. Let's try to do this in a more general setting. Let (M, g) be an oriented Riemannian manifold without boundary. A tangent bundle $\pi : TM \to M$ is a real vector bundle with rank n and a fiber metric g. So (T_pM, g_p) is an n-dimensional real vector space with a positive definite inner product. Then $\mathcal{C}\ell(T_pM, g_p) \otimes \mathbf{C}$ is the complexified Clifford algebra. Hence we obtain $\mathcal{C}\ell(TM) \otimes \mathbf{C}$, which is the $\mathbf{C}$-vector bundle of rank 2^n over M whose fiber over p is $\mathcal{C}\ell(T_pM, g_p) \otimes \mathbf{C}$. Checking of local triviality is left as an exercise.

Suppose S is a bundle of Clifferd modules over M, i.e. S is a $\mathbf{C}$-vector bundle over M such that s_p is a module over $\mathcal{C}\ell(T_pM, g_p)$ (we will see that there are such s that always exist). We need a way to differentiate sections of S, i.e. we need a connection on S.

Definition 3.1.9. We say that S is a *Clifford bundle* if it is equipped with a Hermitian fiber metric h and a h-compatible connection ∇ such that for all $p \in M$, S_p is a module over $\mathcal{C}\ell(T_pM, g_p) \otimes \mathbf{C}$, and

1. the Clifford action of a vector $X_p \in T_pM$ is skew-adjoint with respect to h_p, i.e.

$$h_p(X_p \cdot s_p, t_p) + h(s_p, X_p \cdot t_p) = 0$$

for all $X_p \in T_pM$, $s_p, t_p \in S_p$, and

2. the connection ∇ on S is compatible with the Levi-Civita connection ∇ on TM in the sense that for all $X, Y \in \Gamma(TM)$ and $s, t \in \Gamma(S)$,

$$\nabla_X(Ys) = (\nabla_X Y)s + Y(\nabla_X s).$$

Note that the ∇s are not all the same above.

Definition 3.1.10. The *Dirac operator* $D : \Gamma(S) \to \Gamma(S)$ of S is defined by, for $\{e_1, \ldots, e_n\}$ a local orthonormal frame for TM,

$$\begin{array}{ccccccc} \Gamma(S) & \xrightarrow{\quad\nabla\quad} & \Gamma(T^*M \otimes S) & \xrightarrow[\text{of } \otimes]{\text{mus. iso.}} & \Gamma(TM \otimes S) & \xrightarrow[\text{section}]{\text{Clifford}} & \Gamma(S) \\ s & \longmapsto & \nabla s = e^k \otimes \nabla_{e_k} s & \longmapsto & (e^k)^\# \otimes \nabla_{e_k} s & \longmapsto & g^{k\ell} e_\ell \otimes \nabla_{e_k} s \end{array}.$$

This follows as $(e^k)^\# = g^{kj} e_j \in \Gamma(TM|_U)$, with $g^{\ell k} = g^{k\ell}$. We employed the fact that $\langle (e^k)^\#, e_j \rangle = e^k(e_j) = \delta^k_j$ and $\langle g^{k\ell} e_\ell, e_j \rangle = \delta^k_j$. Moreover, since all the maps used are $\mathbf{C}$-linear, their composition is $\mathbf{C}$-linear.

Remark 3.1.11. Let's compute $D^2 s$ at $p \in M$. Let $\{e_1, \ldots, e_n\}$ be a local orthonormal frame centered at p such that $(\nabla_{e_i} e_j)(p) = 0$ for all j (this is a geodesic frame). We know that $[e_i, e_j] = \nabla_{e_i} e_j - \nabla_{e_j} e_i = 0$, hence $[e_i, e_j]|_p = 0$ for all i, j. So then

$$\begin{aligned} D^2 s &= \sum_{i,j} e_i \nabla_{e_i}(e_j \nabla_{e_j} s) \\ &= \sum_{i,j} e_i \left((\nabla_{e_i} e_j) \cdot (\nabla_{e_j} s) + e_j \nabla_{e_i} \nabla_{e_j} s \right). \end{aligned}$$

Evaluate this at $p \in M$ to get

$$\begin{aligned} (D^2 s)_p &= \sum_{i,j} e_i|_p \left((\nabla_{e_i} e_j)|_p \, (\nabla_{e_j} s)\,\big|_p + e_j|_p \, (\nabla_{e_i} \nabla_{e_j} s)_p \right) \\ &= -\sum_{i=1}^n (\nabla_{e_i} \nabla_{e_i} s)_p + \sum_{i<j} e_i|_p \cdot e_j|_p \left((\nabla_{e_i} \nabla_{e_j} s - \nabla_{e_j} \nabla_{e_i} s)_p - \underbrace{(\nabla_{[e_i, e_j]} s)_p}_{=0} \right) \\ &= \underbrace{-\sum_{i=1}^n (\nabla_{e_i} \nabla_{e_i} s)_p}_{\text{some kind of Laplacian}} + \underbrace{\sum_{i<j} e_i|_p \cdot e_j|_p \cdot F^\nabla(e_i, e_j) s}_{\text{curvature term}}. \end{aligned}$$

We would like to write $-\sum_{i=1}^n \nabla_{e_i} \nabla_{e_i} s$ in a more invariant way.

Definition 3.1.12. For the connection $\nabla : \Gamma(S) \to \Gamma(T^*M \otimes S)$, define the formal *adjoint* $\nabla^* : \Gamma(T^*M \otimes S) \to \Gamma(S)$. We will see that $-\sum_{i=1}^n \nabla_{e_i} \nabla_{e_i} s = \nabla^* \nabla s$. This is the *rough Laplacian*, or *connection Laplacian* on $\Gamma(S)$. This will follow as $\Gamma(S)$, $\Gamma(T^*M \otimes S)$ tave positive-definite Hermitian inner products.

Definition 3.1.13. Let $s, t \in \Gamma(S)$. Define a Hermitian product, with $(h(s,t))_p = h_p(s_p, t_p)$, by

$$\langle\langle s, t\rangle\rangle = \int_M h(s,t)\mathrm{vol}_g.$$

Note that $\langle\langle s, s\rangle\rangle = 0$, so $h(s,s)_p = 0$ for all p. Hence $s_p = 0$ for all p, so $s = 0$. Similarly, we get a positive-definite Hermitian inner product on $\Gamma(T^*M)$ as follows. For $\alpha, \beta \in \Gamma(TM)$, $s, t \in \Gamma(S)$, let

$$\langle\langle \alpha \otimes s, \beta \otimes t\rangle\rangle = \int_M g(\alpha, \beta) h(s,t)\mathrm{vol}_g,$$

so $g(\alpha, \beta) = \alpha_k \beta_\ell g^{k\ell}$ locally. Extend this by linearity and check this is a Hermitian metric. With these metrics, $\Gamma(S)$, $\Gamma(T^*M \otimes S)$ are not complete as normed vector spaces.

3.2 The adjoint and the Hodge star

Definition 3.2.1. Suppose E, F are vector bundles over M with fiber metrics h_E, h_F. Let $P : \Gamma(E) \to \Gamma(F)$ be a linear map. Then $P^* : \Gamma(F) \to \Gamma(E)$ is the formal *adjoint* of P if $\langle\langle Ps, t\rangle\rangle = \langle\langle s, P^*t\rangle\rangle$ for all $s \in \Gamma(E)$ and $t \in \Gamma(F)$.

Moreover, if such an adjoint exists, then it is unique. To see this, suppose that $Q, \widehat{Q}$ satisfy

$$\langle\langle Ps, t\rangle\rangle = \langle\langle s, Qt\rangle\rangle = \left\langle\left\langle s, \widehat{Q}t\right\rangle\right\rangle$$

for all s, t. Then $\left\langle\left\langle s, Qt - \widehat{Q}t\right\rangle\right\rangle = 0$, so take $s = Qt - \widehat{Q}t$. This implies that $Qt = \widehat{Q}t$ for all t, so $Q = \widehat{Q}$.

Proposition 3.2.2. For any vector bundle S with a metric and compatible connection, $\nabla : \Gamma(S) \to \Gamma(T^*M \otimes S)$ has a formal adjoint $\nabla^* : \Gamma(T^*M \otimes S) \to \Gamma(S)$.

Proof: In a local coordinate chart, $\nabla s = dx^j \otimes \nabla_j s$. Let $B \in \Omega^1(M)$, $t \in \Gamma(S)$, so $B \otimes t \in \Gamma(T^*M \otimes S)$. Then, using $\langle\ \rangle$ for the pointwise inner product, and $\langle\langle\ \rangle\rangle$ for the L_2-inner product, we have that

$$\begin{aligned}
\langle \nabla s, B \otimes t\rangle &= \left\langle dx^j \otimes \nabla_j s, B \otimes t\right\rangle \\
&= \left\langle dx^j, B\right\rangle \langle \nabla_j s, t\rangle B_k g^{jk} \left(\nabla_j \langle s, t\rangle - \langle s, \nabla_j t\rangle\right) \\
&= \nabla_j \underbrace{\left(B_k g^{jk} \langle s, t\rangle\right)}_{\text{a vector field } Y} - g^{jk}\left(\nabla_j B_k\right)\langle s, t\rangle - B_k g^{jk} \langle s, \nabla_j t\rangle \\
&= \mathrm{div}(Y) + \underbrace{\left\langle s, -g^{jk}\left(\nabla_j B_k\right) t - B_k g^{jk} \nabla_j t\right\rangle}_{\text{defines a global smooth section of } S}.
\end{aligned}$$

Now we integrate and use the divergence theorem to get that for all s, t, B,

$$\langle\langle \nabla s, B \otimes t\rangle\rangle = 0 + \left\langle\left\langle S, -g^{jk}(\nabla_j B_k)t - B_k g^{jk} \nabla_j t\right\rangle\right\rangle.$$

Hence locally, $\nabla^*(B \otimes t) = -g^{jk}(\nabla_j B_k)t - B_k g^{jk} \nabla_j t$. ■

Remark 3.2.3. Above, we had $t = \nabla_j s$, $B = dx^j$, $B_k = 0$ and $B_j = 1$ for $j \neq k$. Then $\nabla^* \nabla s \in \Gamma(S)$ is given by

$$\begin{aligned}
\nabla^* \nabla s &= \nabla^*(dx^j \otimes \nabla_j s) \\
&= -B_k g^{\ell k} \nabla_\ell \nabla_j s \\
&= -g^{\ell j} \nabla_\ell \nabla_j s \\
&= -\sum_{k=1}^{n} \nabla_{\ell_k} \nabla_{\ell_k} s
\end{aligned}$$

for $\{e_1, \ldots, e_n\}$ orthonormal. Hence $D^2 s = \nabla^*\nabla s + Ks$, where $Ks = \sum_{i<j} e_i \cdot e_j \cdot F^\nabla(e_i, e_j)s$. This is known as the *Bochner–Weitzenböck formula.*

Proposition 3.2.4. The operator $\nabla^*\nabla$ is positive and self-adjoint.

Proof: For self-adjointness, check that $(PQ)^* = Q^*P^*$ and $(P^*)^* = P$, so then $(\nabla^*\nabla)^* = \nabla^*(\nabla^*)^* = \nabla^*\nabla$. For positivity, note that

$$\langle\langle \nabla^*\nabla s, s\rangle\rangle = \langle\langle \nabla s, \nabla s\rangle\rangle = \|\nabla s\|^2 \geqslant 0.$$

■

Proposition 3.2.5. The operators D, D^2, K are all self-adjoint.

Proof: By Bochner–Weitzenböck, it is enough to show that D is self-adjoint. Observe that

$$\begin{aligned}
\langle\langle Ds, t\rangle\rangle &= \int_M \langle e_k \cdot \nabla_{e_k} s, t\rangle \\
&= -\int_M \langle \nabla_{e_k} s, e_k \cdot t\rangle && \text{(by property } \mathbf{1.} \text{ of } S) \\
&= -\int_M \Big(\underbrace{\nabla_{e_k} \langle s, e_k \cdot t\rangle}_{\text{div. of a v.f., so 0 by Stokes}} - \langle s, \nabla_{e_k}(e_k \cdot t)\rangle \Big) && \text{(by metric compatibility)} \\
&= \int_M \langle s, \nabla_{e_k}(e_k \cdot t)\rangle .
\end{aligned}$$

By compatibility in a geodesic frame and by property **2.** of S,

$$\nabla_{e_k}(e_k \cdot t) = \underbrace{(\nabla_{e_k} e_k) \cdot t}_{0 \text{ at } p} + e_k \cdot (\nabla_{e_k} t).$$

Therefore $\langle\langle Ds, t\rangle\rangle = \int_M \langle s, e_k \cdot \nabla_{e_k} t\rangle = \langle\langle s, Dt\rangle\rangle$, so D is self-adjoint. ■

Lemma 3.2.6. $Ds = 0$ iff $D^2 s = 0$.

Proof: The direction $\implies$ is immediate. For the other direction, note that

$$\begin{aligned}
D^2 s = 0 \quad &\implies \quad \big\langle\big\langle D^2 s, s\big\rangle\big\rangle = 0 \\
&\implies \quad \langle\langle Ds, Ds\rangle\rangle = 0 \\
&\iff \quad \|Ds\|^2 = 0 \\
&\iff \quad Ds = 0.
\end{aligned}$$

■

Theorem 3.2.7. [Bochner]
Suppose that the least eigenvalue of K at each point of M is strictly positive. Then there are no non-trivial solutions to $Ds = 0$.

Proof: Pointwise, we can do $\langle K_p s_p, s_p\rangle \geqslant \lambda_{min}(p)|s_p|^2$, if $s_p \neq 0$. Then

$$\frac{\langle K_p s_p, s_p\rangle}{|s_p|^2} \geqslant \lambda_{min}(p) \geqslant 0.$$

For s nowhere-zero, there exists $c > 0$ such that $\frac{\langle K_s, s\rangle}{|s|^2} \geqslant c > 0$. Then

$$\langle\langle D^2 s, s\rangle\rangle = \langle\langle \nabla^*\nabla s, s\rangle\rangle + \langle\langle Ks, s\rangle\rangle$$
$$\|Ds\|^2 = \underbrace{\|\nabla s\|^2}_{\geqslant 0} + \underbrace{\langle\langle Ks, s\rangle\rangle}_{>0},$$

since $s \neq 0$ in some open set by continuity. This implies that $\|Ds\|^2 > 0$, so $Ds \neq 0$. ■

Example 3.2.8. Consider the important example of a Clifford bundle, the complexified exterior bundle $\bigwedge^\bullet(T^*M) \otimes \mathbf{C} \cong \mathcal{C}\ell(TM) \otimes \mathbf{C}$. This is a complex vector bundle that is not a complex vector bundle as an algebra over $\mathbf{C}$. Let $\{e_1, \dots, e_n\}$ be a basis for T_pM and $\{e^1, \dots, e^n\}$ a basis for T_p^*M. The map $e^{i_1} \wedge \cdots \wedge e^{i_k} \mapsto e^{i_1} \cdots e^{i_k}$ is a vector space isomorphism (where multiplication is Clifford multiplication).

Using this isomorphism, we will see that the natural action of $\mathcal{C}\ell(TM) \otimes \mathbf{C}$ on itself makes $\bigwedge^\bullet *(TM) \otimes \mathbf{C}$ into a Clifford bundle. We need to put a metric h and a compatible connection ∇ on $\bigwedge^\bullet(T^*M) \otimes \mathbf{C}$ and check that conditions **1.** and **2.** hold. To do this, we need a brief digression on the Hedge star.

Remark 3.2.9. Let $(V, \langle\, \cdot\, ,\, \cdot\, \rangle)$ be an n-dimensional real oriented positive definite inner product space. This induces positive definite inner products on V^* and $\bigwedge^\bullet(V^*)$. Then $\langle\, \cdot\, ,\, \cdot\, \rangle$ on V^* and $\bigwedge^\bullet(V^*)$ is defined by declaring that $e^{i_1} \wedge \cdots \wedge e^{i_k}$, for $i_1 < \cdots < i_k$ are orthonormal when $e_1, \dots, e_n$ are orthonormal. Alternatively (and equivalently),

$$\langle \alpha, \beta\rangle_{V^*} = \langle \alpha^\sharp, \beta^\sharp\rangle_V \quad \text{and} \quad \langle \alpha^1 \wedge \cdots \wedge \alpha^k, \beta^1 \wedge \cdots \wedge \beta^k\rangle_{\Omega^k(V^*)} = \det\left(\langle \alpha^i, \beta^j\rangle_{V^*}\right)$$

for $\alpha^i, \beta^j \in V^*$. Diagramatically, we then have that

$$\begin{array}{ccccc} \text{an orientation of } V & \xrightarrow{\text{induces}} & \text{an orientation of } V^* & \xrightarrow{\text{induces}} & \text{an orientation of } \bigwedge^n(V^*) \\ \{e_1, \dots, e_n\} & \longrightarrow & \{e^1, \dots, e^n\} & \longrightarrow & \{e^1 \wedge \cdots \wedge e^n\} \end{array},$$

where the last set is a singleton, and all are ariented bases.

Definition 3.2.10. Define the *Hodge star* $*$ by

$$\begin{array}{rccc} *: & \bigwedge^k(V^*) & \to & \bigwedge^{n-k}(V^*) \\ & e^{i_1} \wedge \cdots \wedge e^{i_k} & \mapsto & e^{j_1} \wedge \cdots \wedge e^{j_{n-k}} \end{array},$$

where $\{i_1, \dots, i_k, j_1, \dots, j_{n-k}\} = \{1, \dots, n\}$ and $e^{i_1} \wedge \cdots \wedge e^{i_k} \wedge e^{j_1} \wedge \cdots \wedge e^{j_{n-k}} = e^1 \wedge \cdots \wedge e^n$. Define this on a basis and extend by linearity.

Example 3.2.11. Consider the Hodge star on $\mathbf{R}^3$ with the usual inner product and orientation. Then

$$*(e_1) = e_2 \wedge e_3, \qquad e_1 \wedge e_2 \wedge e_3 = e_2 \wedge (-e_1 \wedge e_3) = e_1 \wedge e_2 \wedge e_3.$$
$$*(e_2) = -e_1 \wedge e_3 = e_2 \wedge e_1,$$

✠ Exercise 3.2.12. We present the following exercises as facts:

1. $*^2 = (-1)^{k(n-k)}$ on $\bigwedge^k(V^*)$
2. $*$ is an isometry, i.e. $\langle *\alpha, *\beta\rangle = \langle \alpha, \beta\rangle$
3. $\alpha \wedge *\beta = \langle \alpha, \beta\rangle\, e^1 \wedge \cdots \wedge e^n = \langle \alpha, \beta\rangle\, \mu$, for μ the volume form (so $|\mu| = 1$)
4. $*1 = u$, $*\mu = 1$

We may now proceed to the global view. Let (M, g) be an oriented Riemannian manifold. Then $* : \Omega^k(M) \mapsto \Omega^{n-k}(M)$.

Proposition 3.2.13. Let $\alpha \in \Omega^1(M)$. Let $\alpha^\sharp \in \Gamma(TM)$ be the metric dual vector field. Let $\omega \in \Omega^k(M)$. Then

$$\alpha^\sharp \lrcorner\, \omega = (-1)^{nk+n} * (\alpha \wedge *\omega) .$$

Corollary 3.2.14. For all $\omega \in \Omega^k(M)$, $\eta \in \Omega^{k-1}(M)$, $\alpha \in \Omega^1(M)$,

$$\left\langle \alpha^\sharp \lrcorner\, \omega, \eta \right\rangle = \langle \omega, \alpha \wedge \eta \rangle .$$

That is, the *interior product* is the adjoint of the exterior product.

Proof: Observe that, for μ the volume form,

$$\begin{aligned}
\left\langle \alpha^\sharp \lrcorner\, \omega, \eta \right\rangle \mu &= (-1)^{nk+n} \langle *(\alpha \wedge *\omega), \eta \rangle \mu && \text{(by proposition)}\\
&= (-1)^{nk+n} \eta \wedge *(*(\alpha \wedge *\omega)) && \text{(by } \mathbf{3.} \text{ above)}\\
&= (-1)^{nk+n}(-1)^{(n-k+1)(k-1)} \eta \wedge \alpha \wedge *\omega && \text{(by } \mathbf{1.} \text{ above)}\\
&= \alpha \wedge \eta \wedge *\omega\\
&= \langle \alpha \wedge \eta, \omega \rangle \mu\\
&= \langle \omega, \alpha \wedge \eta \rangle \mu.
\end{aligned}$$

■

Definition 3.2.15. Define $d^* : \Omega^k(M) \to \Omega^{k-1}(M)$ to be the *formal adjoint* of $d : \Omega^k(M) \to \Omega^{k+1}(M)$. That is, $\langle\langle d\alpha, \beta \rangle\rangle = \langle\langle \alpha, d^*\beta \rangle\rangle$, for all $\alpha \in \Omega^{k+1}(M)$ and $\beta \in \Omega^k(M)$.

✠ Exercise 3.2.16. On $\Omega^k(M)$, $d^* = (-1)^{nk+n+1} * d *$.

Corollary 3.2.17. Similarly to d, $(d^*)^2 = 0$.

Proof:

$$\pm * d * * d * = * d^2 * = 0.$$

■

Proposition 3.2.18. The Clifford action of a 1-form $\alpha \in \Omega^1(M)$ (which is the same as a vector field) on a k-form $\omega \in \Omega^k(M)$ is given by $\alpha \cdot \omega = \alpha \wedge \omega - \alpha^\sharp \lrcorner\, \omega$.

Proof: Both sides of the equation are linear in α, ω (this may be checked on an orthonormal basis of 1-forms $\{e^1, \dots, e^n\}$ at a point). We need to show that $e^j \cdot \omega = e^j \wedge \omega - (e^j)^\sharp \lrcorner\, \omega$, for all j. So write $\omega = e^j \wedge \sigma + \tau$, where σ, τ have no e^js in them. Isomorphically, this is $e^j \cdot \sigma + \tau$. Then

$$\begin{aligned}
e^j \cdot \omega &= e^j \cdot (e^j \cdot \sigma + \tau) = -\sigma + e^j \cdot \tau \xrightarrow{isom.} -\sigma + e^j \wedge \tau,\\
e^j \wedge \omega &= 0 + e^j \wedge \tau,\\
(e^j)^\sharp \lrcorner\, \omega &= (e^j)^\sharp \lrcorner\, (e^j \wedge \sigma + \tau)\\
&= ((e^j)^\sharp \lrcorner\, e^j) \wedge \sigma - e^j \wedge ((e^j)^\sharp \lrcorner\, \sigma) + (e^j)^\sharp \lrcorner\, \tau\\
&= \sigma.
\end{aligned}$$

This follows as

$$(e^j)^\sharp \lrcorner\, e^k = e^k((e^j)^\sharp) = g(e^k, e^j) = \delta^{kj} = \begin{cases} 0 & \text{if } k \neq j \\ 1 & \text{if } i = j \end{cases} .$$

Hence $-\sigma + e^j \wedge \tau = -(e^j)^\sharp \lrcorner\, \omega + e^j \wedge \omega$. ■

Lemma 3.2.19. We know $\bigwedge^\bullet(T^*M)$ has a natural Riemannian metric $g = \langle \cdot\,,\,\cdot \rangle$. Then $\bigwedge^\bullet(T^*M) \otimes \mathbf{C}$, equipped with its natural metric and connection, is a Clifford bundle.

Proof: Define a Hermitian fiber metric h on $\bigwedge^\bullet(T^*M) \otimes \mathbf{C}$ by $h(\alpha, \beta) = \langle \alpha, \overline{\beta} \rangle$. Check that it is actually Hermitian, and that the induced connection ∇ from ∇^{LC} on TM is h-compatible. It remains to check conditions **1.** and **2.** for being a Clifford bundle.

1. Let $\alpha \in \Omega^1_{\mathbf{R}}(M)$, and $\omega, \eta \in \Omega^\bullet(M) \otimes \mathbf{C}$. Then

$$\begin{aligned} h(\alpha \cdot \omega, \eta) &= h(\alpha \wedge \omega - \alpha^\sharp \lrcorner\, \omega, \eta) \\ &= \langle \alpha \wedge \omega - \alpha^\sharp \lrcorner\, \omega, \overline{\eta} \rangle \\ &= \left\langle \omega, \overline{\alpha^\sharp \lrcorner\, \omega} - \overline{\alpha \wedge \eta} \right\rangle \\ &= h(\omega, -(\alpha \wedge \eta - \alpha^\sharp \lrcorner\, \eta)) \\ &= -h(\omega, \alpha \cdot \eta). \end{aligned}$$

2. Check that $\nabla_X(Y \lrcorner\, \omega) = (\nabla_X Y) \lrcorner\, \omega + Y \lrcorner\, (\nabla_X \omega)$ and $\nabla_X \alpha^\sharp = (\nabla_X \alpha)^\sharp$, to get that

$$\begin{aligned} \nabla_X(\alpha \cdot \omega) &= \nabla_X(\alpha \wedge \omega - \alpha^\sharp \lrcorner\, \omega) \\ &= (\nabla_X \alpha) \wedge \omega + \alpha \wedge (\nabla_X \omega) - (\nabla_X \alpha^\sharp) \lrcorner\, \omega - \alpha^\sharp \lrcorner\, (\nabla_X \omega) \\ &= (\nabla_X \alpha) \cdot \omega + \alpha \cdot (\nabla_X \omega). \end{aligned}$$

So $\Omega^\bullet(M) \otimes \mathbf{C}$ is indeed a Clifford bundle. ■

Remark 3.2.20. If $\omega \in \Omega^\bullet(M) \otimes \mathbf{C}$, then $d\omega = \sum_{k=1}^n e_k \wedge \nabla_{e_k} \omega$ for any local frame $\{e_1, \ldots, e_n\}$ and any torsion-free connection ∇ on TM. Also, $d^*\omega = -\sum_{k=1}^n e_k \lrcorner\, \nabla_{e_k} \omega$ for any orthonormal frame $\{e_1, \ldots, e_n\}$ and the Levi-Civita connection. The Dirac operator then is

$$\begin{aligned} D\omega &= \sum_{k=1}^n e_k \cdot \nabla_{e_k} \omega \\ &= \sum_{k=1}^n e_k \wedge \nabla_{e_k} \omega - \sum_{k=1}^n e_k \lrcorner\, \nabla_{e_k} \omega \\ &= d\omega + d^*\omega \\ &= (d + d^*)\omega. \end{aligned}$$

So D is this case is $d + d^* : \Omega^\bullet_{\mathbf{C}}(M) \to \Omega^\bullet_{\mathbf{C}}(M)$, which is called the *Hodge–de Rham operator*.

Then $D^2 = (d + d^*)^2 = d^2 + dd^* + d^*d + (d^*)^2 = dd^* + d^*d = \Delta_d$, the *Hodge Laplacian*, for $\Delta_d : \Omega^k(M) \circlearrowleft$. We will find out on the next assignment that $\Delta_d = \nabla^*\nabla +$ (other stuff).

To get more examples of Clifford bundles, we need to use representation theory.

3.3 A short digression on representation theory

Suppose V is an $n = 2m$-dimensional vector space with a positive definite inner product. Let $C = \mathcal{C}\ell(V) \otimes \mathbf{C}$. We want to understand representations on C. Let S be a complex vector space that is a module over C. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of V, so $\{e_1^{i_1} \cdots e_n^{i_n} \ : \ i_j \in \{0, 1\}\}$ is a basis of C. Then E, the group of order 2^{n+1} consists of all elements $\{\pm e_1^{i_1} \cdots e_n^{i_n} \ : \ i_j \in \{0, 1\}\}$. This is clearly a multiplicative subgroup of C. Denote $\nu = -1 \in E$.

Proposition 3.3.1. There is a 1-1 correspondence between representations of C and representations of E, on which ν acts as -1.

Proof: A representation S of C gives a representation of E, and $\nu = e_i^2$ acts as -1. Then $\nu \cdot s = (e_1 \cdots e_i) \cdot s = -1 \cdot s = -s$. Conversely, a representation of E for which ν acts as -1 induces a representation of C by $\mathbf{C}$-linearity (that is, a surjective homomorphism from the group algebra $\mathbf{C}E$ to C). ■

Let's work out some representations of E. Note ν is a central involution (i.e. $\nu \in \mathrm{Cen}(E)$, the *centralizer* of E) and $\nu^1 = 1 \in E$, hence ν must act as $= 1$ or -1 on any irreducible representation of E. Those irreducible representations on which ν acts as $+1$ are representations of the abelian group $E/(\nu)$ of order 2^n, so there are 2^n of them.

Lemma 3.3.2. The center of E is $\{1, \nu\}$.

Proof: Recall that the *center* of a representation E is the set $\{a \in E \ : \ ag = ga \ \forall \ g \in E\}$. So let $g = e_1^{i_1} \cdots e_n^{i_n} \in E$. Suppose that $i_p = 1$, $i_q = 0$. Then $e_p e_q g = -g e_p e_q = \nu g e_p e_q$ (check this), hence $g \in \mathrm{Cen}(E)$. So only $\mu = e_1 \cdots e_n$ might be in the center. Since $n = 2m$ is even, $e_1 \mu = -\mu e_1 = \nu \mu e_1$, so $\mu \notin \mathrm{Cen}(E)$. ■

Remark 3.3.3. Next, we may count the number of irreducible representations of E by counting conjugacy classes in E. That is, if $g \in \mathrm{Cen}(E)$, then the conjugacy class of g is $\{g\} = \{hgh^{-1} \ : \ h \in E\}$. If $g \notin \mathrm{Cen}(E)$, then the conjugacy class of g is $\{g, g\nu\}$. Then for all $h \in E$, $gh = hg$ or $gh = -hg = hg\nu$. So the number of irreducible representations is the number of conjugacy classes, which is

$$\underbrace{2}_{\{1,\nu\} \text{ in center}} + \underbrace{\frac{2^{n+1}-2}{2}}_{\text{conjugacy classes of the rest}} = \underbrace{2^n}_{\text{where } \nu \text{ acts as } -1} + \underbrace{1}_{\text{where } \nu \text{ acts as } 1}.$$

Hence there exists a unique irreducible representation of C. This representation is called the *spin representation* of C, and denoted by Δ. Recall that the sum of the squares of the dimensions of the irreducible representations of E equals the order of E. Hence the order of E is

$$\begin{aligned} 2^{n+1} &= 2^n(1)^2 + 1(\dim(\Delta))^2, \\ \implies \quad \dim(\Delta)^2 &= 2^{n+1} - 2^n = 2^n(2-1) = 2^n = 2^{2m}. \end{aligned}$$

Moreover, since Δ is the only irreducible representation of C, it is isomorphic to the matrix algebra $\mathrm{End}(\Delta)$. Note that $\dim(\mathrm{End}(\Delta)) = (\dim(\Delta))^2 = (2^m)^2 = 2^n = \dim(C)$.

Remark 3.3.4. We can now construct this representation ∇ explicitly. Note that $V \cong \mathbf{R}^{2m} \cong \mathbf{C}^m$, so we may endow V with a complex structure J. Then $J : V \to V$ is linear so that $J^2 = -I$ and is compatible with $\langle \cdot\, , \cdot \rangle$ in the sense that $\langle Jv, Jw \rangle = \langle v, w \rangle$, so we have orthogonality. Further, we always have that

$$V_{\mathbf{C}} = V \otimes \mathbf{C} = \underbrace{V^{1,0}}_{+i} \oplus \underbrace{V^{0,1}}_{-i},$$

where $V^{1,0}$ and $V^{0,1}$ are eigenspaces of J, complex vector spaces of complex dimension n. Now, consider the exterior algebra $\bigwedge^{\bullet}(V^{1,0}) = S = \Delta$. We can make this into a madule over $C = \mathcal{C}\ell(V) \otimes \mathbf{C}$ as follows: let $\alpha \in \bigwedge^{\bullet}(V^{1,0})$. For $v + w \in V \otimes \mathbf{C}$, so that $v \in V^{1,0})$, $w \in V^{0,1}$, define

$$v \cdot \alpha = \sqrt{2} v \wedge \alpha \qquad \text{and} \qquad w \cdot \alpha = \sqrt{2}\,\overline{w} \lrcorner\, \alpha.$$

These extend to an action of C on $\Delta = \bigwedge^{\bullet}(V^{1,0})$, as $v^2 = w^2 = 0$, and

$$(vw + wv) \cdot \alpha = -2(v \wedge (\overline{w} \lrcorner\, \alpha) + \overline{w} \lrcorner\, (v \wedge \alpha)) = -2v(\overline{w})\alpha = -2 \langle v, w \rangle .$$

Now we put this on a manifold.

Definition 3.3.5. Let (M, g) be a compact oriented Riemannian manifold. Then there exists a Clifford bundle $S \to M$, and M is called a *Spin*^{C} manifold

Remark 3.3.6. We have shown that any almost Hermitian manifold (a compact Riemannion manifold (M, g) with an almost complex structure $J \in \Gamma(\mathrm{End}(TM))$) is a Spin^C manifold. So what is the Dirac operator here? It is

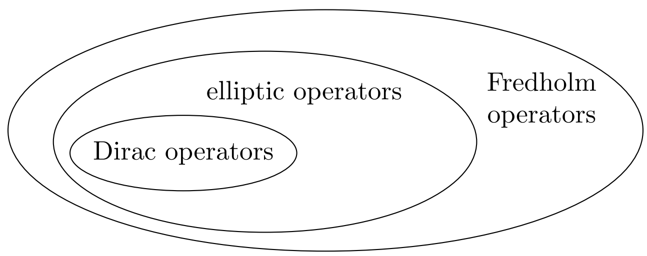
$$D = \sqrt{2}(\overline{\partial} + \overline{\partial}^*) + (\text{other stuff}),$$

where the other stuff vanishes when (M, g) is Kähler. Moreover, in Kähler geometry, $D = \sqrt{2}(\overline{\partial} + \overline{\partial}^*)$ is called the *Hodge–Dolbeault* operator.

Remark 3.3.7. Let us return for a moment to the index theorem. If D is a Dirac operator of a Clifford bundle $S \to M$, then

$$\mathrm{ind}(D) = \dim(\ker(D)) - \dim(\mathrm{coker}(D)) = \int_M \left(\begin{array}{c} \text{forms on } M \text{ representing} \\ \text{characteristic classes of } TM,\ S \end{array} \right).$$

Thi index theorem applies to elliptic operators. We will only prove the one on Dirac operators, but it is enough to prove it in general. We have proper inclusions of operators as in the diagram below:



3.4 Analytic properties of Dirac operators

The analytic properties of Dirac operators and properties of solutions to the heat and wave equations and $Ds = 0$ (which we need for the proof of the index theorem) are described in terms of the Hilbert space of sections of vector bundles. That is, put some metric on E to induce a norm on $\Gamma(E)$ and take the completion to get a Hilbert space.

Start on a torus $T^n = \mathbf{R}^s/2\pi\mathbf{Z}^n = (\mathbf{R}/2\pi\mathbf{Z})^n$, which is a compact oriented Riemannian manifold. We will define Sobolev space of functions on T^n, then use this to define Sobolev spaces of sections $E \to M$.

Definition 3.4.1. Let $f : T^n \to \mathbf{R}$ be Lebesgue integrable. The *Fourier series* for f is the formal series

$$\frac{1}{(2\pi)^{n/2}} \sum_{p \in \mathbf{Z}^n} a_p e^{ip\cdot x} \qquad \text{for} \qquad a_p = \hat{f}_p = \frac{1}{(2\pi)^{n/2}} \int_{T^n} f(x) e^{-ip\cdot x} dx,$$

and $p \cdot x = \sum_{k=1}^n p_k x_k$ is the usual dot product on $\mathbf{R}^n$.

Now we present some results from the theory of Fourier series. All will follow from the fact that $\exp : x \mapsto e^{ip\cdot x}/(2\pi)^{n/2}$ form an orthonormal basis of the Hilbert space $L^2(T^n)$.

Theorem 3.4.2. [Plancherel]
If $f \in L^2(T^n)$, then $\int_{T^n} |f|^2 = \sum_p \left| \hat{f}(p) \right|^2$.

Theorem 3.4.3. [Inversion theorem for L^2]
If $f \in L^2(T^n)$, the Fourier series of f converges in the L^2-norm to f.

Theorem 3.4.4. [Inversion theorem for C^∞]
If $f \in C^\infty(T^n)$, the Fourier series of f converges in the Frechet C^∞-topology (i.e. uniform convergence of all derivatives). In particular, the Fourier series coefficients $\hat{f}(p)$ are rapidly decreasing. This means that, for all $N \in \mathbf{Z}$, there exists $C_N > 0$ such that

$$|\hat{f}(p)| \leqslant C_N \left(1 + |p|^2\right)^N ,$$

where $|p| = p_1^2 + \cdots + p_n^2$.

Definition 3.4.5. Let k be a positive integer. Then the *Sobolev k-inner product* on $C^\infty(T^n)$ is defined by

$$\langle f_1, f_2 \rangle_k = \sum_{p \in \mathbf{Z}^n} \hat{f}_1(p)\hat{f}_2(p)(1 + |p|)^2,$$

for $f_1, f_2 \in C^\infty(TM)$. This converges because $\hat{f}_1(p)$ and $\hat{f}_2(p)$ are rapidly decreasing. The *Sobolev k-norm* in the norm induced by the Sobolev k-inner product.

Definition 3.4.6. The space $L_k^2(T^n) = W^k(T^n)$ is the completion of $C^\infty(TM)$ with respect to this norm.

Notice that by Plancherel, $W^0 = L_0^2 = L^2$, the usual L^2. We will see that we may think of $L_k^2(T^n)$ as the space of functions whose first k derivatives are in L^2.

Proposition 3.4.7. The Sobolev k-norm on $C^\infty(T^n)$ is equivalent to the norm

$$f \mapsto \left(\sum_{|\alpha| \leqslant k} \left\| \frac{\partial f}{\partial x^\alpha} \right\|_{L^2}^2 \right)^{1/2},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}^n_{\geqslant 0}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\frac{\partial f}{\partial x^\alpha} = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$.

Proof: Recall that $\hat{f}(p) = \int_{T^n} f(x) e^{-ip\cdot x} dx/(2\pi)^{n/2}$ and use integration by parts. Let $\hat{f}_\alpha(p) = i^{|\alpha|} p^\alpha \hat{f}(p)$, where $p^\alpha = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$. So by Plancherel,

$$\|f_\alpha\|_{L^2} = \sum_{p \in \mathbf{Z}^n} |\hat{f}_\alpha(p)|^2 = \sum_{p \in \mathbf{Z}^n} |p^\alpha \hat{f}(p)|^2.$$

Next, let $\sum_{|\alpha| \leqslant k} |p|^{2|\alpha|} = a$ and $(1 + |p|^2)^k = b$, both of which are polynomials of degree $2k$ in $p_1, \ldots, p_n$. So there exist $c, C > 0$ such that $ca \leqslant b \leqslant Ca$, hence for any $f \in C^\infty(T^n)$,

$$c \left(\sum_{|\alpha| \leqslant k} \|f_\alpha\|_{L^2}^n \right) \leqslant \|f\|_k^2 \leqslant C \left(\sum_{|\alpha| \leqslant k} \|f_\alpha\|_{L^2}^n \right).$$

■

Proposition 3.4.8. [P1]
Using the norm $\sum_{|\alpha| \leqslant k} \sup_{T^n}\{f_\alpha\} = \|f\|_{C^k}$, $C^k(T^n) \subset W^k(T^n)$, and the inclusion map is continuous.

Proof: Take $f \in C^k(T^n)$, so for all α with $|\alpha| \leqslant k$, f_α is continuous, hence $f_\alpha \in L^2(T^n)$, so $\sum_{|\alpha| \leqslant k} \|f_\alpha\|_{L^2}^2 < \infty$, implying that $f \in W^k(T^n)$. For continuity, check that there exists $c > 0$ such that $\|f\|_{W^k} \leqslant c\|f\|_{C^k}$ for all $f \in C^k(T^n)$. ■

Proposition 3.4.9. [P2 - Sobolev embedding theorem]
If $\ell > \frac{n}{2}$, then $W^{k+\ell}(T^n) \subset C^k(T^n)$, and the inclusion is continuous.

Proof: Using Cauchy–Schwarz, we find that

$$\left(\sum_p |\hat{f}(p)|(1+|p|^2)^{k/2}\right)^2 \leqslant \underbrace{\left(\sum_p |\hat{f}(p)|^2(1+|p|^2)^k(1+|p|^2)^\ell\right)}_{<\infty \text{ if } f\in W^{k+\ell}} \cdot \underbrace{\left(\sum_p (1+|p|^2)^{-\ell}\right)}_{<\infty \text{ if } \ell>\frac{n}{2}}.$$

Hence if $f \in W^{k+\ell}$, then the term on the left is $< \infty$. Then the Fourier series for the first k derivatives of f converge absolutely and uniformly to $f \in C^k$ (check this). ■

Proposition 3.4.10. [P3 - Rellich lemma]
If $k_1 < k_2$, then the inclusion map $W^{k_2}(T^n) \hookrightarrow W^{k_1}(T^n)$ is a compact operator (i.e. it takes bounded sets to precompact sets).

Proof: Let B be the unit ball of W^{k_2}, and $Z = \{f \in W^{k_2} \ : \ \hat{f}(p) = 0 \text{ for } |p| < N\}$. This is a subspace of W^{k_2} of finite codimension. Let $\epsilon > 0$. We claim that we may choose N big enough so that for all $f \in B \cap Z$, $\|f\|_{k_1} < \epsilon$. To see this, observe that

$$\|f\|_{k_1}^2 \left(\sum_p |\hat{f}(p)|^2(1+|p|^2)^k\right) = \underbrace{\left(\sum_p |\hat{f}(p)|^2(1+|p|^2)^k\right)}_{=\|f\|_{k_2}^2,\text{ so }<1} \cdot \underbrace{\left(1+|p|^2\right)^{k_1-k_2}}_{\substack{\leqslant \epsilon \text{ for } |p| \geqslant N \\ \text{for some } N}}.$$

Hence for this N, if $f \in B \cap Z$, then $\hat{f}(p) = 0$ for $|p| \leqslant N$. Hence

$$\|f\|_{k_1}^2 = \sum_{|p|\geqslant N} |\hat{f}(p)|^2(1+|p|^2)^{k_2} \underbrace{(1+|p|^2)^{k_1+k_2}}_{\leqslant\epsilon} \leqslant \epsilon \underbrace{\left(\sum_{|p|\geqslant N} |\hat{f}(p)|^2(1+|p|^2)^{k_2}\right)}_{=\|f\|_{k_2}^2\leqslant 1} \leqslant \epsilon.$$

Now consider the unit ball in $W^{k_2}/Z \cong Z^\perp$. This is compact, hence $Z^\perp$ is finite dimensional, so it can be covered by finitely many balls of radius ϵ, those being $B(g_1, \epsilon), \ldots, B(g_M, \epsilon)$, for $g_j \in B$. Now suppose that $f \in B$, Then $f = f_1 + f_2$, where $f_1 \in Z$ and $f_2 \in Z^\perp$. Then

$$1 \geqslant \|f\|_{k_1}^2 = \|f_1\|_{k_1}^2 + \|f_2\|_{k_1}^2,$$

so $f_1, f_2 \in B$, hence $f_1 \in B \cap Z$ and $f_2 \in B \cap Z^\perp$. Therefore $f_2 \in B(g_j, \epsilon)$ for some j. Then $f - g_j = f_1 + f_2 - g_j$, hence

$$\|f - g_j\|_{k_1}^2 \leqslant \|f_1\|_{k_1}^2 + \|f_2 - g_j\|_{k_1}^2 \leqslant \epsilon^2 + \epsilon^2 = 2\epsilon.$$

So $f \in B(g_j, \sqrt{2}\epsilon)$. Hence B can be covered by finitely many balls of radius $\sqrt{2}\epsilon$ in the W^k-norm, so B is precompact in the W^k-norm. ■

Let us now consider some corollaries to the fact that $\|f_\alpha\|$ is equal to $\left(\sum_{|\alpha|\leqslant k} \|f_\alpha\|_{L^2}^2\right)^{1/2}$.

Corollary 3.4.11. Multiplication by a C^∞ function on T^n is a bounded linear operator on each Sobolev space. That is, if $f \in C^\infty(T^n)$, then $m_f : W^k \to W^k$, given by $m_f(h) = fh$, is bounded linear operator.

Corollary 3.4.12. A linear differential operator P of order ℓ acts bounded linearly from W^k to $W^{k-\ell}$. That is, for $h_\alpha \in C^\infty(T^n)$,

$$P = \sum_{|\alpha|\leqslant\ell} h_\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}.$$

Corollary 3.4.13. Let $f \in L^2(T^n)$ with $\mathrm{supp}(f)$ on a compact subset K. Let $U \subset T^n$ be open with $K \subset U$. Let $\varphi : U \to f(U) \subset T^n$ be a diffeomorphism. Then $f \in W^k$ iff $f \circ \varphi \in W^k$.

Proof: By the symmetry of the statement, it is enough to show that we can estimate L^2 norms of derivatives of $f \circ \varphi$ in terms of L^2 norms of the derivatives of f (up to order k). However, by the chain rule, for $y = \varphi(x)$,

$$\frac{\partial}{\partial x^\alpha}(f \circ \alpha) \quad \text{“equals”} \quad \sum \left(\frac{\partial f}{\partial y^\beta} \circ \varphi \right) \frac{\partial y^\beta}{\partial x^\alpha}.$$

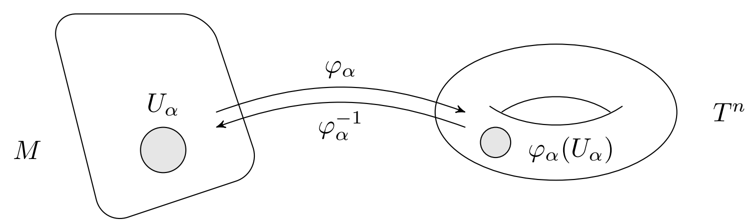
So to compute the L^2 norm, we integrate, and the change of variables formula will give us a $|\det(\mathrm{Jac}(\varphi))|$. Then φ is smooth and T^n is compact, implying that all the terms $\partial y^\beta / \partial x^\alpha$ and $|\det(\mathrm{Jac}(\varphi))|$ are bounded. Hence, for $|\alpha| \leqslant c$,

$$\left\| \frac{\partial}{\partial x^\alpha}(f \circ \varphi) \right\|_{L^2}^2 \leqslant c \left(\sum_{|\alpha| \leqslant k} \left\| \frac{\partial f}{\partial \alpha^k} \right\|_{L^2}^2 \right).$$

■

3.5 General Sobolev spaces

Let M be a compact smooth oriented manifold. Let $\{U_\alpha\}$ be a cover of M by domains of (a finite number of) charts. Let $\{p_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Let φ_α be a diffeomorphism of U_α into the open set $\varphi_\alpha(U_\alpha) \subset T^n$.



Definition 3.5.1. The *Sobolev k-norm* on $C^\infty_{\mathbf{C}}(M)$ is given by

$$\|f\|_k = \sum_\alpha \| \underbrace{(\rho_\alpha f) \circ \varphi_\alpha^{-1}}_{\in\, C^\infty(T^n)} \|_k,$$

where $\|\cdot\|_k$ on the right is the $W^k(T^n)$ norm. Note this definition depends on the choices of charts $\{(U_\alpha, \varphi_\alpha)\}$ and $\{\rho_\alpha\}$. Homever, by the three corollaries above, if we make different choices, we replace the norm by an equivalent one. Hence we define $W^k(M)$ to be the completion of $C^\infty_{\mathbf{C}}(M)$ with respect to this norm, which then makes it well-defined as a topological vector space.

Definition 3.5.2. Let $\pi : E \to M$ be a vector bundle. Cover M by a finite number of charts $(U - \alpha, \varphi_\alpha)$ such that $\varphi_\alpha(U_\alpha) \subset T^n$, and WLOG $E|_{U_\alpha}$ trivializes. If $s \in \Gamma(E)$, then over U_α, $s = s^a e_a$ for $\{e_1, \ldots, e_n\}$ a local frame over U_α corresponding to the trivializations $s^a \in C^\infty_{\mathbf{C}}(U_\alpha)$. Define

$$\|s\|_k = \sum_\alpha \|(\rho_\alpha s) \circ \varphi_\alpha^{-1}\|_k \quad \text{where} \quad \|s^a e_a\|_k^2 = \sum_{a=1}^n \|s^a\|_k^2.$$

This defines $W^k(E)$ as the k-Sobolev space of sections of E, or the completion of $\Gamma(E)$ with respect to $\|\cdot\|_k$. Also, note that $W^k(M) = W^k(M \times \mathbf{C})$.

Remark 3.5.3. Propositions P1, P2, and P3 still apply to $W^k(E)$. On Assignment 3 you will show that the general section can be reduced to the special case $W^k(T^n)$.

Corollary 3.5.4. If $s \in W^k(E)$ for all $k \in \mathbf{N}$, then $s \in \Gamma(E)$, so s is smooth. That is,

$$\bigcap_{k\in\mathbf{N}} C^k(E) = \bigcap_{k\in\mathbf{N}} W^k(E) = \Gamma(E).$$

Suppose S is a Clifford bundle over M, and $D : \Gamma(S) \to \Gamma(S)$ is the associated Dirac operator, so

$$Ds = \sum_{i=1}^{n} e_i \nabla_{e_i} s = \sum_{i=1}^{n} \left(f_i \frac{\partial s}{\partial x^i} + g_i s \right).$$

Then, since D is a 1st order linear differential operator, there exists $c > 0$ such that $\|Ds\|_0 \leqslant C\|s\|_1$. This is also true in general for any 1st order linear differential operator. For Dirac operators (and more generally, for elliptic operators), some kind of partial converse holds.

Theorem 3.5.5. [Gårding's inequality]
For all $k \geqslant 0$, there exists $c_k > 0$ such that for all $s \in \Gamma(S)$,

$$\|s\|_{k+1} \leqslant c_k(\|Ds\|_k + \|s\|_k).$$

Proof: This will be done by induction. Let $k = 0$. By a partition oy unity, WLOG we may setrict to the domain U of a coordinate chart such that $\overline{U}$ is compact, so smooth things on U will be bounded (check this). So let's do it on a chart. Recall the Bochner–Weitzenböck formula $Ds = \nabla^*\nabla s + \kappa s$, where $\kappa s = F^{\nabla}_{ij} \cdot e_i \cdot e_j \cdot s$. Before we start, recall that for $x \in \Gamma(TM|_U)$,

$$|X \cdot s|^2 = \langle X \cdot s, X \cdot s\rangle = -\langle X \cdot (x \cdot s), s\rangle = -\langle (X \cdot X) \cdot s, s\rangle = |X|^2|s|^2.$$

So take the L^2 inner product of the Bochner–Weitzenböck formula with s to get

$$\begin{aligned} \langle\langle D^2 s, s\rangle\rangle &= \langle\langle \nabla^*\nabla s, s\rangle\rangle + \langle\langle \kappa s, s\rangle\rangle \\ &\| \\ \|Ds\|^2 &= \|\nabla s\|^2 + \langle\langle \kappa s, s\rangle\rangle \leqslant \|\nabla s\|^2 + M\|s\|^2, \end{aligned}$$

for some $M > 0$. Then by Cauchy–Schwarz,

$$|\langle\langle \kappa s, s\rangle\rangle| = \left| F^{\nabla}_{ij} \langle\langle e_i \cdot e_j \cdot s, s\rangle\rangle \right| \leqslant M\|e_i \cdot e_j \cdot s\| \, \|s\| = M\|s\|^2,$$

so

$$\|\nabla s\| \leqslant \|Ds\| + C\|s\|. \tag{3}$$

In local coordinates, $\nabla_i s = \frac{\partial s}{\partial x^i} + A_i s$, where s is a vector-valued function and $A_i \in \Gamma(\mathrm{End}(S|_U))$. Then

$$\begin{aligned} \|\nabla s\|^2 &= \int g^{ij} \langle \nabla_i s, \nabla_j s\rangle \,\mathrm{vol}_M \\ &= \int g^{ij} \left\langle \frac{\partial s}{\partial x^i} + A_i s, \frac{\partial s}{\partial x^j} + A_j s \right\rangle \mathrm{vol}_M, \end{aligned}$$

and since there exists $c_2 > 0$ such that $g^{ij} > c_2\delta^{ij}$,

$$\begin{aligned} &> c_2 \int \sum_i \left| \frac{\partial s}{\partial x^i} + A_i s \right|^2 \mathrm{vol}_M \\ &= c_2 \sum \left\| \frac{\partial s}{\partial x^i} + A_i s \right\|^2 \\ &\geqslant c_2 \sum_i \left(\left\| \frac{\partial s}{\partial x^i} \right\|^2 - \|A_i s\|^2 \right) \\ &= c_2 \sum_i \left(\left\| \frac{\partial s}{\partial x^i} \right\|^2 + \frac{\|s\|^2}{n} - \frac{\|s\|^2}{n} - \|A_i s\|^2 \right). \end{aligned}$$

Hence $\|\nabla s\|_0^2 \geqslant c_2\|s\|_1^2 + c_2\|s\|_0^2 - c_2\|A_i s\|_0^2$, so there exists $\tilde{c} > 0$ such that $\|A_i s\|_0 + \tilde{c}\|s\|$. Therefore, for $c_3 = c_2(1+\tilde{c})$,

$$\|\nabla s\|^2 \geqslant c_2\|s\|_1^2 - c_3\|s\|_0. \tag{4}$$

Now combine (3) and (4) to get that

$$\begin{aligned}\|s\|_1^2 &\leqslant M\|\nabla s\|_0^2 + \widetilde{M}\|s\|_0^2 \\ &\leqslant \widetilde{\widetilde{M}}\left(\|s\|_0^2 + \|Ds\|_0^2 + 2\|s\|_0\|Ds\|_0\right) + \widetilde{M}\|s\|_0^2 \\ &\leqslant \widetilde{\widetilde{\widetilde{M}}}\left(\|s\|_0 + \|Ds\|_0\right)^2.\end{aligned}$$

Therefore $\|s\|_1 \leqslant c(\|s\|_0 + \|Ds\|_0)$. This cempletes the base case. For the inductive case, assume we have the case for $k-1$. As before, we are on U with $\overline{U}$ compact. For the equivalence of norms, $\|s\|_{k+1} \leqslant A_1(\sum_{i=1}^n \|\frac{\partial s}{\partial x^i}\|_k)$ for some $A_1 > 0$. By the inductive hypothesis,

$$\|\partial_i s\|_k \leqslant c_{k-1}\left(\|\partial_i n\|_{k-1} + \|D\partial_i s\|_{k-1}\right), \tag{5}$$

where $\partial_i s = \frac{\partial s}{\partial x^i}$. But ∂_i is a 1st order linear differential operator, so by Corollary 2, there exists $A_2 > 0$ such that

$$\|\partial_i s\|_{k-1} \leqslant A_2\|s\|_k. \tag{6}$$

Also, $[D, \partial_i]$ is a 1st order linear differential operator, with

$$[D,\partial_i]s = D(\partial_i s) - \partial_i(D-) = \sum_j e_j \cdot (\partial_j \cdot \partial_i s) - \partial_i \cdot \left(\sum_j e_j \cdot \partial_j s\right) = -\sum_j (\partial_i \cdot e_j) \cdot \partial_j s.$$

Further, there exists $A_3 > 0$ such that

$$\|[D,\partial_i]s\|_{k-1} \leqslant A_3\|s\|_k, \tag{7}$$

so by the triangle inequality and (6) and (7),

$$\|D\partial_i s\|_{k-1} \leqslant \|\partial_i Ds\|_{k-1} + \|[D,\partial_i]s\|_{k-1} \leqslant A_2\|Ds\|_k + A_3\|s\|_k. \tag{8}$$

This implies that

$$\begin{aligned}\|s\|_{k+1} &\leqslant A_1\left(\sum_i \|\partial_i s\|_k\right) \\ &\leqslant A_1 c_{k-1}\sum_i \left(\|\partial_i s\|_{k-1} + \|D\partial_i s\|_{k-1}\|_{k-1}\right) && \text{by (5)} \\ &\leqslant A_1 c_{k-1}\left(nA_2\|s\|_k + nA_2\|Ds\|_k + nA_3\|s\|_k\right) && \text{by (6) and (8)} \\ &= nA_1c_{k-1}((A_2+A_3)\|s\|_k + A_2\|Ds\|_k).\end{aligned}$$

Hence $\|s\|_{k+1} \leqslant c(\|s\|_k + \|Ds\|_k)$. ■

To study the Dirac operator, we think of H as an unbounded ("not necessarily bounded") operator on $L^2(S) = W^0(S)$.

Definition 3.5.6. Let H be a Hilbert space. An *unbounded operator* on H is a linear map P defined on some dense subspace $\operatorname{dom}(P)$ of H to H (the map need not be continuous). The *graph* of an unbounded operator P is

$$\Gamma_P = \{(s, Ps) \ : \ s \in \operatorname{dom}(P)\} \subset H \oplus H,$$

where the inclusion is as a subspace. In this class, we will take P to be the Dirac operator D, H to be $L^2(S) = W^0(S)$ and $\operatorname{dom}(P) = \Gamma(S)$, the space of smooth sections.

Lemma 3.5.7. The closure $\overline{\Gamma}_D$ of Γ_D is also a graph, i.e. there exists $Y \supset \mathrm{dom}(D)$ a subspace of $L^2(S)$, and a linear map $\overline{D} : Y \to L^2(S)$ such that $\overline{\Gamma}_D = \{(s, \overline{D}s) \ : \ s \in Y\} = \Gamma_{\overline{D}}$, where $Y = \mathrm{dom}(\overline{D})$.

Proof: If not, there exists $(0,t) \in \overline{\Gamma}_D$ with $t \neq 0$ (because if not, then there exists (s,t_1), $(s,t_2) \in \overline{\Gamma}_D$ with $t_1 \neq t_2$ and the closure of a subspace is a subspace). So there exists a sequence $s_k \in \Gamma(S)$ with $s_j \to 0$ in $L^2(S)$ and $Ds_j \to t \neq 0$ in $L^2(S)$. But for all $u \in \Gamma(S)$,

$$\begin{array}{c} \langle\langle Ds_j, u\rangle\rangle \to \langle\langle t, u\rangle\rangle \\ \| \\ \langle\langle s_j, Du\rangle\rangle \to \langle\langle 0, Du\rangle\rangle = 0, \end{array}$$

hence $\langle\langle t, u\rangle\rangle = 0$ for all $u \in \Gamma(S)$, so $t = 0$. ∎

Hence we have an unbounded operator $\overline{D} : \mathrm{dom}(\overline{D}) \to L^2(S)$, where $\mathrm{dom}(\overline{D})$ is the set of all $s \in L^2(S)$ such that there exists a sequence $s_j \in \Gamma(S)$ for which $s_j \to s$ and Ds_j converges in L^2. But by Gårding (which was $\|s\| \leqslant c(\|s\|_0 + \|Ds\|_0)$), we see that $\mathrm{dom}(\overline{D}) = W^1(S)$, so $\overline{D} : W^1(S) \to W^0(S)$. Now, suppose that $s, t \in \Gamma(S)$ with $Ds = t$. Then for all $u \in \Gamma(S)$, we have $\langle\langle Du, s\rangle\rangle = \langle\langle u, Ds\rangle\rangle = \langle\langle u, t\rangle\rangle$, so this expression makes sense for all $s, t \in W^0 = L^2$.

Definition 3.5.8. If $s, t \in W^0(S)$ are such that $\langle\langle Du, s\rangle\rangle = \langle\langle u, t\rangle\rangle$ for all $u \in \Gamma(S)$,we say that the equality $Ds = t$ is *satisfied weakly.*

Definition 3.5.9. A bounded linear map $A : L^2(S) \to L^2(S)$ is called a *smoothing operator* if there exists a smooth section $\kappa \in \pi_1^*(S) \otimes \pi_2^*(S^*)$, where $\pi_i : M \times M \to M$ is projection onto the ith factor such that

$$(As)(x) = \int_M \underbrace{\kappa(x,y) \underbrace{s(y)}_{\in S_y}}_{\in S_x} \mathrm{vol}_y,$$

where $\kappa(x,y) \in S_x \otimes S_y^* = \mathrm{End}(S_y, S_x)$ and vol_y means integrating only with respect to the variable y. The map κ is called the *kernel* of A, and it will be a main player in the proof of the index theorem. Note that by differentiation under the integral sign (since κ is smooth), we have that $\mathrm{Im}(A) \subset \Gamma(S) \subset W^0(S)$, i.e. the image of a smoothing operator consists of smooth sections.

Definition 3.5.10. Let S be a Clifford bundle. A *mollifier* for S is a family F_ϵ, $\epsilon \in (0,1)$ of self-adjoint smoothing operators on $L^2(S)$ such that

a. $\{F_\epsilon \ : \ \epsilon \in (0,1)\}$ is a bounded family of operators on $L^2(S)$ (i.e. there exists $c > 0$ such that $\|F_\epsilon(s)\|_0 \leqslant c\|s\|_0$ for all $\epsilon \in (0,1)$),

b. $\{[D, F_\epsilon] \ : \ \epsilon \in (0,1)\}$ extends to a bounded family of operators on $L^2(S)$ (i.e. there exists $c > 0$ such that $\|[D, F_\epsilon]s\|_0 \leqslant c\|s\|_0$ for all $s \in \Gamma(S)$ and $\epsilon \in (0,1)$), and

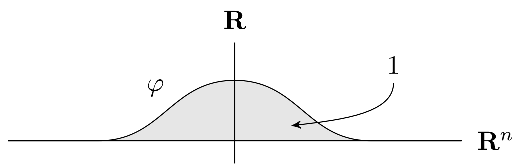
c. $F_\epsilon \to 1_{L^2(S)}$, the identity operator, as $\epsilon \to 0$ in the weak topology of operators on $L^2(S)$ (i.e. for all $s, t \in L^2(S)$, $\langle\langle F_\epsilon s, t\rangle\rangle_0 \xrightarrow{\epsilon \to 0} \langle\langle s, t\rangle\rangle_0$).

Lemma 3.5.11. Mollifiers exist.

Proof: Self-adjointness follows by replacing F_ϵ by $(F_\epsilon + F_\epsilon^*)/2$ (check this). As usual, by partitions of unity, we can restrict to a single chart U such that $\overline{U}$ is compact. Let ρ_α be a partition of unity. Notice that (check the details)

$$[D, F_\epsilon] = \sum_{\alpha,\beta} [\rho_\alpha D, \rho_\beta F_\epsilon] = \sum_{\alpha,\beta} \rho_\alpha \rho_\beta [D, F_\epsilon] = \sum_{\alpha,\beta} (\rho_\alpha (D\rho_\beta) F_\epsilon - \rho_\beta (F_\epsilon \rho_\alpha) D)$$

is bounded on a chart U such that $\overline{U}$ is compact. Now choose a smooth function $\varphi : \mathbf{R}^n \to \mathbf{R}$ with $\varphi \geqslant 0$, compact support, and radially symmetric with $\int_{\mathbf{R}^n} \varphi = 1$, as below.



Let $\varphi_\epsilon(x) = \epsilon^{-n}\varphi(x/\epsilon)$. Define F_ϵ by

$$(F_\epsilon(s))(x) = (\varphi_\epsilon * s)(x) = \int \varphi_\epsilon(x-y)s(y)\mathrm{vol}_y = \frac{1}{\epsilon^n}\int \varphi\left(\frac{x-y}{\epsilon}\right)s(y)\mathrm{vol}_y,$$

where $*$ is the convolution operator. Then $\|\varphi_\epsilon * s\|_{L^2} \leqslant \|\varphi_\epsilon\|_{L^1}\|s\|_{L^2} = \|s\|_{L^2}$, which follows as $\|\varphi_\epsilon\|_{L^1} = 1$ since $\int_{\mathbf{R}^n}\varphi = 1$, and is called *Young's inequality.* This proves **a.**, and **c.** is a standard fact about mollifiers (see any text on distribution theory).

For **b.**, note that the Dirac operator is a sum of 1st order and 0th order operators, so if h is a smooth function, then by the triangle inequality and **a.**, with c independent of ϵ, we have that

$$\|[h, F_\epsilon]s\|_0 = \|hF_\epsilon s - F_\epsilon hs\|_0 \leqslant c\|s\|_0.$$

It remains to show that $[h\frac{\partial}{\partial x^j}, F_\epsilon]$ is uniformly bounded for all j. To see this, observe that

$$\left(h\frac{\partial}{\partial x^j}F_\epsilon s\right)(x) = \frac{1}{\epsilon^{n+1}}\int h(x)\partial_j\varphi\left(\frac{x-y}{\epsilon}\right)s(y)\mathrm{vol}_y, \tag{9}$$

by the chain rule and as $\frac{\partial}{\partial x^j}\varphi(\frac{x-y}{\epsilon}) = \frac{1}{\epsilon}(\partial_j\varphi)(\frac{x-y}{\epsilon})$. Moreover,

$$\begin{aligned}\left(F_\epsilon\left(h\frac{\partial}{\partial x^j}s\right)\right)(x) &= \frac{1}{\epsilon^n}\int\varphi\left(\frac{x-y}{\epsilon}\right)h(y)(\partial_j s)(y)\mathrm{vol}_y \\ &= \frac{-1}{\epsilon^n}\int\varphi\left(\frac{x-y}{\epsilon}\right)(\partial_j h)(y)s(y)\mathrm{vol}_y + \frac{1}{\epsilon^{n+1}}\int h_y(\partial_j\varphi)\left(\frac{x-y}{\epsilon}\right)s(y)\mathrm{vol}_y. \end{aligned}\tag{10}$$

The first term on the right side of (10) has L^2-norm equal to $\|\varphi_\epsilon\|_{L^1}\|(\partial_j h)s\|_{L^2} \leqslant c\|s\|_0$, where $\|\varphi_\epsilon\|_{L^1} = 1$, hence

$$\begin{aligned}\left\|\left[D, h\frac{\partial}{\partial x^j}\right]s\right\|_0 &\leqslant c\|s\|_0 + \frac{1}{\epsilon^{n+1}}\int\left|(\partial_j\varphi)\left(\frac{x-y}{\epsilon}\right)\right||h(x)-h(y)||s(y)|\mathrm{vol}_y \\ &\leqslant c\|s\|_0 + \frac{\tilde{c}}{\epsilon^{n+1}}\int\left|(\partial_j\varphi)\left(\frac{x-y}{\epsilon}\right)\right||x-y||s(y)|\mathrm{vol}_y,\end{aligned}$$

by the mean value theorem and Cauchy–Schwarz. Now the second term is the convolution of $|s|$ with the function $g_\epsilon : x \to \frac{\tilde{c}}{\epsilon^{n+1}}|(\partial_j\varphi)(\frac{x}{\epsilon})||x|$. Again, using the standard convolution estimate, the 2nd term is $\|g_\epsilon\|_{L^1}\|s\|_{L^2}$. But $\|g_\epsilon\|_{L^1}$ is independent of ϵ by change of variables, hence $\|[h\frac{\partial}{\partial x^j}, F_\epsilon]s\|_0 \leqslant c\|s\|_0$, independently of ϵ. ∎

We will now apply the existence of mollifiers.

Proposition 3.5.12. [Strong proposition]
Suppose that $s, t \in L^2(S) = W^0(S)$, and that $Ds = t$ weakly. Then $s \in W^1(S) = \mathrm{dom}(\overline{D})$ and $\overline{D}s = t$ (this is a *strong* solution).

Proof: Let F_ϵ be a mollifier. Let $s_\epsilon = F_\epsilon s \in \Gamma(S)$. If $u \in \Gamma(S)$, then $\langle\langle Ds_\epsilon, u\rangle\rangle_0 = \langle s_\epsilon, Du\rangle_0$ since D is self-adjoint, and $\langle s_\epsilon, Du\rangle_0 = \langle\langle s, F_\epsilon Du\rangle\rangle_0$ since F_ϵ is self-adjoint. Hence

$$\begin{aligned}
\langle\langle Ds_\epsilon, u\rangle\rangle_0 &= \langle\langle s, F_\epsilon Du\rangle\rangle_0 \\
&= \langle\langle s, DF_\epsilon u\rangle\rangle_0 + \langle\langle s, [F_\epsilon, D]u\rangle\rangle_0 \\
&= \langle\langle t, F_\epsilon u\rangle\rangle_0 + \langle\langle s, [F_\epsilon D]u\rangle\rangle_0 \\
&\leqslant \|t\|_0\|F_\epsilon u\|_0 + \|s\|_0\|[F_\epsilon, D]u\|_0 && \text{(since } Ds = t \text{ weakly)} \\
&\leqslant \|c_1\|u_0 + c_2\|u\|_0 && \text{(by } \mathbf{a.} \text{ and } \mathbf{b.}) \\
&= c\|u\|_0.
\end{aligned}$$

Therefore $|\langle\langle Ds_\epsilon, u\rangle\rangle_0| \leqslant c\|u\|_0$ for all $u \in \Gamma(S)$ for some $c > 0$ dependent on s, t and independent of ϵ. Therefore $\|Ds_\epsilon\| \leqslant c$ uniformly in ϵ. Now recall Gårding, $\|v\|_1 \leqslant c(\|v\|_0 + \|Dv\|_0)$ for all $v \in \Gamma(S)$, and apply to $v = s_\epsilon$ to get

$$\|s_\epsilon\|_1 \leqslant c(\|s_\epsilon\|_0 + \|Ds_\epsilon\|_0) \leqslant \tilde{c}$$

independent of ϵ, by **a.**. And, $\|s_\epsilon\|_0 = \|F_\epsilon s\|_0 \leqslant c\|s\|_0$. Hence there exists a sequence $\epsilon_j \to 0$ such that $s_{\epsilon_j} \to w \in W^1(S)$ W^1-weakly, because a closed ball in the Hilbert space $W^1(S)$ is weakly compact. Also, by Rellich's lemma, by possibly passing to a subsequence, $s_{\epsilon_j} \to \tilde{w}$ in the norm topology of $W^0(S)$ in the L^2-norm. By **c.**, $s_{\epsilon_j} \xrightarrow{L^2} s$ as $\epsilon_j \to 0$. Hence $s = w \in W^1(S) = \mathrm{dom}(\overline{D})$ and $\overline{D}s = t$. ■

Now we generalize.

Proposition 3.5.13. Let $k \geqslant 1$ and $s, t \in W^k(S)$. Suppose that $\overline{s} = t$. Then $s \in W^{k+1}(S)$.

Proof: Again, it is enough to reduce this to a chart U such that $\overline{U}$ is compact. By the previous strong proposition, $\overline{D}s = t$ iff $Ds = t$ weakly, so the strong proposition is the $k = 0$ case of this base case. For the inductive step, note that $\partial_j = \frac{\partial}{\partial x^j}$ extends to a bounded linear map from W^k to W^{k-1}. Since $Ds = t$ weakly,

$$D(\partial_j s) = \partial_j(Ds) + [D, \partial_j]s \underset{\text{weakly}}{=} \partial_j t + [D, \partial_j]s.$$

But the right side is in W^{k-1} since $[D, \partial_j]$ is first order. By the inductive hypothesis, $\partial_j s \in W^k$ for all j, implying that $s \in W^{k+1}$. ■

We are now almost ready to understand the spectral theory of D.

3.6 Spectral theory

Lemma 3.6.1. Let $H = L^2(S) = W^0(S)$ and $\Gamma = \Gamma_D$ be the graph of D. Let $J : H \oplus H \to H \oplus H$ be given by $J(s, t) = (t, -s)$, so $J^2 = \mathrm{id}_{H\oplus H}$. Then there exists an orthogonal direct sum decomposition $H \oplus H = \overline{\Gamma} \oplus J(\overline{\Gamma})$.

Proof: Suppose that $s, t \in \Gamma^\perp$. Then for all $u \in \Gamma(S)$, we have that

$$\begin{aligned}
\langle\langle (s,t), (u, Du)\rangle\rangle = 0 \quad &\Longrightarrow \quad \langle\langle s, u\rangle\rangle + \langle\langle t, Du\rangle\rangle = 0 \\
&\Longrightarrow \quad s + Dt = 0 \ \text{ weakly} \\
&\Longrightarrow \quad D(-t) = s \ \text{ weakly.}
\end{aligned}$$

By the strong proposition, $-t \in W^1(S)$, so $(-t, s) = J(s, t) \in \overline{\Gamma}$. Hence $(s, t) \in J(\overline{\Gamma})$. ■

Remark 3.6.2. Now define a map Q as follows. Let $s \in H = L^2(S)$, and let $(Qs, \overline{D}(Qs))$ be the orthogonal projection of $(s, 0)$ onto $\overline{\Gamma}$ in $H \oplus H$. Here $Qs \in \mathrm{dom}(\overline{D}) = W^1(S) \subsetneq W^0(S)$. Then for some $t \in W^1(S)$,

$$(s, 0) = (Qs, \overline{D}(Qs)) + (-\overline{D}t, t),$$

where the two terms on the rigt are orthogonal. Therefore $s = Qs - \overline{D}t$ and $0 = \overline{D}(Qs) + t$. So

$$\begin{aligned}\|s\|_0^2 &= \|(Qs, \overline{D}(Qs))\|_0^2 + \|(-\overline{D}t, t)\|_0^2 \\ &= \|Qs\|_0^2 + \|\overline{D}(Qs)\|_0^2 + \|\overline{D}t\|_0^2 + \|t\|_0^2,\end{aligned}$$

therefore $\|Qs\|_0^2 \leqslant \|s\|_0$ and $\|\overline{D}(Qs)\|_0 \leqslant \|s\|_0$. Now we have that $c = Qs - \overline{D}(-\overline{D}Qs) = (I + \overline{D}^2)Qs = s$, and $(I + \overline{D}^2)(Q) = I$. Hence Q is injective and self-adjoint. Next apply Gårding, $\|u\|_1 \leqslant c(\|u\|_0 + \|Du\|_0)$, to $u = Qs$. So $\|Qs\|_1 \leqslant c(\|Qs\|_0 + \|\overline{D}(Qs)\|_0) \leqslant c\|s\|_0$, so $Q : W^0(S) \to W^1(S)$ is bounded. By Rellich, $W^1(S) \hookrightarrow W^0(S)$ is compact, so $Q : W^0(S) \to W^0(S)$ is compact.

Theorem 3.6.3. [Spectral theorem for compact self-adjoint operators on a Hilbert space] Let $Q : H \to H$ be a compact, injective, self-adjoint positive operator. Then $H = \bigoplus_{n=1}^{\infty} E_{\mu_n}$, which is an orthogonal decomposition, and E_{μ_n} is an eigenspace of Q with eigenvalue μ_n, each E_{μ_n} is finite-dimensional, and the eigenvelaues are discrete, strictly positive and tend to 0, i.e. $\mu_1 > \mu_2 > \cdots$ with $\lim_{n\to\infty}[\mu_n] = 0$. That is, any $v \in H$ is

$$v = \sum_{n=1}^{\infty} v_n \qquad \text{where} \qquad Qv_n = \mu_n v_n,$$

and the sum converges in the H-norm.

Theorem 3.6.4. There is a direct sum decomposition of $H = L^2(S)$ into a sum of countably many orthogonal subspaces H_λ. Each H_λ is a finite-dimensional space of smooth sections, and is an eigenspace for D with eigenvalue λ. The λs form a discrete subset of $\mathbf{R}$.

Proof: Let s be an eigenvector for Q, so $Qs = \mu s$ for some $\mu > 0$. Then $s = \frac{1}{\mu} Qs \in W^1(S)$, so there exists t such that

$$(s, 0) = (Qs, \overline{D}(Qs)) + (-\overline{D}t, t) = \mu(s, \overline{D}s) + (-\overline{D}t, t),$$

so $\mu s - \overline{D}t = s$ and $\mu\overline{D}s + t = 0$. Now note that the eigenvalues of Q are in $(0, 1]$, because $\|Qs\| \leqslant \|s\|$ in the previous calculation. Next, rearrange to get $(\mu - 1)s = \overline{D}t$ and $t = -\mu\overline{D}s$. Let $\lambda^2 = (1 - \mu)/\mu$ and $u = -t/(\mu\lambda)$, so that

$$\overline{D}s = -\frac{1}{\mu} t = \lambda u \qquad \text{and} \qquad \overline{D}u = -\frac{1}{\mu\lambda}\overline{D}t = \frac{1-\mu}{\mu\lambda} s = \lambda s.$$

Now, since $\overline{D}s = \lambda u$ and $\overline{D}u = \lambda s$, $u + s$ and $u - s$ are eigenvectors of $\overline{D}$ with eigenvalues $\lambda, -\lambda$, respectively. Check also that this works for $\lambda = 0$.

So H can be written as a direct sum of countably many (necessarily orthogonal) eigenspaces of $\overline{D}$, each eigenspace a finite-dimensional subspace of $W^1(S)$. We need to show the eigenvectors of D are in $\Gamma(S)$. So for $\overline{D}s = \lambda s = u$, by Gårding

$$\|s\|_{k+1} \leqslant c_k(\|s\|_k + \|\overline{D}s\|_k) = c_k(\|s\|_k + |\lambda|\|s\|_k) \leqslant \mu_k\|s\|_k,$$

for all k. This is called *bootstrapping*, and gives us that $s \in W^k(S)$ for all $k \geqslant 1$, so $s \in \Gamma(S)$. ■

Let $\sigma(D) = \{\lambda_n \ : \ n \in \mathbf{Z}\}$ be the *spectrum* of D. Let $f : \sigma(D) \to \mathbf{C}$ be bounded. Then we can define a bounded operator on $L^2(S) = W^0(S)$ by letting $f(D)$ be multiplication by $f(\lambda)$ on E_λ the λ-eigenspace of D. That is, if $s = \sum_n c_n s_n$, then $f(D)s = \sum_n c_n f(\lambda_n) s_n$. This $f(D)$ is clearly a bounded linear map, i.e. $|f(\lambda)| \leqslant M$ for all $\lambda \in \sigma(D)$, so $\|f(D)s\|_0 \leqslant M\|s\|_0$.

Proposition 3.6.5.

1. The map $f \mapsto f(D)$ is a unital $*$-ring homomorphism from bounded forms on $\sigma(D)$ to bounded linear maps on $W^0(S)$. On functions it is $f^* = \overline{f}$ an involution, and on bounded linear maps of $W^0(S)$ it is f^* the formal adjoint involution. The map acts by

$$\begin{array}{rcl} (f, g) & \mapsto & f(D)g(D) \\ f + g & \mapsto & f(D) + g(D) \end{array} \qquad \begin{array}{rcl} 1 & \mapsto & \text{id} \\ f^* & \mapsto & (f(D))^* \end{array}.$$

2. If $A : W^0(S) \to W^0(S)$ is linear and commutes with D, then A commutes with $f(D)$ for any such f.
3. $f(D) : \Gamma(S) \to \Gamma(S)$.

Proof: **1.** If $s = \sum_n c_n s_n$, then

$$g(D)f(D) = g(D)\left(\sum_n c_n f(\lambda_n) s_n\right) = \sum_n c_n g(\lambda_n) f(\lambda_n) s_n = (gf)(D)(s).$$

Similarly, $f^*(D)s = \sum_n c_n \overline{f}(\lambda_n) s_n$ and

$$\langle\langle f(D)s, t\rangle\rangle_0 = \left\langle\left\langle \sum_n c_n f(\lambda_n) s_n, \sum_m b_m s_m \right\rangle\right\rangle_0 = \sum_n c_n f(\lambda_n)\overline{b_n} = \sum_n c_n \overline{\overline{f}(\lambda_n) b_n} = \langle\langle s, f^*(D)t\rangle\rangle .$$

Hence $f^*(D) = (f(D))^*$.

2. This is clear. Check that $f(D)As = Af(D)s$, where $AD = DA$ (i.e. A preserves eigenspaces of D).

3. Let $s \in \Gamma(S)$, so $D^k s \in W^0(S)$ for all $k \geqslant 0$. Hence $\overline{D}^k f(D)- = f(D)D^k s \in W^0(S)$. By the strong solution proposition, $f(D) \in W^k(S)$ for all $k \geqslant 0$. Hence $f(D)s \in \Gamma(S)$. ■

3.7 Hodge theorem

Consider a special case. Let $V^0, V^1, \ldots, V^n$ be finite-dimensional $\mathbf{C}$-vector spaces with Hermitian inner products. Suppose we are given linear maps $P_i : V^i \to V^{i+1}$ such that $P_i \circ P_{i+1} = 0$ for all $i \geqslant 1$:

$$0 \xrightarrow{\;0\;} V^0 \xrightarrow{\;P_0\;} V^1 \xrightarrow{\;P_1\;} \cdots \xrightarrow{\;P_{N-1}\;} V^{N-1} \xrightarrow{\;P_N\;} V^N \xrightarrow{\;0\;} 0$$

This is a complex of vector spaces, with $\mathrm{Im}(P_{i-1}) \subset \ker(P_i)$. Define the ith cohomology of this complex to be the complex vector space $H^i(V^\bullet, P_\bullet) = \ker(P_i)/\mathrm{Im}(P_{i-1})$. Note that $H^i = 0$ iff $\ker(P_i) = \mathrm{Im}(P_{i-1})$. In this case, we say that the complex is *exact* at V^i.

Definition 3.7.1. The *Euler characteristic* of the complex above is

$$\chi(V^\bullet, P_\bullet) = \sum_{i=0}^{n} (-1)^i \dim(H^i).$$

Definition 3.7.2. Take the complex above and consider the dual maps:

$$0 \underset{0}{\overset{0}{\rightleftarrows}} V^0 \underset{P_0^*}{\overset{P_0}{\rightleftarrows}} V^1 \underset{P_1^*}{\overset{P_1}{\rightleftarrows}} \cdots \underset{P_{N-1}^*}{\overset{P_{N-1}}{\rightleftarrows}} V^{N-1} \underset{P_N^*}{\overset{P_N}{\rightleftarrows}} V^N \underset{0}{\overset{0}{\rightleftarrows}} 0$$

The map $P_i^* : V^{i+1} \to V^i$ is the formal adjoint, defined by the Hermitian inner product as $\langle v, P_i^* w\rangle = \langle P_i v, w\rangle$ for all $v \in V^i$ and $w \in V^{i+1}$. Define $\mathcal{H}^i(V^\bullet, P_\bullet) = \ker(P_i) \cap \ker(P_{i-1}^*) \subset V^i$. This is called the *subspace of P-harmonic elements of* V^i. Note that $\mathrm{Im}(P_{i-1}) \perp \mathrm{Im}(P_i^*)$ in V^i. This follows as $P_i \circ P_{i-1} = 0$ and

$$\langle P_{i-1}v, P_i^* w\rangle = \langle P_i P_{i-1} v, w\rangle = \langle 0, w\rangle = 0.$$

Theorem 3.7.3. [Finite-dimensional Hodge theorem]
1. The induced map $\pi_j : \mathcal{H}^j \to H^j$ is an isomorphism.
2. Given the objects

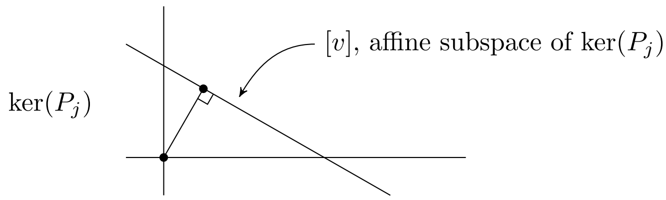
$$V = \bigoplus_{i=0}^{N} V^i \quad , \quad P = \bigoplus_{i=0}^{N} P_i \quad , \quad P^* = \bigoplus_{i=0}^{N} P_{i-1}^* \quad , \quad \Delta_P = (P + P^*)^2,$$

where $\Delta_P : V \to V$ is the Laplacian associated to P, the space $\mathcal{H}^\bullet(V) = \bigoplus_{i=0}^{N} \mathcal{H}^i = \ker(\Delta_P)$.
3. The cohomology of $V(V^\bullet, P_\bullet)$ vanishes iff Δ_P is an isomorphism.

Proof: **1.** We need to show that π_j is injective and surjective. So suppose that $v \in \ker(\pi_j)$. Then $P_j v = 0$ and $P_{j-1}^* v = 0$ and $[v] = 0$ in H^j. So $v = P_{j-1} w$ for some $w \in V^{j-1}$. Hence $P_{j-1}^* P_{j-1} w = 0$, implying that

$$\left\langle P_{j-1}^* P_{j-1} w, w \right\rangle = 0 \quad \Longrightarrow \quad |P_{j-1} w|^2 = 0.$$

Hence $v = 0$ so the map is injective. For surjectivity, we use finite-dimensionality. We need to show every $v \in \ker(P_j)$ is cohomologous to some $u \in \ker(P_j) \cap \ker(P_{j-1}^*)$. That is, if $v \in \ker(P_j)$, there exists $u \in \ker(P_j) \cap \ker(P_{j-1}^*)$ such that $[v] = [u]$, so $H^j \ni [v] = \pi_j(u)$. The cohomology class $[v]$ is given by $[v] = \{v + P_{j-1} w \ : \ w \in V^{j+1}\}$.



By finite dimensionality, there exists a point $v \in [v]$ closest to the origin (we also need completeness). Consider the function

$$\begin{array}{rccl} f_w : & \mathbf{R} & \to & [0, \infty) \\ & t & \mapsto & \mathrm{dist}(u + tP_{j-1} w, 0)^2 = |u + tP_{j-1} w|^2 \end{array}$$

for $w \in V^{j-1}$. By construction of u, $f_w(t) \geqslant f_w(0)$ for all t. Hence $f_w'(0) = 0$ and

$$\begin{aligned} |u + tP_{j-1} w|^2 &= \langle u + tp_{j-1} w, u + tP_{j-1} w \rangle \\ &= |u|^2 + t \langle p_{j-1} w, u \rangle + t \langle u, P_{j-1} w \rangle + t^2 |P_{j-1} w|^2 \\ &= |u|^2 + 2t\mathrm{Re}(\langle P_{j-1} w, u \rangle) + t^2 |P_{j-1} w|^2. \end{aligned}$$

So $f_w'(0) = 0 = 2\mathrm{Re}(\langle P_{j-1} w, 0 \rangle) = 0$ for all $w \in V^{j-1}$, so $2\mathrm{Re}(\left\langle w, P_{j-1}^* u \right\rangle) = 0$ for all such w. Let $w = P_{j-1}^* u \in V^{i-1}$, so that $|P_{j-1}^* u|^2 = 0$, meaning that $P_{j-1}^* u = 0$.

2. Let $v = \bigoplus_{i=0}^N v_i \in V$. Then $v \in \mathcal{H}$ iff $v_i \in \mathcal{H}^i$ for all i, iff $P_i v_i = 0$ and $P_{i-1}^* v_i = 0$ for all i.

3. Note that $(P + P^*)(v) = \bigoplus_{i=0}^N (P_{i-1} v_{i-1} + P_i^* v_{i+1})$, where every element in the direct sum is in V^i. Now note that $(P + P^*)(v) = 0$ iff $P_{i-1} v_{i-1} + P_i^* v_{i+1} = 0$ for all i, where the two terms are orthogonal, as $\mathrm{Im}(P_{i-1}) \perp \mathrm{Im}(P_i^*)$. Hence $v \in \ker(P + P^*)$ iff $P_{i-1} v_i = 0$ and $P_i^* v_{i+1} = 0$ for all i, iff $v_i \in \mathcal{H}^i$. So then $\Delta_P = (P + P^*)^2$, so $\ker(P + P^*) \subset \ker(\Delta_P)$. Notice $(P + P^*)^2 = P^2 + PP^* + P^*P + (P^*)^2 = PP^* + P^*P$. Let $v \in \ker(\Delta_P)$. Then

$$\begin{aligned} \Delta_P v = 0 \quad &\Longrightarrow \quad \langle \Delta_P v, v \rangle = 0 \\ &\Longrightarrow \quad \langle PP^* v + P^*P v, v \rangle = 0 \\ &\Longrightarrow \quad |P^* v|^2 + |P v|^2 = 0, \end{aligned}$$

so $v \in \ker(P + P^*)$. ■

We now move to the more general version of the Hodge theorem. Let (M, g) be a compact oriented Riemannian manifold. Let $S^0, S^1, \ldots, S^N$ be Hermitian vector bundles over M. Suppose for all i, $P_i : \Gamma(S^i) \to \Gamma(S^{i+1})$ is a 1st order linear differential operator with $P_i \circ P_{i-1} = 0$ (and $P_0 = P_N = 0$).

Definition 3.7.4. We say that the complex $0 \longrightarrow \Gamma(S^0) \xrightarrow{P_0} \Gamma(S^1) \xrightarrow{P_1} \cdots \xrightarrow{P_{N-1}} \Gamma(S^N) \longrightarrow 0$ is a *Dirac complex* if $S = \bigoplus_{i=0}^N S^i$ is a Clifford bundle and $D = P + P^* = \sum_{i=0}^N (P_i + P_{i-1}^*)$ is the Dirac operator for S, so $D : \Gamma(S) \to \Gamma(S)$.

Example 3.7.5. Consider $S^i = \bigwedge^i(T^*M) \otimes \mathbf{C}$ and $\Gamma(S^i) = \Omega^i_{\mathbf{C}}(M)$. Take $P_i = d : \Omega^i_{\mathbf{C}}(M) \to \Omega^{i+1}_{\mathbf{C}}(M)$. We have already seen that $\bigwedge^\bullet(T^*M) = \bigoplus_{i=0}^N \bigwedge^i(T^*M)$ is a Clifford bundle with Dirac operator $D = d + d^*$. Also, $\Delta_d = (d+d^*)^2 = dd^* + d^*d$.

Theorem 3.7.6. [(More) general Hodge theorem]

1. The map $\pi_j : \mathcal{H}^j \to H^j$ is an isomorphism.

2. The space $\mathcal{H}^i$ is finite-dimensional.

3. $\ker(D) = \ker(\Delta_P)$.

Proof: The injectivity of π_j is exactly as in the previous Hodge theorem. For surjectivity, consider the extended complex:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathcal{H}^0 & \xrightarrow{\;0\;} & \mathcal{H}^1 & \xrightarrow{\;0\;} & \cdots & \xrightarrow{\;0\;} & \mathcal{H}^n & \longrightarrow & 0 \\
 & & \rho_0 \uparrow\downarrow i_0 & & \rho_1 \uparrow\downarrow i_1 & & & & \rho_N \uparrow\downarrow i_N & & \\
0 & \longrightarrow & \Gamma(S^0) & \xrightarrow[P_0]{} & \Gamma(S^1) & \xrightarrow[P_1]{} & \cdots & \xrightarrow[P_{N-1}]{} & \Gamma(S^N) & \longrightarrow & 0
\end{array}$$

For all $j = 1, \ldots, N$, we have defined maps $\rho_j : \Gamma(S^j) \to \mathcal{H}^j$ by orthogonal projection onto $\mathcal{H}^j$, as $W^0(S^j) = \mathcal{H}^j \oplus (\mathcal{H}^j)^\perp$. Note that $\rho_j \circ i_j = 1$ and $i_j \circ \rho_j = 1 - f(D)$, where $f(\lambda) = \begin{cases} 1 & \lambda \neq 0 \\ 0 & \lambda = 0 \end{cases}$. Let $g(\lambda) = \begin{cases} \lambda^{-2} & \lambda \neq 0 \\ 0 & \lambda = 0 \end{cases}$ and define $G = g(D^2)$ (this is called *Green's operator* for D), a bounded linear map $W^0(S) \to W^0(S)$. Notice that $D^2G = \begin{cases} 1 & \lambda \neq 0 \\ 0 & \lambda = 0 \end{cases}$, so $D^2G = f(D) = 1 - i_j \circ p_j$. Also notice that

$$D^2P = (PP^* + P^*P)P = PP^*P = P(PP^* + P^*P) = PD^2,$$

so P commutes with D^2. Hence P commutes with $G = g(D^2)$, and

$$D^2G = PP^*G + P^*PG = PP^*G + P^*GP = PK + KP$$

for $K = P^*G$. Let $w \in H^j(S^\bullet, P_\bullet)$, so $w = [u]$ with $u \in \Gamma(S^j)$ and $Pu = 0$. Then $u = 1 \cdot u = i_j(\rho_j(u)) + PKu + KPu$, where the first term is in $\mathcal{H}^j$, the second is in $\mathrm{Im}(P)$, and the third vanishes. So $[u] = [i_j(\rho_j(u))] = w$ and $w = \pi_j(i_j(\rho_j(u)) \in \mathcal{H}^j$.

2. Let $v_i \in \mathcal{H}^i$, so $P_i v_i = 0$ and $P^*_{i-1} v_i = 0$, meaning that $(P + P^*)v_i = 0$. So $\mathcal{H}^i = \ker(D) \cap \Gamma(S^i)$ (we already know that $\ker(D) \subset \Gamma(S)$), and $\ker(D)$ is finite-dimensional because E_0 is the eigenspace of $\lambda = 0$. So $\mathcal{H}^i$ is finite-dimensional.

3. This is done exactly as in the previous version of the theorem. ∎

Corollary 3.7.7. The cohomology of a Dirac complex over a compact oriented manifold is finite-dimensional.

Remark 3.7.8. Define $S^{even} = \bigoplus_{k \text{ even}} S^k$ and $S^{odd} = \bigoplus_{k \text{ odd}} S^k$. We know that $P, P^*, D : S^{even} \to S^{odd}$ and $S^{odd} \to S^{even}$. Let $D_+ : S^{even} \to S^{odd}$ and $D_- : S^{odd} \to S^{even}$. We claim that $D_+^* = D_-$. To see this, note that $\langle\langle D_+ s, t\rangle\rangle = \langle\langle s, D_- t\rangle\rangle$. Next define $\mathcal{H}^{even}$, $\mathcal{H}^{odd}$, H^{even}, $\mathcal{H}^{odd}$ analogously. From the Hodge theorem, $\ker(D_+) = \mathcal{H}^{even}$ and $\ker(D_-) = \mathcal{H}^{odd}$. Hence the Euler characteristic is given by

$$\begin{aligned}
\chi &= \sum_{i=0}^N (-1)^i \dim(H^i) \\
&= \dim(H^{even}) - \dim(H^{odd}) \\
&= \dim(\mathcal{H}^{even}) - \dim(\mathcal{H}^{odd}) \\
&= \dim(\ker(D_+)) - \dim(\ker(D_-)).
\end{aligned}$$

This is the first example of an index theorem. It is the difference in dimensions of finite-dimensional kernels of Dirac-type operators.

4 The index theorem

4.1 Fredholm operators

Definition 4.1.1. Let B_1, B_2 be Banach spaces. A bounded linear map $P : B_1 \to B_2$ is called *Fredholm* if the following conditions are satisfied:

a. $\ker(P)$ is finite-dimensional,

b. $\mathrm{Im}(P)$ is closed in B_2, and

c. $\mathrm{coker}(P) = B_2/\mathrm{Im}(P)$ is finite-dimensional.

If P is Fredholm, define the *index* of P to be

$$\mathrm{ind}(P) = \dim(\ker(P)) - \dim(\mathrm{coker}(p)) \in \mathbf{Z}.$$

Remark 4.1.2. Knowing something about the index of an operator tells you how much it fails to be a bijection. Indeed,

$$\begin{aligned}
&\text{if } \mathrm{ind}(P) > 0,\ P \text{ is not injective,}\\
&\text{if } \mathrm{ind}(P) < 0,\ P \text{ is not surjective,}\\
&\text{if } \mathrm{ind}(P) = 0,\ P \text{ is an isomorphism iff } \ker(P) = 0\\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{iff } \mathrm{coker}(P) = 0.
\end{aligned}$$

In practice, an index theorem is combined with vanishing-type theorems, which is a Bochner-Weitzenböck argument to find both dimensions.

Example 4.1.3. It is useful to know when an operator $P : B_1 \to B_2$ is surjective to apply the Banach space implicit function theorem, to conclude that some subsets defined by P have smooth structure.

Now we will show that our Dirac operator is Fredholm. Note that $P \to \mathrm{ind}(P)$ is constant on connected components of the space of Fredholm operators. It is stable under compact perturbations, i.e. $\mathrm{ind}(P+T) = \mathrm{ind}(P)$ for $T : B_1 \to B_2$ compact.

Lemma 4.1.4. Let (M, g) be a compact oriented Riemannian manifold and S a Clifford bundle with a Dirac operator D. The bounded (shown before) map $D : W^1(S) \to W^0(S)$ is Fredholm.

Proof: We already saw that $\ker(D)$ is finite-dimensional. We claim that $\mathrm{Im}(D) = (\ker(D))^\perp$ in $W^0(S)$. If so, we are done, because $W^0(S) = \mathrm{Im}(D) \oplus \ker(D)$. Any orthogonal complement is closed, and the just given statement says that $\mathrm{coker}(D) \cong \ker(D)$ is finite-dimensional.

So suppose that $s \in \ker(D)$. Let $t \in W^1(S)$, so there exists a sequence $\{t_j\} \in \Gamma(S)$ such that $t_j \to t$ in the $W^1(S)$-norm, so $Dt_j \to Dt$ in the $W^0(S)$-norm. But then $\langle\langle Dt_j, s\rangle\rangle_0 = \langle\langle t_j, Ds\rangle\rangle_0 = 0$, since $s \in \ker(D)$. Hence $\langle\langle Dt, s\rangle\rangle_0 = 0$, so $\mathrm{Im}(D) \subset (\ker(D))^\perp$.

Conversely, let $t \in \ker(D)^\perp$. Define f on $\sigma(D)$ by $f(0) = 0$ and $f(\lambda) = 1/\lambda$, so $Df(D)t = t$, as $t \notin \ker(D)$. But $t \in W^0(S)$, so the strong solution theorem says that $f(D)t \in W^1(S)$. Therefore $t = D(f(D)t) \in \mathrm{Im}(D)$, so $\ker(D)^\perp = \mathrm{Im}(D)$. ■

Remark 4.1.5. Let $D : W^1(S) \to W^0(S)$ be the Dirac operator of a Clifford bundle. Then

$$\mathrm{ind}(D) = \dim(\ker(D)) - \dim(\mathrm{coker}(D)) = \dim(\ker(D)) - \dim(\ker(D)) = 0.$$

So $\mathrm{ind} : W^1(S) \to W^0(S)$ is always zero. To get an interesting (non-zero) index, we need to introduce additional structure, a $\mathbf{Z}/2\mathbf{Z}$-grading, a.k.a. a *superstructure*.

Definition 4.1.6. A Dirac operator D is *graded* (or *supersymmetric*) if it comes from a Dirac complex of length 2, i.e.

$$0 \longrightarrow \Gamma(S^0) \underset{P^*}{\overset{P}{\rightleftarrows}} \Gamma(S^1) \longrightarrow 0,$$

so $S = S^0 \oplus S^1$ and $D = P + P^* = \left[\begin{smallmatrix} 0 & P^* \\ P & 0 \end{smallmatrix}\right]$ with respect to the grading $S = S^0 \oplus S^1$. The operator $\mathcal{E} = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$ is called the *grading operator*, so we see that S^0 is the $+1$-eigenspace of $\mathcal{E}$ and S^1 is the -1-eigenspace of $\mathcal{E}$.

Remark 4.1.7. Observe that $D : \Gamma(S) \to \Gamma(S)$ is graded iff $D\mathcal{E} + \mathcal{E}D = 0$. Indeed,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 2A & 0 \\ 0 & -2A \end{bmatrix}.$$

Since D is Fredholm, $P : \Gamma(S^1) \to \Gamma(S^1)$ is also Fredholm. We leave it as an exercise to show that $\mathrm{ind}(P) = \dim(\ker(P)) - \dim(\mathrm{coker}(P))$ and $\mathrm{coker}(P) = \ker(P^*)$ as before. To show this, use $P^* : \Gamma(S^1) \to \Gamma(S^0)$, where $\Gamma(S^1) = \mathrm{Im}(P) \oplus \ker(P^*)$. From before, we will still have

$$d : \Omega^{even}(M) \to \Omega^{odd} \qquad \text{and} \qquad d^* : \Omega^{odd}(M) \to \Omega^{even}.$$

Moreover, as $D = d + d^*$, the Euler characteristic of M is given by $\chi = \mathrm{ind}(d) = \dim(\ker(P)) - \dim(\ker(P^*))$. Hence the Euler characteristic is the index of a Fredholm operator.

Theorem 4.1.8. [Atiyah–Singer theorem for Dirac operators]
Let (M, g) be a compact, oriented Rimenannian manifold such that D is graded and $D = P + P^*$. Then

$$\mathrm{ind}(P) = \int_M \hat{A}(TM) = \left\langle \hat{A}(TM), [M] \right\rangle,$$

is probably true, where

$$\hat{A} = \sqrt{\det\left(\frac{\sinh\left(\frac{iF^\nabla}{2\pi}\right)}{\frac{iF^\nabla}{2\pi}}\right)}$$

is the *A-hat genus* and $[\,\cdot\,]$ is the fundamental class.

Proof: The idea behind the proof is a finite-dimensional linear algebra result, which we will extend to our setting. Let U_+, U_- be finite-dimensinoal complex vector spaces with Hermitian metrics $\langle \cdot\, , \cdot \rangle$, with $n = \dim(U_+)$, $m = \dim(U_-)$. Let $P : U_+ \to U_-$ be a linear operator.

First we claim that $\mathrm{Im}(P) = \ker(P^*)^\perp$ and $\mathrm{Im}(P^*) = \ker(P)^\perp$. This follows as $\langle Pv, w \rangle = \langle v, P^* w \rangle$, so $w \in \ker(P^*)$ iff $w \in \mathrm{Im}(P)^\perp$. So $U_+ = \ker(P) \oplus \mathrm{Im}(P^*)$ and $U_- = \ker(P^*) \oplus \mathrm{Im}(P)$, meaning P induces an isomorphism $P : \mathrm{Im}(P^*) \xrightarrow{\cong} \mathrm{Im}(P)$. Then

$$\begin{aligned} \mathrm{ind}(P) &= \dim(\ker(P)) - \dim(\mathrm{coker}(P)) \\ &= \dim(\ker(P)) - \dim(\ker(P^*)) \\ &= n - \dim(\mathrm{Im}(P)) - (m - \dim(\mathrm{Im}(P^*))) \\ &= n - m \\ &= dim(U_+) - \dim(U_-). \end{aligned}$$

■

Let's now give another more complicated proof, which generalizes to infinite dimensions.

Proof: Let $U = U_+ \oplus U_-$ and $D = \left[\begin{smallmatrix} 0 & P^* \\ P & 0 \end{smallmatrix}\right] : U \to U$. Notice that $D^* = D$ and

$$\langle D(u,v), (s,t) \rangle = \langle (P^* v, Pu), (s,t) \rangle = \langle P^* v, s \rangle + \langle Pu, t \rangle = \langle (u,v), P(s,t) \rangle.$$

We can compute the self-adjoint operator $D^2 = \left[\begin{smallmatrix} P^*P & 0 \\ 0 & PP^* \end{smallmatrix}\right]$ is non-negative, so $\left\langle D^2 x, x \right\rangle \geqslant 0$, meaning that $\sigma(D^2) \in [0, \infty)$. Let $\mu \geqslant 0$, so then $\ker(D^2 - \mu I) = \ker(D - \sqrt{\mu} I) \oplus \ker(D + \sqrt{\mu} I)$. Let $\mathcal{E} = \left[\begin{smallmatrix} I_{U_+} & 0 \\ 0 & I_{U_-} \end{smallmatrix}\right]$ be a grading. Then D^2 commutes with $\mathcal{E}$, so they are simultaneously diagonalizable for any eigenvalue μ of D^2, i.e. $E_\mu = E_\mu^+ \oplus E_\mu^-$, the sum of the $+1$-eigenspace and -1-eigenspace. To finish the proof, we need the following lemma. ■

Lemma 4.1.9. For all non-zero $\mu \in \sigma(D^2)$, $\dim(E_\mu^+) = \dim(E_\mu^-)$. More precisely, $P|_{E_\mu^+} : E_\mu^+ \xrightarrow{\cong} E_\mu^-$.

Proof: First note that $[D, D^2] = 0$, so E_μ is D-invariant, i.e. $D(E_\mu) \subset E_\mu$. Suppose $\mu \neq 0$ and $v \in \ker(D|_{E_\mu})$. Then

$$Dv = 0 \quad \Longrightarrow \quad D^2 v = \mu v = 0,$$

and for $\mu \neq 0$, it must be that $v = 0$. So $D|_{E_\mu} : E_\mu \xrightarrow{\cong} E_\mu$, and $D = \left[\begin{smallmatrix} 0 & P^* \\ P & 0 \end{smallmatrix}\right]$, meaning that

$$P : E_\mu^+ \xrightarrow{\cong} E_\mu^- \qquad \text{and} \qquad P^* : E_\mu^- \xrightarrow{\cong} E_\mu^+,$$

proving the claim. Now we have that $U_+ = \bigoplus^\perp_{\mu \in \sigma(D^2)} E_\mu^+$ and $U_- = \bigoplus^\perp_{\mu \in \sigma(D^2)} E_\mu^-$, and by the claim,

$$\dim(U_+) - \dim(U_-) = \dim(E_0^+) - \dim(E_0^-) = \dim(\ker(P)) - \dim(\ker(P^*)) = \mathrm{ind}(P).$$

■

Definition 4.1.10. Let $T : U \to U$ be an operator. The *supertrace* of T, denoted $\mathrm{str}(T)$, is defined to be

$$\mathrm{str}(T) = \mathrm{Tr}(A) - \mathrm{Tr}(D) = \mathrm{Tr}(T|_{U_+}) - \mathrm{Tr}(T|_{U_-}) \;\text{ where }\; T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} + \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} = T_{even} + T_{odd}.$$

So $\mathrm{str}(T) = \mathrm{str}(T_{even})$, as well as $\mathrm{str}(T) = \mathrm{Tr}(\mathcal{E}T) = \mathrm{Tr}(\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right])$.

Remark 4.1.11. With the above definition, for any $t \geqslant 0$,

$$\mathrm{str}\left(e^{-tD^2}\right) = \sum_{\mu \in \sigma(D^2)} e^{-t\mu}(\dim(E_\mu^+) - \dim(E_\mu^-)).$$

By the previous lemma, these cancel in pairs except when $\mu = 0$ (also called *supersymmetry*). So we see that

$$\begin{aligned} \mathrm{str}(e^{-tD^2}) &= \dim(E_0^+) - \dim(E_0^-) \\ &= \dim(\ker(P^*P)) - \dim(\ker(PP^*)) \\ &= \dim(\ker(P)) - \dim(\ker(P^*)) \\ &= \mathrm{ind}(P). \end{aligned}$$

The last expression is smooth in t and independent of t, so

$$\mathrm{ind}(P) = \lim_{t \to 0^+} \left[\mathrm{str}(e^{-tD^2})\right] = \mathrm{str}\left(\lim_{t \to 0^+} \left[e^{-tD^2}\right]\right) = \mathrm{str}(1_U) = \dim(U_+) - \dim(U_-).$$

The above illustrates the main idea that we will follow: D is a Dirac operator for a Clifford bundle $S = S^+ \oplus S^-$. Write $\mathrm{ind}(D_+ : W^1(S^+) \to W^0(S^-)) = \mathrm{str}(e^{-tD^2})$ (which still needs to be defined). We will show that the right side is independent of t. The index theorem will then be obtained by equating $t \to \infty$ and $t \to 0$ on the right hand side.

Now, recall that if D is a Dirac operator, for all $s \in \Gamma(S)$ we have that $\|s\|_{k+1} \leqslant c_k(\|Ds\|_k + \|s\|_k)$, where c_k depends on k, M, S but not $s \in \Gamma(S)$. Also, since D is first order linear, there exists $c > 0$ such that $\|Ds\| \leqslant c_k \|s\|_{k+1}$ for all $s \in W^{k+1}(S)$ and $s \in \Gamma(S)$. Since D is mth order linear, $\|D^m s\|_k \leqslant c\|s\|_{k+m}$ for some $c \geqslant 0$.

Proposition 4.1.12. There exists a constant $c = c(m, k, M, S)$ such that $\|s\|_{k+m} \leqslant c(\|D^m s\|_k + \|s\|_k)$ for all $s \in \Gamma(S)$.

Corollary 4.1.13. Define a norm $\|s\|_\sim = \|s\|_0 + \|D^m s\|_0$. Then the previous statements imply that $\| \cdot \|_\sim$ is equivalent to $\| \cdot \|_m$.

Proof: Observe that

$$\frac{1}{c}\|s\|_m \leqslant \|s\|_0 + \|D^m s\|_0 \leqslant c\|s\|_m.$$

■

4.2 The heat kernel

Definition 4.2.1. Given vector bundles as on the left, define the *box tensor* operator $\boxtimes$ as on the right.

$$\begin{array}{ccc} & M\times M & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ M & & M \\ \pi_1(p,q)=p & & \pi_2(p,q)=q \end{array} \qquad \begin{array}{rcl} S\boxtimes S^* & = & \pi_1^*(S)\otimes\pi_2^*(S^*) \\ (S\boxtimes S^*) & = & S_p\boxtimes S_q^* = \mathrm{End}(S_q,S_p) \end{array}$$

Recall that a smoothing operator A_k for S is determined by a smooth section k of $S\boxtimes S^*$ by

$$(A_k s)(p) = \int_M \overbrace{\underbrace{k(p,q)}_{\in\ \mathrm{End}(S_q,S_p)} \quad \underbrace{s(q)}_{\in\ S_q}}^{\in\ S_p} \mathrm{vol}_q.$$

The integrand is a smooth function on M taking vectors in S_p. Note that $A_k s\in\Gamma(S)$ even if $s\in W^0(S)$. The map k is also called an *integral kernel* for s.

Example 4.2.2. Consider the following examples of integral kernels.

· Let $s\in\Gamma(S)$, $\alpha\in\Gamma(S^*)$. Then $s\boxtimes\alpha\in\Gamma(S\boxtimes S^*)$ is defined by $(s\boxtimes\alpha)_{(p,q)} = s_q\otimes\alpha_q\in S_p\otimes S_q^*$, which is a decomposable integral kernel.

· There exists a conjugate linear map $\Gamma(S)\to\Gamma(S^*)$, given by $s\mapsto s^*$, such that $s^*(t)=\langle t,s\rangle$ for all $t\in S$ (this is a pointwise fiber metric). Also, $(\lambda s)^*=\overline{\lambda}s^*$ for all $\lambda\in C^\infty_{\mathbf{C}}(M)$. Hence if $s,t\in\Gamma(S)$, then $s\boxtimes t^*\in\Gamma(S\boxtimes S^*)$. Next, let $k\in\Gamma(S\boxtimes S^*)$ be an integral kernel for S. For all $s\in W^0(S)$, $(A_k s)(p)=\int_M k(p,q)s_q\mathrm{vol}_q$ and $A_k : W^0(S)\to\Gamma(S)$ is linear.

· Let $\lambda\in\sigma(D)$ and let $m_\lambda=\dim(E_\lambda)=\dim(\ker(D-\lambda I))$, the *multiplicity* of λ. The space E_λ is the λ-eigenspace of D, which is finite-dimensional and consists of smooth sections. Let P_λ be the orthogonal projection onto E_λ, with $P_\lambda : W^0(S)\to E_\lambda$. Fix an orthonormal (with respect to thet $W^0(S)$-norm) basis $s_1,\dots,s_{m_\lambda}$ of E_λ and define $k_\lambda=\sum_{j=1}^{m_\lambda} s_j\boxtimes s_j^*$. Now let $t\in\Gamma(S)$, for which

$$(A_{k_\lambda}t)(p) = \int_M k_\lambda(p,q)t(q)\mathrm{vol}_q = \sum_{j=1}^{m_\lambda} s_j(p)\int\langle t(q),s_j(q)\rangle\,\mathrm{vol}_q = (P_\lambda t)(p).$$

So $A_{k_\lambda}=P_\lambda$ and $P_\lambda t=\sum_{j=1}^{m_\lambda} a_j s_j$, where $a_\ell=\langle\langle P_\lambda t,s_\ell\rangle\rangle=\langle\langle t,s_\ell\rangle\rangle$.

Let's generalize the last example further. Let $I\subset[0,\infty)$ be a compact interval. Define $k_I=\sum_{\lambda\in\sigma(D)}k_\lambda\in\Gamma(S\boxtimes S^*)$. The smoothing operator A_{k_I} corresponding to k_I is $A_{k_I}=P_I$, the orthogonal projection onto

$$E_I = \bigoplus_{\substack{\lambda\in\sigma(D)\\|\lambda|\in I}} E_\lambda = \bigoplus_{\lambda\in\sigma(D)} H_{\lambda^2},$$

where E_λ are the eigenspaces of D and H_{λ^2} are the eigenspaces of D^2. If $I=\{\lambda\}$, then $E_I=E_\lambda=H_{\lambda^2}$. Define also

$$P_I = \bigoplus_{\substack{\lambda\in\sigma(D)\\|\lambda|\in I}} P_\lambda \qquad\text{and}\qquad d_I=\dim(E_I)=\sum_{\substack{\lambda\in\sigma(D)\\|\lambda|\in I}} m_\lambda.$$

Let $s_1,\dots,s_{d_I}$ be an orthonormal basis of E_I. Then $k_I=\sum_{j=1}^{d_I}s_j\boxtimes s_j^*$.

Theorem 4.2.3. Let $r = \mathrm{rank}(S)$. Then there exists $c > 0$ such that

a. $d_I \leqslant c^2 r\mathrm{vol}(M^n)(1+b^{2\ell})$, for all $I \subset [a,b] \subset [0,\infty)$, where $\ell = \lfloor \frac{n}{2} \rfloor + 1$. This is a bound on $\dim(E_I)$ in terms of $1+b^{2\ell}$.

b. For any $j \geqslant 0$, there exists a constant $B_j > 0$ such that for any $[a,b] \subset [0,\infty)$, we have that $\|k_I\|_{C^j} \leqslant B_j(1+b^{2\ell+2j})$. This is a bound on the C^j-norm of k_I in terms of $1+b^{2\ell+2j}$.

Proof: For all $j \geqslant 0$, there exists $M_j > 0$ such that $\|s\|_j \leqslant M_j(\|s\|_0 + \|D^j s\|_0)$. From the Sobolev embedding theorem, if $\ell > j + \frac{n}{2}$, there exists $A = A(\ell, j)$ such that

$$\|s\|_{C^j} \leqslant A\|s\|_{W^\ell} \leqslant AM_\ell(\|s\|_0 + \|D^\ell s\|_0). \tag{11}$$

Let $I = [a,b]$. For any $s \in W^0(S)$, set $s_I = P_I s \in \Gamma(S)$. Let $u \in \Gamma(S)$, so then for $u_I = \sum_{|\lambda| \in I} c_\lambda s_\lambda$, we have that $D^\ell u_I = \sum_{|\lambda|\in I} c_\lambda \lambda^\ell s_\lambda$ and

$$\|D^\ell u_I\|_0^2 = \sum_{|\lambda|\in I} |\lambda|^{2\ell}|c_\lambda|^2 \leqslant |b|^{2\ell} \sum_{|\lambda|\in I} |c_\lambda|^2 = b^{2\ell}\|u_I\|_0^2.$$

Now we see that

$$\|u_I\|_0^2 + \|D^\ell u_I\|_0^2 \leqslant (1+b^{2\ell})\|u_I\|_0^2 \leqslant (1+b^{2\ell})\|u\|_0^2. \tag{12}$$

Let $\ell_j = \lfloor \frac{n}{2} \rfloor + j + 1$. Then (11) and (12) for $j = 0$ and $s = u_I$ give

$$\|u_I\|_{C^0} \leqslant B_0(1+b^{2\ell_0})^{1/2}\|u\|_0, \tag{13}$$

for all $u \in W^0(S)$, passing to the limit. Next, fix a point $p_0 \in M$ and $w_{p_0} \in S_{p_0}$ of unit length. Define $V_{p_0,w_{p_0}}(q) = \sum_{i=1}^{d_I} \langle w_{p_0}, s_i(p_0)\rangle\, s_i(q)$. This means that $v_{p_0,w_{p_0}} \in \Gamma(S)$. Also, $u_I(p_0) = \sum_{i=1}^{d_I} \langle\langle u, s_i\rangle\rangle\, s_i(p_i)$ for all $u \in W$. Further, notice that

$$\langle u_I(p_0), w_{p_0}\rangle = \sum_{i=1}^{d_I} \langle\langle u, s_i\rangle\rangle \langle s_i(p_0), w_{p_0}\rangle = \left\langle\left\langle u, v_{p_0,w_{p_0}}\right\rangle\right\rangle. \tag{14}$$

Hence for all $u \in W^0(S)$,

$$\left\langle\left\langle u, v_{p_0,w_{p_0}}\right\rangle\right\rangle = \langle u_I(p_0), w_{p_0}\rangle \leqslant |u_I(p_0)| \leqslant \|u_I\|_{C^0} \leqslant B_0(1+b^{2\ell_0})^{1/2}\|u\|_0,$$

where the first equality follows from (14) and the first inequality from Cauchy–Schwarz. Since this holds for all u, take $u = V_{p_0,w_{p_0}}$. Then $\|v_{p_0,w_{p_0}}\|_0^2 \leqslant B_0(1+b^{2\ell_0})^{1/2}\|v_{p_0,w_{p_0}}\|_0$, hence

$$\|v_{p_0,w_{p_0}}\|_0 \leqslant B_0(1+b^{2\ell_0})^{1/2}. \tag{15}$$

Since $s_1, \ldots, s_{d_I}$ is an orthonormal basis of E_I, $\|v_{p_0,w_{p_0}}\|_0^2 = \sum_{i=1}^{d_I} |\langle w_{p_0}, s_i(p_0)\rangle|^2$. Let $e_1, \ldots, e_r$ be an orthonormal basis of s_{p_0}, with respect to $\langle\,\cdot\,,\,\cdot\,\rangle_{p_0}$. Then

$$\begin{aligned}\sum_{k=1}^r \|v_{p_0,e_k}\|^2 &= \sum_{k=1}^r \sum_{i=1}^{d_I} |\langle e_k, s_i(p_0)\rangle|^2 \\ &= \sum_{i=1}^{d_I} \sum_{k=1}^r |\langle e_k, s_i(p_0)\rangle|^2 \\ &= \sum_{i=1}^{d_I} |s_i(p_0)|^2,\end{aligned}$$

hence $\sum_{i=1}^{d_I} |s(p_0)|^2 \leqslant rB_0^2(a+b^{2\ell})$, by (15), for any $p_0 \in M$. Now integrate over M for

$$\int_M \sum_{i=1}^{d_I} |s_i|^2 \mathrm{vol} = \sum_{i=1}^{d_I} \|s_i\|^2 = \dim(E_I) = d_I \leqslant rB_0^2 \mathrm{vol}(\mu)(a+b^{2\ell_0}),$$

which proves **a.**. For **b.**, use (11) for $\|s\|_{C^j} \leqslant \widetilde{M}_j(1+b^{2\ell_j})^{1/2}\|s\|_0$, where $\ell_j = \lfloor \frac{n}{2} \rfloor + j + 1$. Then $k_I = \sum_{i=1}^{d_I} s_i \boxtimes s_i^*$, and

$$\begin{aligned}\|k_I\|_{C^j} &\leqslant \sum_{i=1}^{d_I} \|s_i\|_{C^j}^2 \\ &\leqslant \widetilde{M}_j^2(1+b^{2\ell_j})d_I \\ &\leqslant \widetilde{M}_j^2(1+b^{2\ell_j})B_0^2 r(1+b^{2\ell_0})\mathrm{vol}(M) \\ &\leqslant N_j(1+b^{2\ell_0+\ell_j}),\end{aligned}$$

and $\ell_0 + \ell_j = 2(\lfloor \frac{n}{2} \rfloor + 1 + j)$, which is what we wanted. ■

Recall that if $f : \sigma(D) \to \mathbf{C}$ is bounded, we can define $f(D) : W^0(S) \to W^0(S)$ bounded linear with $f(D) = \sum_{\lambda \in \sigma(D)} f(\lambda) P_\lambda s$.

Proposition 4.2.4. Suppose $f : \mathbf{R} \to \mathbf{R}$ is continuous with rapid decay at infinity. That is, suppose that $\lim_{|\lambda| \to 0}[|\lambda|^k f(\lambda)] = 0$ for all $k \geqslant 0$. Then $f(D)$ is a smoothing operator A_{k_f} associated to the integral kernel $k_f = \sum_{\lambda \in \sigma(D)} f(\lambda) k_\lambda$, where $A_{k_\lambda} = P_\lambda$.

<u>*Proof:*</u> For all $n \geqslant 1$, set

$$k_{f,n} = \sum_{\substack{\lambda \in \sigma(D) \\ |\lambda| \in [n-1,n]}} f(\lambda) k_\lambda = \sum_{|\lambda| \in [n-1,n]} f(\lambda) k_\lambda,$$

where $k_\lambda = 0$ if $\lambda \notin \sigma(D)$. This is an integral kernel, i.e. $k_{f,n} \in \Gamma(S \boxtimes S^*)$. Now, we claim that $\sum_{n \geqslant 1} k_{f,n} = k_f$ converges in $C^j(S \boxtimes S^*)$ for any $j \geqslant 0$. This shows that k_f is smooth. To prove this claim, we first let

$$d_n = \dim(E_{[n-1,n]}) \qquad \text{and} \qquad f_n = \sup_{|t| \in [n-1,n]} \{|f(t)|\}$$

for all n. Note that $f_n < \infty$ because f is continuous. Then

$$\begin{aligned}\|k_{f,n}\|_{C^j} &\leqslant \left(\sup_{|t| \in [n-1,n]} \{|f(t)|\} \right) \left(\sum_{|\lambda| \in [n-1,n]} \|k_\lambda\|_{C^j} \right) \\ &\leqslant f_m d_m B_j \left(1 + m^{2\ell+2j}\right) \\ &\leqslant f_m M \left(1 + m^{4\ell+2j}\right).\end{aligned}$$

Since f is rapidly decaying, $\sum_{m=1}^\infty f_m(1+m^{4\ell+2j}) < \infty$, so $k_{f,m} \xrightarrow{m \to \infty} k_f$ in the C^j norm, for all j. This proves the claim and proves the proposition. ■

Definition 4.2.5. Let $f_t(\lambda) = e^{-t\lambda^2}$. This is rapidly decreasing for any $t > 0$, so by the previous proposition, $f_t(D) = e^{-tD^2}$ is a smoothing operator with integral kernel $h_t = \sum_{\lambda \in \sigma(D)} e^{-t\lambda^2} k_\lambda$. The collection $\{h_t \ : \ t > 0\}$ is called the *heat kernel* of D.

Definition 4.2.6. Define a bundle $\pi : \mathbf{R}_{\geqslant 0} \times M \times M \to M \times M$ by $\pi(t,p,q) = (p,q)$, projection onto the second factor. Define

$$\widehat{S \boxtimes S^*} = \pi^*(S \boxtimes S^*),$$

which is a bundle over $\mathbf{R}_{\geqslant 0} \times M \times M$, whose fibers are $\widehat{S \boxtimes S^*}_{(t,p,q)} = (S \boxtimes S^*)_{(p,q)} = \mathrm{End}(S_q, S_p)$.

Proposition 4.2.7. With respect to the definitions above,

- **a.** the heat kernel $\{h_t \;:\; t > 0\}$ defines a smooth section of $\widehat{S \boxtimes S^*}$ by $(t, p, q) \mapsto h_t(p, q) \in \mathrm{End}(S_q, S_p)$,
- **b.** for any fixed $q \in M$, $\frac{\partial}{\partial t} h_t(p, q) + D_p^2 h_t(p, q) = 0$, i.e. h satisfies the heat equation on S, and
- **c.** if $s \in \Gamma(S)$, then $\lim_{t\to 0^+}[\|s_t - s\|_{C^0}] = 0$ for $s_t = e^{-tD^2} s = A_{h_t} s$.

<u>*Proof:*</u> For any $N > 0$, set $h_{t,N}(p, q) = \sum_{|\lambda|\leqslant N} e^{-t\lambda^2} k_\lambda(p, q)$, recalling that $k_\lambda(p, q) = \sum_{j=1}^{m_\lambda} s_j(p) \boxtimes s_j^*(q)$. Then $h_{t,N}$ is a smooth function of p, q for any N. The previous proposition showed that $h_{t,N} \xrightarrow{N\to\infty} h_t$ in the C^k norm, for any k, and is uniformly integrable on compact subsets of $\mathbf{R}_{\geqslant 0}$. Also note that

$$D_p^2 k_\lambda = \lambda^2 k_\lambda \qquad \text{implying} \qquad \left(\frac{\partial}{\partial t} + D_p^2\right) h_{t,N} = 0$$

for all N. Now, the integral kernel $D_p^2 h_{t,N}$ converges in any C^k to the smooth integral kernel associated to the rapidly decaying function $\lambda \to \lambda^2 e^{-t\lambda^2}$ (and this too is uniformly convergent in t on compact subsets of $\mathbf{R}_{\geqslant 0}$). Hence $h_t = \lim_{N\to\infty}[h_{t,N}]$ is C^k in p, q, t for all k (i.e. a C^k function of t, p, q by uniform convergence on compact subsets). Hence $h : \mathbf{R}_{\geqslant 0} \times M \times M \to \widehat{S \boxtimes S^*}$ is a smooth section and

$$\left(\frac{\partial}{\partial t} + D_p^2\right) h_t = 0.$$

This proves **a.** and **b.**. For **c.**, let $s \in \Gamma(S)$ and $s_t = e^{-tD^2}$. Write $s = A_{h_t} s = \sum_{\lambda\in\sigma(D)} s_\lambda$, for $s_\lambda = P_\lambda s$. Let $m \geqslant 0$, for which

$$\|D^m s\|_0^2 \leqslant \sum_{\lambda\in\sigma(D)} |\lambda|^m \|s_\lambda\|_0^2 < \infty,$$

since $D^m s \in \Gamma(S)$. Therefore

$$\|D^m(s_t - s)\|_0^2 = \sum_{\lambda\in\sigma(D)} \left(e^{-t\lambda^2} - 1\right)^2 |\lambda|^{2m} \|s_\lambda\|_0^2.$$

Consider the functions $\varphi_t : \sigma(D) \to \mathbf{R}$ given by $\varphi_t(\lambda) = (e^{-t\lambda^2} - 1)^2 |\lambda|^{2m} \|s_\lambda\|_0^2$. Equip $\sigma(D)$ with the canonical discrete topology. Apply the dominated convergence theorem to interchange the sum and the limit. Then

$$\lim_{t\to 0^+}\left[\|D^m(s_t - s)\|_0^2\right] = \sum_{\lambda\in\sigma(D)} \lim_{t\to 0^+}\left[\left(e^{-t\lambda^2} - 1\right)^2 |\lambda|_0^{2m} \|s_\lambda\|_0^2\right] = 0$$

for all $m \geqslant 0$. Hence

$$\lim_{t\to 0^+}\left[\|D^m(s_t - s)\|_0^2 + \|s_t - s\|_0^2\right] = 0.$$

Last time, we showed that the norm $\|D^m s\|_0^2 + \|s\|_0^2$ is equivalent to the $W^m(S)$ norm. Hence $\lim_{t\to 0^+}[\|s_t - s\|_{W^m(S)}] = 0$ for any $m \geqslant 0$. So by the Sobolev embedding theorem, $\lim_{t\to 0^+}[\|s_t - s\|_{C^k}] = 0$ for all k, proving **c.**. ■

Lemma 4.2.8. [Uniqueness of solutions to heat kernel]
Let $\widehat{S}$ denote tha pullback of $S \to M$ to $[0, \infty) \times M = \mathbf{R}_{\geqslant 0} \times M$. For any $s_0 \in \Gamma(S)$, the initial value problem

$$\left(\frac{\partial}{\partial t} + D^2\right)(s(t, p)) = 0 \tag{16}$$

where $s(0, p) = s_0(p)$ for all $p \in M$, admits a unique solution, which is a continuous section of $\widehat{S}$ on $[0, \infty) \times M$ and smooth on $(0, \infty) \times M$.

Proof: Assume that $s(t,p)$ is a solution of (16). Let $s_t = s(t, \cdot\,)$ be the restriction of s to $\{t\} \times M$. For uniqueness, we need to show that if $s_0 = 0$,then $s_t = 0$ for all $t \geqslant 0$. Then

$$\begin{aligned}\frac{\partial}{\partial t}\|s_t\|_0^2 &= \left\langle\left\langle \frac{\partial}{\partial t}s_t, s_t\right\rangle\right\rangle_0 + \left\langle\left\langle s_t, \frac{\partial}{\partial t}s_t\right\rangle\right\rangle_0 \\ &= \langle\langle -D^2 s_t, s_t\rangle\rangle_0 + \langle\langle s_t, -D^2 s_t\rangle\rangle_0 \\ &= -2\|Ds_t\|_0^2 \\ &\leqslant 0.\end{aligned}$$

So $\|s_t\|_0^2 \leqslant \|s_0\|_0^2 = 0$, meaning that $s_t = 0$ for all $t \geqslant 0$. ■

Above we showed that $s_t = e^{-tD^2}s_0$ is a solution, so $s_t = \int_M h_t(p,q)s_0(q)\mathrm{vol}_q$ and

$$\begin{aligned}D_p^2 s_t &= \int_M (D_p^2 h_t(p,q))s_0(q)\mathrm{vol}_q \\ &= -\int_M \frac{\partial}{\partial t}h_t(p,q)s_0(q)\mathrm{vol}_q \\ &= -\frac{\partial}{\partial t}s_t.\end{aligned}$$

The fact that the solution is smooth on $(0,\infty)\times M$ and continuous on $[0,\infty)\times M$ follows from similar facts about h_t, which we will fix next time.

Theorem 4.2.9. [Main theorem]
The heat kernel $(h_t)_{t>0}$ is the unique smooth section $(k_t)_{t>0}$ of $\widehat{S \boxtimes S^*}$ satisfying:

- **a.** $k_t(p,q)$ satisfies $(\frac{\partial}{\partial t} + D_p^2)k_t(p,q) = 0$ for all $q \in M$, and
- **b.** if $s \in \Gamma(S)$, then $\lim_{t\to 0^+}[\|A_{k_t}s - s\|_{C^0}] = 0$.

Proof: We already know that h_t satisfies **a.** and **b.**, so we only need to show it is unique. Firstly, we know that $s_t = e^{-tD^2}s_0$ satisfies $(\frac{\partial}{\partial t} + D^2)s_t = 0$, where $s_t|_{t=0} = s_0$, and that this solution is unique. So suppose we have a family of integral kernels $(k_t)_{t>0}$ satisfying **a.** and **b.**, for which we would like to show $k_t = h_t$. Let $s_0 \in \Gamma(S)$, $t > 0$, and set $w_t = A_{k_t}s_0$. Let $v_t = w_{t+\epsilon} = A_{k_{t+\epsilon}}s_0$. By hypothesis, v_t satisfies $(\frac{\partial}{\partial t} + D^2)V_t = 0$, where $v_0 = w_\epsilon = A_{k_\epsilon}s_0$. By uniqueness of solutions to the heat equation,

$$A_{k_{t+\epsilon}}s_0 = v_t = e^{-tD^2}w_\epsilon.$$

Let $\epsilon \to 0$, so then $w_\epsilon \to s_0$ in C^0 by hypothesis **b.**. Then finally $A_{k_t}s_0 = e^{-tD^2}s_0$ for any s_0, so $A_{k_t} = A_{h_t}$, implying that $k_t = h_t$. ■

Recall that we want to prove the index theorem by mimicking the baby case - for $P : U_+ \to U_-$ linear, $\mathrm{ind}(P) = \mathrm{str}(e^{-tD^2}) = \dim(U_+) - \dim(U_-)$, where $D = \left[\begin{smallmatrix} 0 & P^* \\ P & 0\end{smallmatrix}\right]$ and $U = U_+ \oplus U_-$.

Theorem 4.2.10. [Atiyah–Singer index theorem]
Let $D : \Gamma(S) \to \Gamma(S)$ be a graded Dirac operator. Let $(h_t)_{t>0}$ be the heat kernel of D. Then

$$\mathrm{ind}(P) = [\hat{A}(TM)]([M]) = \int_M \hat{A}(TM).$$

Theorem 4.2.11. [McKean–Singer formula]
Let M be a compact oriented Riemannian manifold, S a Clifford bundle, D a Dirac operator of S, assumed to be graded as $D = \left[\begin{smallmatrix} 0 & P^* \\ P & 0\end{smallmatrix}\right]$, with $S = S^+ \oplus S^-$ and $P : \Gamma(S^+) \to \Gamma(S^-)$. Let h_t be the integral kernel of e^{-tD^2} for $t > 0$ (i.e. the heat kernel of D). Then

$$\mathrm{ind}(D^+) = \mathrm{ind}(P) = \int_M \mathrm{str}(h_t(p,p))\mathrm{vol}_p$$

for all $t > 0$, where $D^+ = D|_{\Gamma(S)}$. Note that this is independent of t.

Proof: Write $D^2 = \left[\begin{smallmatrix} P^*P & 0 \\ 0 & PP^* \end{smallmatrix}\right] = \begin{bmatrix} \Delta_+ & 0 \\ 0 & \Delta_- \end{bmatrix}$, so

$$\begin{array}{rcl} \Delta_+ & = & P^*P : \Gamma(S^+) \to \Gamma(S^+) \\ \Delta_- & = & PP^* : \Gamma(S^-) \to \Gamma(S^-) \end{array}.$$

Let $\mu \geqslant 0$ be an eigenvalue of D^2. Let

$$N_\mu^+ = \dim(\underbrace{\ker(\Delta_+ - \mu I)}_{=\, H_\mu^+}) \qquad \text{and} \qquad N_\mu^- = \dim(\underbrace{\ker(\Delta_- - \mu I)}_{=\, H_\mu^-}),$$

where $H_\mu^\pm$ is the μ-eigenspace of D^2, restricted to $\Gamma(S^\pm)$. Let $s_{\mu,1}^\pm, \ldots, s_{\mu,N_\mu^\pm}^\pm$ be an orthonormal basis of $H_\mu^\pm$, orthonormal with respect to the $W^0(S)$-norm. Then $\{s_{\mu,j}^\pm \ : \ j = 1, \ldots, N_\mu^\pm, \ \mu \in \sigma(D^2)\}$ is an orthonormal basis of $W^0(S)$. Express

$$e^{-tD^2} = \sum_{\mu \in \sigma(D^2)} e^{-t\mu} \left(P_{H_\mu^+} - P_{H_\mu^-} \right),$$

where $P_{H_\mu^\pm}$ is the orthogonal projection onto $H_\mu^\pm$. We already saw that $P_{H_\mu^\pm} = A_{k_{P_\mu^\pm}}$, so the smoothing operator associatos to the integral kernel by

$$k_{P_\mp^\pm}(p,q) = \sum_{j=1}^{N_\mu^\pm} s_{\mu,j}^\pm(p) \boxtimes (s_{\mu,j}^\pm)^*(q).$$

Also, $H_\mu^+ \oplus H_\mu^- = H_\mu = E_{\sqrt{\mu}} \oplus E_{-\sqrt{\mu}}$, where E represents an eigenspace of D. Hence $k_{P_\mu^+} + k_{P_\mu^-} = k_{-\sqrt{\mu}} + k_{\sqrt{\mu}}$, and so further

$$h_t(p,q) = \sum_{\mu \in \sigma(D^2)} e^{-t\mu} \left(k_{P_\mu^+}(p,q) + k_{P_\mu^-}(p,q) \right),$$

and

$$\begin{aligned} \operatorname{str}\left(k_{P_\mu}(p,p)\right) &= \operatorname{Tr}\left(k_{P_\mu^+}(p,p)\right) - \operatorname{Tr}\left(k_{P_\mu^-}(p,p)\right) \\ &= \operatorname{Tr}\left(\sum_{j=1}^{N_\mu^+} s_{\mu,j}^+(p) \boxtimes \left(s_{\mu,j}^+\right)^*(p) - \sum_{j=1}^{N_\mu^-} s_{\mu,j}^-(p) \boxtimes \left(s_{\mu,j}^-\right)^*(p) \right). \end{aligned}$$

When $p = q$, then $(S \boxtimes S^*)_{(p,p)} = S_p \boxtimes S_p^* = (S \otimes S^*)_p$. We need to know how to compute $\operatorname{Tr}(v \otimes v^*)$ for $v \in (V, \langle \cdot, \cdot \rangle)$ a hermitian vector space. So let $e_1, \ldots, e_r$ be an orthonormal basis of V, with respect to $\langle \cdot, \cdot \rangle$, with $P : V \to V$ and $\operatorname{Tr}(P) = \sum_{j=1}^r \langle P(e_j), e_j \rangle$. Then

$$\begin{aligned} \operatorname{Tr}(v \otimes v^*) &= \sum_{j=1}^r \langle (v \otimes v^*)(e_j), e_j \rangle \\ &= \sum_{j=1}^r \langle v^*(e_j) v, e_j \rangle \\ &= \sum_{j=1}^r v^j \langle v, e_j \rangle \\ &= \sum_{j=1}^r v^j v^j \\ &= |v|^2 \\ &= \langle v, v \rangle, \end{aligned}$$

so $\mathrm{Tr}(s^+_{\mu,j}(p) \boxtimes (s^+_{\mu,j})^*(p)) = |s^+_{\mu,j}(p)|^2$, where the norm is the pointwise norm on S^+_p from the fiber metric. Hence

$$\mathrm{str}(k_{P_\mu}(p,p)) = \sum_{j=1}^{N^+_\mu} \left| s^+_{\mu,j}(p) \right|^2 - \sum_{j=1}^{N^-_\mu} \left| s^-_{\mu,j}(p) \right|^2,$$

$$\mathrm{str}(h_t(p,p)) = \sum_{\mu \in \sigma(D^2)} e^{-t\mu} \left(\sum_{j=1}^{N^+_\mu} \left| s^+_{\mu,j}(p) \right|^2 - \sum_{j=1}^{N^-_\mu} \left| s^-_{\mu,j}(p) \right|^2 \right).$$

So

$$\begin{aligned} \int_M \mathrm{str}(h_t(p,p))\mathrm{vol}_p &= \sum_{\mu \in \sigma(D^2)} e^{-t\mu} \left(\sum_{j=1}^{N^+_\mu} \int_M \left| s^+_{\mu,j}(p) \right|^2 \mathrm{vol}_p - \sum_{j=1}^{N^-_\mu} \int_M \left| s^-_{\mu,j}(p) \right|^2 \mathrm{vol}_p \right) \\ &= \sum_{\mu \in \sigma(D^2)} e^{-t\mu} \left(N^+_\mu - N^-_\mu \right), \end{aligned}$$

where the reduction occurs since the integrals are all 1 on an orthonormal basis. Now, just like in the baby index theorem, $D|_{H^+_\mu} : H^+_\mu \to H^-_\mu$. If $s \in H_\mu$, then $Ds = 0$, so $D^2 s = \mu s = 0$. Hence if $\mu \neq 0$, $s = 0$. Therefore $D|_{H^+_\mu} : H^+_\mu \to H^-_\mu$ is an isomorphism if $\mu > 0$. This means that

$$\begin{aligned} \int_M \mathrm{str}(h_t(p,p))\mathrm{vol}_p &= N^+_0 - N^-_0 \\ &= \dim(\ker(\Delta_+)) - \dim(\ker(\Delta_-)) \\ &= \dim(\ker(P)) - \dim(\ker(P^*)) \\ &= \mathrm{ind}(P). \end{aligned}$$

■

Remark 4.2.12. The strategy we will use to prove the index theorem is to find a way to calculate $\int_M \mathrm{str}(h_t(p,p))\mathrm{vol}_p$, which is independent of t for $t > 0$. We will see that even though $h_t(p,q)$ can't be computed exactly, by finding an approximate expression for $h_t(p,p)$, the expression $\int_M \mathrm{str}(h_t(p,q))\mathrm{vol}_p$ can be computed exactly.

4.3 Approximating the heat kernel

Definition 4.3.1. Let B be a Banach space and $f : (0,\infty) \to B$ a function. We say that a formal series $\sum_{k=0}^\infty a_k(t)$, with $a_k(0,\infty) \to B$, is an *asymptotic expansion* for f near $t = 0$, and denote this by $f(t) \sim \sum_{k=0}^\infty a_k(t)$, if for every positive integer N, there exists $\ell_N \in \mathbf{Z}_{>0}$ such that if $\ell \geqslant \ell_N$, then there exists $c = c(\ell, N) > 0$ and $\tau = \tau(\ell, N) > 0$ such that

$$\left\| f(t) - \sum_{k=0}^{\ell} a_k(t) \right\|_B \leqslant c(\ell, N) t^N$$

for all $t \in (0, \tau(\ell, N)]$. In words, given N, f minus a sufficiently large partial sum is $O(t^N)$ for t sufficiently small.

Example 4.3.2. Note that an asymptotic expansion for f need not converge to f at $t = 0$ in any sense. Indeed, Consider $B = \mathbf{R}$ or $\mathbf{C}$ and $f : (-\epsilon, \epsilon) \to B$ smooth. Take the Taylor series of f at $t = 0$, given by $\sum_{k=0}^\infty t^k f^{(k)}(0)/k!$. We know from Taylor's theorem this is an easy asymptotic expansion for f near $t = 0$. But the Taylor series does not converge to f unless f is analytic at $t = 0$.

Theorem 4.3.3. [Main theorem - Asymptotic expansion of heat kernel]
Let $(h_t)_{t>0}$ be the heat kernel for a graded Dirac operator D. Let dist: $M \times M \to [0, \infty)$ be the geodesic distance function on $M \times M$ determined by G. Let $n = \dim(M)$. For any $t > 0$, define

$$\rho_t(p,q) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{\operatorname{dist}(p,q)^2}{4t}\right).$$

Then:

a. There exists an asymptotic expansion for h_t of the form $h_t(p,q) \sim \rho_t(p,q)(\Theta_0(p,q) + t\Theta_1(p,q) + t^2\Theta_2(p,q) + \cdots)$, where

$$\rho_t(p,q)\left(\sum_{k=0}^{\infty} t^k \Theta_k(p,q)\right) = \sum_{k=0}^{\infty} a_k(t),$$

for $\Theta_k \in \Gamma(S \boxtimes S^*)$ for all $k \in \mathbf{Z}_{\geqslant 0}$.

b. The expansion is valid in the Banach space $C^j(S \boxtimes S^*)$ for any $j \geqslant 0$. It may be differentiated formally with respect to t, p, q to obtain asymptotic expansions for the corresponding derivatives of $h_t(p,q)$.

c. The sections $\Theta_k(p,q)$ (along the diagonal) and their derivatives with respect to p are described by universal algebraic expressions involving the metric g on M, the fiber metric h on S, and the connections and their derivatives. Also, $\Theta_0(p,p) = \mathrm{id}_{S_p}$.

To prove the main theorem, we need a criterion for recognizing an asymptotic expansion of h_0.

Definition 4.3.4. Let $m \in \mathbf{Z}_{>0}$. An *approximate heat kernel of order* m for a Dirac operator D is a t-independent section $\tilde{h}_t(p,q)$ of $S \boxtimes S^*$ that is C^1 in t, C^2 in p, q, and satisfies
a. for all $s \in \Gamma(S)$, $\lim_{t\to 0^+}[\|A_{\tilde{h}_t} s - s\|_{C^m}] = 0$, i.e. $\tilde{h}_t$ converges to a δ-function in C^m, and
b. for all $p, q \in U$, all $t > 0$, and $r_t(p,q)$ a section of $S \boxtimes S^*$ that descends continuously on t,

$$\left(\frac{\partial}{\partial t} + D_p^2\right)\left(\tilde{h}_t(p,q)\right) = t^m r_t(p,q).$$

Proposition 4.3.5. Suppose we have a sequence $\Theta_k \in \Gamma(S \boxtimes S^*)$, for $k \in \mathbf{Z}_{\geqslant 0}$ such that for any $m \in \mathbf{Z}_{>0}$, there exists $J_m \in \mathbf{Z}_{>0}$ such that for any $J \geqslant J_m$, the integral kernel

$$\tilde{h}_t(p,q) = \rho_t(p,q)\left(\sum_{k=0}^{J} t^k \Theta_k(p,q)\right)$$

is an approximate heat kernel of order m. Then the formal power series $\rho_t(p,q)(\sum_{k=0}^{J} t^k\Theta_k(p,q))$ is an asymptotic expansion for the heat kernel, in the sense of parts **a.** and **b.** in the main theorem.

You will complete the proof to this in Assignment 5. This proposition says that to prove **a.** and **b.** of the main theorem, it is enough to find an approximate heat kernel of order $m > 0$ for any m of the form $\tilde{h}_t^J(p,q)$.

Lemma 4.3.6. Let f_t be a section of S that is C^2 in $p \in M$, continuous in $t > 0$. Then there exists a smooth section s_t of S, differentiable at $t > 0$, with $s_0 = 0$, such that $(\frac{\partial}{\partial t} + D^2)s_t = f_t$, i.e. it solves the inhomogeneous heat equation. In fact, then $s_t = \int_0^t e^{-(t-u)D^2} f_u du$, i.e.

$$s_t(p) = \int_0^t \left(\int_M h_{t-u}(p,q) f_u(q) \mathrm{vol}_q\right) du. \tag{17}$$

<u>*Proof:*</u> Uniqueness is exactly as in the homogeneous case. If s_t, $\tilde{s}_t$ are two such solutions, then $s_t - \tilde{s}_t$ solves $(\frac{\partial}{\partial t} + D^2)(\,\cdot\,) = 0$ with initial value 0, meaning that $s_t - \tilde{s}_t = 0$. For existence, note that (17) is smooth in p

and differentiable in t. We need to check that (17) satisfies the inhomogeneous heat equation. We find that

$$\begin{aligned}\frac{\partial}{\partial t}s_t(p) &= \int_0^t \left(\int_M \frac{\partial}{\partial t}\left(h_{t-u}(p,q)\right) f_u(q)\mathrm{vol}_q\right) du + \lim_{u\to t}\left[\int_m h_{t-u}(p,q)f_u(q)\mathrm{vol}_q\right] \\ &= \int_0^t \left(\int_M \left(-D_p^2\right)\left(h_{t-u}(p,q)\right) f_u(q)\mathrm{vol}_q\right) + \lim_{\epsilon\to 0}\left[\int_M h_\epsilon(p,q)f_{t+\epsilon}(q)\mathrm{vol}_q\right] \\ &= -D_p^2 s_t(p) + f_t(p)\end{aligned}$$

by the δ-function properties of the heat kernel and continuity of f_t in t. ■

Corollary 4.3.7. For any $j \geqslant 0$, there exists $c_j > 0$ such that $\|s_t\|_j \leqslant tc_j(\sup_{0\leqslant u\leqslant t}\{\|f_u\|_j\})$.

Proof: Recall that $s_t = \int_0^t e^{-(t-u)D^2} f_u du$, so

$$\|s_t\|_j \leqslant t\left\|e^{-(t-u)D^2} f_u\right\|_j \leqslant tc_j\|f_u\|_j \leqslant tc_j \sup_{0\leqslant u\leqslant t}\{\|f_u\|\},$$

because $e^{-\epsilon D^2} : W^j(S) \to W^j(S)$ is uniformly bounded for all $\epsilon > 0$. ■

Proposition 4.3.8.

a. Let h_t be the heat kernel for D. For every $m > 0$, there exists $m' > m$ such that if $\tilde{h}_t$ is an approximate heat kernel of order m', then $h_t(p,q) - \tilde{h}_t(p,q) = t^m e_t(p,q)$, where e_t is a C^m section of $S \boxtimes S^*$, depending continuously on $t \geqslant 0$.

b. $(\frac{\partial}{\partial t} + D_p^2)\tilde{h}_t(p,q) = t^m r_t(p,q)$, where r_t is C^m in p,q and continuous in t.

Proof: Take $m' > m + \dim(M)/2$. By the definition above of arroximate heat kernel of order m', $\tilde{h}_t(p,q)$ tends to a δ-function as $t \to 0^+$, proving part **a.**. For **b.**, Let $v_t(p,q)$ be the unique solution (for fixed q) to the inhomogeneous heat equation, i.e.

$$\left(\frac{\partial}{\partial t} + D_p^2\right) v_t(p,q) = -t^m r_t(p,q),$$

with $v_0(p,q) = 0$. Then $\tilde{h}_t(p,q) + v_n(p,q)$ has δ-form properties as $t \to 0^+$. Also, $(\frac{\partial}{\partial t} + D_p^2)(\tilde{h}_t(p,q) + v_n(p,q)) = 0$, hence by the result characterizing heat kernels, $\tilde{h}_t + v_t = h_t$, and

$$\|s_t\|_j \leqslant tc_j\left(\sup_{0\leqslant u\leqslant t}\{\|f_u\|_j\}\right). \tag{18}$$

By (18), we find that

$$\|v_t\| \leqslant c_j \sup_{0\leqslant u\leqslant t}\{\|u^{m'} r_u(p,q)\|\} \leqslant Bt^{m'+1}.$$

Define e_t by $v_t = t^m e_t$, so $\|t^m e_t\|_j = \|v_t\|_j \leqslant Bt^{m'+1}$, but also $\|t^m e_t\|_j = t^m\|e_t\|_j$, hence

$$\|e_t\|_j \leqslant Bt^{(m'-m)+1}.$$

■

Fix $q \in M$ and take normal coordinates $\{x^1, \ldots, x^n\}$ centered at q. Let $\rho = \rho_t(p,q)$ as above. Let $p = (x^1, \ldots, x^n)$, so then $\mathrm{dist}(p,q)^2 = \sum_{j=1}^n (x^i)^2 = r^2$. Hence in these coordinates, $\rho = \frac{1}{(4\pi t)^{n/2}} e^{-r^2/4t}$.

Lemma 4.3.9. In the described chart,

a. $\nabla\rho = -\frac{\rho}{2t} r\frac{\partial}{\partial r}$ for ∇ as a function of p on (M,g), and

b. $\frac{\partial\rho}{\partial t} + \Delta\rho = \frac{r\rho}{4gt}\frac{\partial g}{\partial r}$, where $\det(g_{ij}) = g$ is a smooth function on the domain of the chart.

Proof: For part **a.** notice that

$$d\rho = \frac{1}{(4\pi t)^{n/2}} e^{-r^2/4t}\left(-\frac{2r}{4t}dr\right) \qquad \text{and} \qquad \nabla\rho = (d\rho)^\sharp = -\frac{\rho r}{2t}(dr)^\sharp = -\frac{\rho r}{2t}\nabla r,$$

by Riemannian geometry that gives us $\nabla r = \frac{\partial}{\partial r}$. For part **b.**, note that $\Delta f = -\mathrm{div}(\nabla f) = \nabla^*\nabla f$, where $\nabla^* = -\mathrm{div}$, a vector field. Recall that $\mathrm{div}(fX) = f\mathrm{div}(X) + \langle \nabla f, X\rangle$, so

$$\begin{aligned}\Delta\rho &= -\mathrm{div}(\nabla\rho)\\ &= \mathrm{div}\left(\frac{\rho}{2t}\left(r\frac{\partial}{\partial r}\right)\right)\\ &= \frac{\rho}{2t}\mathrm{div}\left(r\frac{\partial}{\partial r}\right) + \left\langle \nabla\left(\frac{\rho}{2t}\right), r\frac{\partial}{\partial r}\right\rangle\\ &= \frac{\rho}{2t}\mathrm{div}\left(r\frac{\partial}{\partial r}\right) + \frac{-\rho}{4t^2}\left\langle r\frac{\partial}{\partial r}, r\frac{\partial}{\partial r}\right\rangle,\end{aligned}$$

and because $|\frac{\partial}{\partial r}| = 1$, we have that

$$\frac{-\rho}{4t^2}\left\langle r\frac{\partial}{\partial r}, r\frac{\partial}{\partial r}\right\rangle = \frac{-\rho r^2}{4t^2}\left|\frac{\partial}{\partial r}\right|^2 = \frac{-\rho r^2}{4t^2}.$$

If $Y = Y^i\frac{\partial}{\partial x^i}$ in local coordinates, then $\mathrm{div}(Y) = \frac{1}{\sqrt{g}}\sum_j \frac{\partial}{\partial x_j}(Y^i\sqrt{g})$. Since $\frac{\partial}{\partial r} = \frac{X^i}{r}\frac{\partial}{\partial x^i}$ and $r\frac{\partial}{\partial r} = X^i\frac{\partial}{\partial x^i}$ and $Y^i = X^i$, we have that

$$\begin{aligned}\mathrm{div}\left(r\frac{\partial}{\partial r}\right) &= \frac{1}{\sqrt{g}}\sum_j \frac{\partial}{\partial x^j}\left(X^j\sqrt{g}\right)\\ &= \frac{1}{\sqrt{g}}\sum_j\left(rg + \frac{X^j}{2\sqrt{g}}\frac{\partial g}{\partial x^j}\right)\\ &= n + \frac{r}{2g}\frac{\partial g}{\partial r}.\end{aligned}$$

Hence $\Delta\rho = \frac{\rho}{2t}(n + \frac{r}{2g}\frac{\partial g}{\partial r}) - \frac{\rho r^2}{4t^2}$, and $\frac{\partial\rho}{\partial t} = (-\frac{n}{2t} + \frac{r^2}{4t^2})\rho$, so

$$\nabla\rho + \frac{\partial\rho}{\partial t} = \frac{\rho r}{4gt}\frac{\partial g}{\partial r}.$$

■

Lemma 4.3.10. Let $t \in C^\infty(M)$, D be the Dirac operator. Then
- **a.** $[D, f]s = D(fs) - f(Ds) = (\nabla f)\cdot s$ and
- **b.** $[D^2, f]s = D^2(fs) - f(D^2 s) = (\Delta f)s - 2\nabla_{\nabla f}s$.

Proof: Choose an orthonormal geodesic frame $\{e_1, \ldots, e_n\}$ centered at q, so the e_is are orthonormal eigenvalues in the chart. This means that $\nabla_{x_i}e_i = 0$ for all $x_i \in \Gamma(T_qM)$. For part **a.** observe that

$$\begin{aligned}D(fs) &= \sum_i e_i\cdot\nabla_{e_i}(fs)\\ &= \sum_i \cdot\left((\nabla_{e_i}f)s + f(\nabla_{e_i}s)\right)\\ &= \left(\sum_i(\nabla_{e_i}f)\,e_i\right)s + f\nabla s\\ &= (\nabla f)\cdot s + fDs.\end{aligned}$$

For part **b.**, note first that

$$
\begin{aligned}
D^2(fs)\big|_q &= \sum_{i,j} e_i \cdot \left(\nabla_{e_i}\left(\nabla_{e_j} \cdot \left(\nabla_{e_j} f s\right)\right)\right)\Bigg|_q \\
&= \sum_{i,j} e_i \cdot \left(\nabla_{e_i}\left(\nabla_{e_j} \cdot \left(\left(\nabla_{e_j} f\right) s + f\left(\nabla_{e_j} s\right)\right)\right)\right)\Bigg|_q \\
&= \sum_{i,j} e_i \cdot e_j \cdot \left(\left(\nabla_{e_i}\nabla_{e_j} f\right) s + \left(\nabla_{e_j} f\right)\left(\nabla_{e_i} s\right) + \left(\nabla_{e_i} f\right)\left(\nabla_{e_j} s\right) + f\nabla_{e_i}\nabla_{e_j} s\right)\Bigg|_q .
\end{aligned}
$$

We know that $\nabla_{e_j} e_j = 0$ at q, and $\nabla_{e_i}\nabla_{e_j} f - \nabla_{e_j}\nabla_{e_i} f = 0$ at q, as well as that

$$
e_i(e_j f) - e_j(e_i f) = [e_i, e_j]f = \underbrace{\left(\nabla_{e_i} e_j - \nabla_{e_j} e_i\right)}_{0 \text{ at } q} f.
$$

Therefore we find that

$$
\sum_{i,j} e_i \cdot e_j \cdot \left(\left(\nabla_{e_i}\nabla_{e_j} f\right) s\right)\Bigg|_q = \left(-\sum_{i=1}^{n} \nabla_{e_i}\nabla_{e_i} f\right) s\Bigg|_q = \left(\nabla^*\nabla f\right) s\big|_q = (\Delta f)s\big|_q .
$$

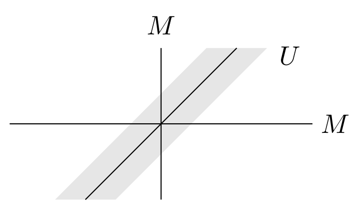
The last term reduces to $fD^2 s$. And finally, the two middle terms become

$$
\sum_{i,j} e_i \cdot e_j \left(\left(\nabla_{e_i} f\right)\left(\nabla_{e_j} s\right) + \left(\nabla_{e_j} f\right)\left(\nabla_{e_i} s\right)\right) = -2\sum_{i,j}^{n} \left(\nabla_{e_i} f\right)\left(\nabla_{e_i} s\right) = -2\nabla_{\nabla f} s.
$$

■

We will use the above two lemmas to derive a recursive procedure for solving ODEs in normal coordinates, to obtain Θ_ks. The idea is to find smooth sections $\Theta_k(p,q)$ of $S \boxtimes S^*$ such that for any $m > 0$, the partial sum $\rho_t(p,q)(\sum_{k=0}^{J} t^k \Theta_k(p,q))$ is an approximate heat kernel of order m for sufficiently large J.

Proposition 4.3.11. It suffices to construct $\Theta_k(p,q)$ for p near q, i.e. in an open neighborhood U of the diagonal in $M \times M$.



This will happen because $\rho_t \to 0$ faster than any power of t as $t \to 0$, so we can cut off our definition of $\Theta_k(p,q)$ smoothly to 0 outside U.

<u>*Proof:*</u> Again, fix normal coordinates $\{x^1, \ldots, x^n\}$ centered at q. Let $\rho = 1/(4\pi t)^{n/2}\exp(-r^2/4t)$ as before.

Let s be a section of $S \boxtimes S_q^*$ (for q fixed), i.e. $s(\,\cdot\,, q)$ is a section of $S \boxtimes S^*$. By the above two lemmas,

$$\begin{aligned}\frac{1}{\rho}\left(\frac{\partial}{\partial t} + D^2\right)(\rho s) &= \frac{1}{\rho}\left(\frac{\partial}{\partial t}s + \rho\frac{\partial s}{\partial t} + \rho D^2 s + (\Delta\rho)s - 2\nabla_{\nabla\rho}s\right)\\ &= \frac{1}{\rho}\left(\left(\frac{\partial}{\partial t}\rho + \Delta\rho\right) + \rho\left(\frac{\partial}{\partial t} + D^2\right)s - 2\nabla_{-\frac{\rho r}{2t}\frac{\partial}{\partial r}}\rho\right)\\ &= \frac{1}{\rho}\left(\left(\frac{r\rho}{4gt}\frac{\partial g}{\partial r}\right)s + \rho\left(\frac{\partial}{\partial t} + D^2\right)s + \frac{\rho r}{t}\nabla_{\frac{\partial}{\partial r}}s\right)\\ &= \left(\frac{r}{4gt}\frac{\partial g}{\partial r}\right)s + \left(\frac{\partial}{\partial t} + D^2\right)s + \frac{1}{t}\nabla_{r\frac{\partial}{\partial r}}s.\end{aligned}$$

■

Remark 4.3.12. Define $H = \frac{1}{\rho}(\frac{\partial}{\partial t} + D^2)(p\,\cdot\,)$, and call it the *conjugate heat operator*. Above we have shown that

$$H(s) = \left(\frac{\partial}{\partial t} + D^2\right)s + \frac{1}{t}\nabla_{r\frac{\partial}{\partial r}}s + \left(\frac{r}{4gt}\frac{\partial g}{\partial r}\right)s.$$

We want to find $\rho(\sum_k t^k u_k) = s$ which solves the heat equation, i.e. $(\frac{\partial}{\partial t} + D^2)s = 0$ iff $H(\sum_k t^k u_k) = 0$. So let $s = t^k u$ with u independent of t. Then

$$\begin{aligned}H(t^k u) &= kt^{k-1}u + t^k D^2 u + t^{k-1}\nabla_{r\frac{\partial}{\partial r}}u + t^{k-1}\left(\frac{r}{4g}\frac{\partial}{\partial r}\right)u\\ &= t^{k-1}\left(\nabla_{r\frac{\partial}{\partial r}} + k + \frac{r}{4g}\frac{\partial g}{\partial r}\right)u + t^k D^2 u.\end{aligned}$$

Let's formally try to solve $(\frac{\partial}{\partial t} + D^2)(\rho_t s) = 0$ with $s \sim u_0 + tu_1 + t^2u_2 + \cdots$. Equating powers of t,

$$\nabla_{r\frac{\partial}{\partial r}}u_k + \left(k + \frac{r}{4g}\frac{\partial g}{\partial r}\right)u_k = -D^2u_{k-1}, \tag{19}$$

which is an ODE for u_k along the radial geodesic emanating from q in terms of u_{k-1}. Next, introduce an "integrating factor" $r^k g^{1/4}$ for

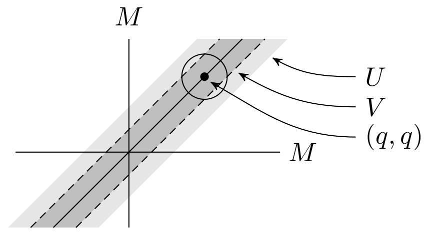
$$\nabla_{\frac{\partial}{\partial r}}\left(r^k g^{1/4}u_k\right) = kr^{k-1}g^{1/4}D^2u_k + \frac{1}{4}r^k g^{-3/4}\frac{\partial g}{\partial r}u_k + r^{k-1}g^{1/4}\nabla_{r\frac{\partial}{\partial r}}u_k - \left(k + \frac{k}{4g}\frac{\partial g}{\partial r}\right)u_k - D^2u_{k-1},$$

which implies that

$$\nabla_{\frac{\partial}{\partial r}}\left(r^k g^{1/4}u_k\right) = -r^{k-1}g^{1/4}D^2u_{k-1}, \tag{20}$$

where $u_{-1} = 0$. For $k = 0$, u_0 is uniquely determined by its initial value $u_0(0) = \mathrm{id}|_{S_q}$. The origin 0 corresponds to the point $q \in M$, because we want $\Theta_0(q,q) = \mathrm{id}_{S_q}$. For $k \geqslant 1$, (20) determines u_k in terms of u_{k-1} up to a constant multiple of a term of order r^{-k} near $r = 0$, because $(c_k r^{-k} g^{-1/4})(r^k g^{1/4} = c_k$ is constant. Since we want a smooth solution u_k as $r \to 0$, we must have $c_k \to 0$. So all the ks are uniquely determined from $u_0(0)$ and the demanding of continuity at $r = 0$.

Proposition 4.3.13. Choose an open neighborhood U of the diagonal in $M \times M$ such that every point in U lies in a normal coordinate chart centered at $(q,q) \in M \times M$. Define $\Theta_k(q,q)$ to be the section of $S \boxtimes S^*$ over U represented in normal coordinates by $U_k(x^1, \ldots, x^n)$ defined above. neighborhood U of the diagonal in $M \times M$.



Fix a smooth neighborhood $V \subset U$ and $\varphi : M \times M \to [0, \infty)$ smooth such that $\varphi(p, q) = 1$ for all $(p, q) \in V$, $\varphi(p, q) = 0$ for all $(p, q) \in M \times M \setminus U$, and smooth in between. Now define Θ_k on all of $M \times M$ by $\varphi(p, q)\Theta_k(p, q) \in \Gamma(S \boxtimes S^*)$. For $J > 0$, let

$$\tilde{h}_t^J(p, q) = \varphi(p, q)\rho_t(p, q)\left(\sum_{k=0}^{J} t^k \Theta_k(p, q)\right).$$

Let $m > 0$. Then $\tilde{h}_t^J$ is an opproximate heat kernel of order m for J sufficiently large.

Proof: Since $\Theta_0(p, p) = \mathrm{id}_{S_p}$ if $s \in \Gamma(S)$, then $\lim_{t \to 0^+}[nA_{\tilde{h}_t^J} s - s\|_{C^0}] = 0$ by the δ-functions property of ρ_t. That is, $\lim_{t \to 0^+}[\rho_t]$ has the δ-function property. And,

$$\Theta_0(p, q) = \underbrace{\Theta_0(p, p)}_{= \ \mathrm{id}_{S_p}} + \underbrace{\Theta_0(p, q) - \Theta_0(p, p)}_{\to 0 \text{ as } t \to 0}.$$

That was the first condition to be an approximate heat kernel. Also we need to show it approximately solves the heat equation. By construction of the u_ks, we have that

$$\left(\frac{\partial}{\partial t} + D_p^2\right)\left(\tilde{h}_t^J(p, q)\right) = t^J \rho_t(p, q) e_t^J(p, q)$$

for $t > 0$, where e_t^J is a smooth section of $S \boxtimes S^*$, which is continuous in $t \geqslant 0$. This follows as the terms up to t^{J-1} in $\tilde{h}_t^J$ are killed by the heat operator on the new diagonal, and all that's left in a neighborhood of the diagonal is $t^J D_p^2(\varphi \rho_t \Theta_J)$. Now let $m > 0$. If $J > m + \frac{n}{2}$, then $t^J \rho_t(p, q) \to 0$ in the C^m-norm as $t \to 0$. So for $J > m + \frac{n}{2}$, $\tilde{h}_t^J(p, q)$ is an approximate heat kernel of order m. Hence by Assignment 5 question 2, $\rho_t(\sum_{k=0}^{\infty} t^k \Theta_k(p, q))$ is an asymptotic expansion of $h_t(p, q)$ as required. ■

Remark 4.3.14. Finally, for the last part of the main theorem on the asymptotic expansion, for all $k \geqslant 0$, $\Theta_k(p, p)$ can be expressed as an algebraic expression involving matrices, connections, and their derivatives. It follows by induction on the form

$$\nabla_{r\frac{\partial}{\partial r}} u_k + \left(k + \frac{r}{4g}\frac{\partial g}{\partial r}\right) u_k = -D^2 u_{k-1}.$$

In practice, nobodoy really does this, but this shows that it exists.

Remark 4.3.15. Recall the McKean–Singer formula, which said that for $D = \left[\begin{smallmatrix} 0 & P^* \\ P & 0 \end{smallmatrix}\right]$ with $P : \Gamma(S^+) \to \Gamma(S^-)$ and $P = D^+ = D|_{\gamma(S^+)}$, we have that

$$\mathrm{ind}(P) = \mathrm{ind}(D^+) = \int_M \mathrm{str}(h_t(p, p))\mathrm{vol}_p,$$

which is independent of $t > 0$. For ρ_t as previously, we showed that

$$h_t(p, p) \sim \rho_t(p, p)\left(\sum_{k=0}^{\infty} t^k \Theta_k(p, p)\right) = \frac{1}{(4\pi t)^{n/1}}\left(\sum_{n=0}^{\infty} t^k \Theta_k(p, p)\right),$$

so then $\mathrm{str}(h_t(p,p)) \sim \frac{1}{(4\pi t)^{n/2}} \sum_{k=0}^{\infty} t^k \mathrm{str}(\Theta_k(p,p))$. We also know that the integral of $\mathrm{str}(h_t(p,p))$ over M is independent of t. So (check this), $\int_m \Theta_k(p,p)\mathrm{vol}_p = 0$ unless $k = n/2$.

Corollary 4.3.16.

$$\mathrm{ind}(D^+) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \dfrac{1}{(4\pi)^{n/2}} \displaystyle\int_M \mathrm{str}(\Theta_{n/2}(p,p))\mathrm{vol}_p & \text{if } n \text{ is even} \end{cases}.$$

Moreover, $\Theta_{n/2}(p,p)$ is an algebraic expression in metrics, connections, and their derivatives.

Now, the last step is to find a way to rewrite the expression for $\mathrm{ind}(D^+)$ more invariantly, i.e. in terms of characteristic classes. Before we can talk about the rescaling trick of Getzler to compute the index, we need to gefine the Bochner–Weitzenböck formula.

Recall that if S is a Clifford bundle and D is a Dirac operator, $D^2 s = \nabla^*\nabla s + k\cdot s$, where $k = \sum_{i<j} e_i \cdot e_j \cdot F_{ij}^{\nabla}$, for $e_1, \ldots, e_n$ a local orthonormal frame of M, and F^{∇} the curvature 2-form of the connection on S. Let $c : TM \to \mathrm{End}(S)$ denote Cliford multiplication, i.e. if $Z_p \in T_pM$ and $s_p \in S_p$, then $c(Z_p) \in \mathrm{End}(S_p)$ such that $c(Z_p)s_p = Z_p \cdot s_p$. We see that $c : \Gamma(TM) \to \Gamma(\mathrm{End}(S))$ by $(c(Z)s)_p = c(Z_p)s_p$.

4.4 Curvature

Lemma 4.4.1. Let $X, Y, Z \in \Gamma(TM)$. Then $[F^{\nabla}(X,Y), c(Z)] = c(R(X,Y), Z)$, where R is the Riemann curvature tensor of the metric g on M. That is, the curvature R of (M,g) measures the failure of F^{∇} to be a 2-form-valued endomorphism of S in the category of $\mathcal{C}\ell(M) \otimes \mathbf{C}$-modules.

Proof: This identity is pointwise on M. Let $p \in M$ and $\{e_1, \ldots, e_r\}$ be an orthonormal geodesic frame centered at p. That is, $(\nabla_X e_i)_p = 0$ for all X, as $[e_i, e_j]|_p = 0$. By linearity, it is enough to assume $X = e_i$, $Y = e_j$, and $Z = e_k$. Then

$$\begin{aligned} \nabla_{e_i}(\nabla_{e_j}(e_k \cdot s)) &= \nabla_{e_i}((\nabla_{e_j} e_k) \cdot s + e_k \cdot (\nabla_{e_j} s)) \\ &= (\nabla_{e_i}\nabla_{e_j} e_k) \cdot s + e_k \cdot (\nabla_{e_i}\nabla_{e_j} s). \end{aligned}$$

Now take the difference of the last line with itself with i and j switched to get

$$F^{\nabla}(e_i, e_j)(e_k \cdot s) = (R(e_i, e_j)e_k) \cdot s + e_k \cdot (F^{\nabla}(e_i, e_j)s).$$

■

Definition 4.4.2. Recall that $R(e_i, e_j)e_k = r_{ijk}^{\ell} e_\ell = \sum_{\ell=1}^{n} R_{ijk\ell} e_\ell$ in an orthonormal frame. The *Riemann endomorphism* R^S of the Clifford bundle S is defined to be the $\mathrm{End}(S)$-valued 2-form given by

$$R^S(X,Y) = \frac{1}{4} \sum_{k,\ell} c(e_k)c(e_\ell)R(X,Y,e_k,e_\ell),$$

in an orthonormal frame, independent of the orthonormal frame.

Lemma 4.4.3.

$$[R^S(X,Y), c(Z)] = c(R(X,Y)Z).$$

Proof: It is enough to show the result for $X = e_i$, $Y = e_j$, $Z = e_m$, where $\{e_1, \ldots, e_n\}$ is an orthonormal frame. Then

$$\begin{aligned} R^S(e_i,e_j)c(e_m) - c(e_m)R^S(e_i,e_j) &= \frac{1}{4}\sum_{k,\ell} R_{ijk\ell}\left(c(e_k)c(e_\ell)c(e_m) - c(e_m)c(e_k)c(e_\ell)\right) \\ &= \frac{1}{4}\sum_{k,\ell} c\left(e_k e_\ell e_m - e_m e_k e_\ell\right) \\ &= \frac{1}{4}\sum_{k,\ell} R_{ijk\ell} c\left([e_k e_\ell, e_m]\right). \end{aligned}$$

If $k = \ell$, then $[e_k e_\ell, e_m] = [-1, e_m] = 0$. If k, ℓ, m are all distinct, then $e_k e_\ell e_m - e_m e_k e_\ell = 0$. The only remaining case is $m = k \neq \ell$ and $m = \ell \neq k$. Then

$$\begin{aligned} R^S(e_i, e_j)c(e_m) - c(e_m)R^S(e_i, e_j) &= \frac{1}{4}\sum_{\ell=1}^{n} R_{ijm\ell}c\left(e_m e_\ell e_m - e_m^2 e_\ell\right) + \frac{1}{4}\sum_{\substack{k=1 \\ k\neq m}}^{n} R_{ijkm}c\left(e_k e_m^2 - e_m e_k e_m\right) \\ &= \frac{1}{4}\sum_{\ell=1}^{n} R_{ijm\ell}c(2e_\ell) + \frac{1}{4}\sum_{k=1}^{n} R_{ijkm}c(-2e_k) \\ &= \sum_{\ell=1}^{n} R_{ijm\ell}c(e_\ell) \\ &= c(R(e_i, e_j)e_m). \end{aligned}$$

■

Definition 4.4.4. The *twisting curvature* of the Clifford bundle S is denoted by F^S and is defined by $F^\nabla = R^S + F^S$.

Corollary 4.4.5. For all X, Y, Z, $[F^S(X,Y), c(Z)] = 0$. Hence $F^S(X,Y)$ is a $\mathcal{C}\ell(TM) \otimes \mathbf{C}$-linear endomorphism of S.

This now allows us to rewrite the Bochner–Weitzenböck formula.

Lemma 4.4.6. In an orthonormal frame,

$$\sum_{i,j,k} R_{ijk\ell}c(e_i e_j e_k) = -2\sum_j (\mathrm{Ric})_{\ell j}c(e_j),$$

where Ric is the $(2,0)$ Ricci tensor.

Proof: If i, j, k are distinct, then $e_i e_j e_k - e_j e_k e_i = e_k e_i e_j$. But $R_{ijk\ell} + R_{jki\ell} + R_{kij\ell} = 0$ by the 1st Bianchi identity, so those terms vanish, as do the $i = j$ terms. Only $i = k \neq j$ and $i \neq k = j$ remain. So

$$\begin{aligned} \sum_{\substack{j,k \\ j\neq k}} R_{kjk\ell}c(e_k e_j e_k) + \sum_{\substack{i,k \\ i\neq k}} R_{ikk\ell}c(e_i e_k e_k) &= \sum_{j,k} R_{kjk\ell}c(e_j) + \sum_{i,k} R_{ikk\ell}c(e_i) \\ &= -2\sum_{j,k} R_{jkk\ell}c(e_j) \\ &= -2\sum_j (\mathrm{Ric})_{j\ell}c(e_j). \end{aligned}$$

■

Corollary 4.4.7. For $\hat{F}^S = \sum_{i<j} F^S(e_i, e_j)e_i \cdot e_j$ and $K = \sum_{i<j} F^\nabla(e_i, e_j)e_i \cdot e_j$ the scalar curvature,

$$D^2 = \nabla^*\nabla + \tfrac{1}{4}K + \hat{F}^S.$$

Proof: We already know that $D^2 = \nabla^*\nabla + K$. We just need to show that $\sum_{i<j} e_i \cdot e_j R^s(e_ie_j) = \frac{1}{4}K$. Well,

$$\begin{aligned}\sum_{i<j} e_i \cdot e_j R^S(e_i,e_j) &= \frac{1}{4}\sum_{i<j}\sum_{k,\ell} R(e_i,e_j,e_k,e_\ell)e_i\cdot e_j\cdot e_k\cdot e_\ell \\ &= \frac{1}{8}\sum_{i,j,k,\ell} R_{ijk\ell}e_i\cdot e_j\cdot e_k\cdot e_\ell \\ &= \frac{-2}{8}\sum_{j,\ell}(\mathrm{Ric})_{\ell j}e_j\cdot e_\ell \\ &= \frac{-1}{4}\sum_{j=1}^{n}(\mathrm{Ric})_{jj}e_j^2 \\ &= \frac{1}{4}\sum_{j}(\mathrm{Ric})_{jj} \\ &= \frac{1}{4}K.\end{aligned}$$

■

Recall that a Clifford bundle S has the property that each fiber S_p over $p \in M$ is a representation of $\mathcal{C}\ell(TM)\otimes\mathbf{C}$. From now on, we will assume that

1. $n = 2m$ is always even (otherwise $\mathrm{ind}(D^+) = 0$), and
2. S_p is an irreducible representation of $\mathcal{C}\ell(T_pM)\otimes\mathbf{C}$.

We assume **2.** because if we don't, then the algebra is much messier. Moreover, for the special cases of Chern–Gauss–Bonnet, the signature theorem, Hirzebruch–Riemann–Roch, the assumption holds. Note that for the first two, $S = \bigwedge^\bullet(T^*M)\otimes\mathbf{C}$ and $D = d + d^*$, but they have different splittings. For the last theorem, $S = \bigwedge^{0,\bullet}(T^*M)$ and $D = \overline{\partial} + \overline{\partial}^*$.

Recall that there is exactly one non-trivial irreducible representation Δ of $\mathcal{C}\ell(\mathbf{R}^{2m})\otimes\mathbf{C}$ of dimension $2^{2m} = 2^n$ (up to isomorphism). For us, S will be of rank 2^n, as the main example will have $S = \bigwedge^\bullet(T^*M)\otimes\mathbf{C}$, which has rank 2^n.

Example 4.4.8. Recall the definition of a $\mathbf{Z}/2\mathbf{Z}$-graded (or supersymmetric) Clifford bundle $S = S^+ \oplus S^-$. For example, we have $S = \bigwedge^\bullet(T^*M)\otimes\mathbf{C}$. If $v \in \Gamma(TM)$ and $\alpha \in \Gamma(S)$, then $v\cdot\alpha = v\wedge\alpha - v\lrcorner\,\alpha$.

Definition 4.4.9. Let S be $\mathbf{Z}/2\mathbf{Z}$ graded. given $A, B \in \mathrm{End}(S)$, define the *supercommutator* of A and B with respect to S by

$$[A,B]_S = \begin{cases}[A,B] & \text{if } A \text{ or } B \text{ are even}\\ \{A,B\} = AB + BA & \text{if } A \text{ and } B \text{ are odd}\end{cases}.$$

An endomorphism A is even if $A(S^\pm) \subset S^\pm$, i.e. $A = \left[\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right]$. Similarly, A is odd if $A(S^\pm) \subset S^\mp$, i.e. $A = \left[\begin{smallmatrix}0 & \delta\\ \gamma & 0\end{smallmatrix}\right]$.

Lemma 4.4.10. If $P = [A,B]_S$ is a supercommutator, then $\mathrm{str}(P) = 0$.

Proof: There are 3 cases to check. First note that by writing $A = A^+ + A^-$ and $B = B^+ + B^-$, we have that

$$[A,B]_S = [A^+,B^+] + [A^+,B^-] + [A^-,B^+] + \{A^-,B^-\}.$$

Case 1: A and B are both even. Then $[A,B]_S = [A,B]$ and

$$A = \begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix} \quad , \quad B = \begin{bmatrix}\gamma & 0\\ 0 & \delta\end{bmatrix} \quad \text{implies} \quad [A,B] = \begin{bmatrix}[\alpha,\gamma] & 0\\ 0 & [\beta,\gamma]\end{bmatrix},$$

so $\mathrm{str}([A,B]_S) = \mathrm{str}([A,B]) = \mathrm{Tr}([\alpha,\gamma]) - \mathrm{Tr}([\beta,\delta]) = 0$.
Case 2: One of A, B is even, one is odd. Then $[A,B]_S = [A,B] = \left[\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right]$, so $\mathrm{str}([A,B]_S) = 0$.
Case 3: A and B are both odd. Then for A, B as above,

$$[A,B]_S = \{A,B\} = \begin{bmatrix} \alpha\gamma + \delta\beta & 0 \\ 0 & \beta\delta + \gamma\delta \end{bmatrix}.$$

Hence $\mathrm{str}([A,B]_S) = \mathrm{Tr}(\alpha\gamma + \delta\beta) - \mathrm{Tr}(\beta\delta + \gamma\alpha) = 0$. ∎

Remark 4.4.11. Consider $\alpha \in \Gamma(\mathcal{C}\ell(TM) \otimes \mathbf{C})$, so $\alpha_p \in \mathcal{C}\ell(T_pM) \otimes \mathbf{C} \cong \mathrm{End}(S_p)$, with $\alpha_p(s_p) = \alpha_p \cdot s_p$ the Clifford action. Then $\mathrm{str}(\alpha) \in C^\infty_{\mathbf{C}}(M)$. Let $e_1, \dots, e_n$ be a local orthonormal frame of M. Let $I = \{i_1 < \cdots < i_k \ : \ i_j \in \{1, \dots, n\} \ \forall \ j\}$. Define $e_I = e_{i_1} \cdots e_{i_k}$. Then

$$\alpha = \sum_{\substack{\text{all multi-} \\ \text{indices } I}} \alpha_I e_I \qquad \text{and} \qquad \mathrm{str}(\alpha) = \sum_{\substack{\text{all multi-} \\ \text{indices } I}} \alpha_I \mathrm{str}(e_I).$$

Note that at some fixed I,

$$[e_a, e_a \cdot e_I]_S = e_a \cdot e_a \cdot e_I - (-1)^{k+1} e_a \cdot e_I \cdot e_a = e_a^2 \cdot e_I - (-1) e_a^2 \cdot e_I = -2e_I,$$

so $e_I = [e_a, -\frac{1}{2} e_a \cdot e_I]_S$. Hence $\mathrm{str}(e_I) = 0$ unless $I = \{1, 2, \dots, n\}$.

Remark 4.4.12. Let $\Gamma = e_1 \cdots e_n$. What is $\mathrm{str}(\Gamma)$? To compute this, recall that we found an explicit realization of Δ, the unique non-trivial irreducible representation of $\mathcal{C}\ell(\mathbf{R}^{2m}) \otimes \mathbf{C}$. We described Δ as follows: Let J be a complex structure on $\mathbf{R}^{2m} = V$. Then

$$V_{\mathbf{C}} = V \otimes \mathbf{C} = V^{1,0} \oplus V^{0,1} \qquad \text{and} \qquad \Delta = \textstyle\bigwedge^\bullet (V^{1,0}),$$

where $V^{1,0}$ and $V^{0,1}$ are the $\pm i$-eigenspaces of J, respectively, and Δ is given as a complex vector space of dimension $2^{2m} = 2^n$. Then Δ becomes a $\mathcal{C}\ell(V) \otimes C$-module by letting, for all $\alpha \in \bigwedge^\bullet(V^{1,0})$, $v + w \in V \otimes \mathbf{C}$ with $v \in V^{1,0}$ and $w =\in V^{0,1}$ (so $\overline{w} \in V^{1,0}$),

$$v \cdot \alpha = \sqrt{2} v \wedge \alpha \qquad \text{and} \qquad w \cdot \alpha = -\sqrt{2}\, \overline{w} \lrcorner \alpha.$$

This satisfies $x \cdot y \cdot \alpha + y \cdot x \cdot \alpha = -2 \langle x, y \rangle \alpha$ for all $x, y \in V \otimes \mathbf{C}$. Next, choose a basis $e_1, \dots, e_m, Je_1, \dots, Je_m$ of V. Let

$$\begin{array}{rcl} v_j & = & \frac{1}{\sqrt{2}}(e_j - iJe_j) \in V^{1,0} \\ w_j & = & \frac{1}{\sqrt{2}}(e_j + iJe_j) = \overline{v}_j \in V^{1,0} \end{array} \qquad \text{and} \qquad \begin{array}{rcl} e_j & = & (v_j + w_j)/\sqrt{2} \\ Je_j & = & i(v_j - w_j)/\sqrt{2} \end{array}.$$

Then $e_j \cdot \alpha = v_j \wedge \alpha - \overline{w}_j \lrcorner \alpha = v_j \wedge \alpha - v_j \lrcorner \alpha$. For $\alpha \in \Delta$, $(Je_j) \cdot \alpha = i(v_j \wedge \alpha + v_j \lrcorner \alpha)$, so letting $\alpha = v_{i_1} \wedge \cdots \wedge v_{i_k} \in \Delta$, we then have that

$$v_j \wedge (v_j \lrcorner \alpha) = \begin{cases} \alpha & \text{if } j \in \{i_1, \dots, i_k\} \\ 0 & \text{if } j \notin \{i_1, \dots, i_k\} \end{cases} \qquad \text{and} \qquad v_j \lrcorner (v_j \wedge \alpha) = \begin{cases} 0 & \text{if } j \in \{i_1, \dots, i_k\} \\ \alpha & \text{if } j \notin \{i_1, \dots, i_k\} \end{cases}.$$

Hence $\Gamma \cdot \alpha = (e_1 \cdot Je_1) \cdots (e_m \cdot Je_m) \cdot \alpha$, where $\Gamma = e_1 \cdot Je_1 \cdots e_m \cdot Je_m$ has the standard orientation. Also,

$$e_j \cdot (Je_j) \cdot \alpha = e_j \cdot (i(v_j \wedge \alpha + v_j \lrcorner \alpha)) = i(v_j \wedge (v_j \lrcorner \alpha) - v_j \lrcorner (v_j \wedge \alpha)),$$

so $\Gamma \cdot \alpha = i^m (-1)^{m-k} \alpha$, where $m - k$ is the number of v_js not in $\{i_1, \dots, i_k\}$. Hence

$$\mathrm{str}(\Gamma) = \sum_J (-1)^{|J|} \langle \Gamma \cdot v_J, v_J \rangle = \sum_J (-1)^{|J|} \left\langle i^m (-1)^{m-|J|} v_J, v_J \right\rangle = \sum_J (-1)^m = (-i)^m 2^m = (-2i)^m.$$

This allows us to conclude that if $\alpha = \sum_I \alpha_I e_I$, then $\mathrm{str}(\alpha) = \alpha_{1 \cdots n} \mathrm{str}(\Gamma) = \alpha_{1 \cdots n}(-2i)^m$, i.e. only the top degree component of α contributes to the supertrace. Hence we need to find a method (it will be Getzler's method) of picking the top degree part $\alpha_{1 \cdots n}\Gamma$ of $\alpha \in \mathcal{C}\ell(V) \otimes \mathbf{C}$. To introduce this method, we need to discuss graded and filtered algebras.

4.5 Graded and filtered algebras

Definition 4.5.1. Let A be an algebra over $\mathbf{C}$ (a vector space with associated multiplication). We say that A is $\mathbf{Z}$*-graded* if $A = \bigoplus_{m\in\mathbf{Z}} A^m$, for A^m a subspace of A and $A^m \cdot A^n \subset A^{m+n}$. If $\alpha \in A^m$, we say that α has degree m.

Example 4.5.2. The space $A = \mathbf{C}[x]$ is $\mathbf{Z}$-graded, with $A^m = \left\{\begin{smallmatrix} \{0\} & \text{if } m<0 \\ \mathrm{span}\{x^m\} & \text{if } m\geqslant 0 \end{smallmatrix}\right.$.

Similarly, $A = \bigwedge^\bullet(V)$, the exterior algebra of a vector space V, is $\mathbf{Z}$-graded by $A^m = \bigwedge^m(V)$.

Definition 4.5.3. Let A be an algebra over $\mathbf{C}$. Then A is a *filtered algebra* if there exists a family of subspaces A_m of A, for all $m \in \mathbf{Z}$, with $A_m \subset A_{m+1}$, $A_{mn} \subset A_{m+n}$, and $A = \bigcup_{m\in\mathbf{Z}} A_m$.

Example 4.5.4. The space $A = \mathcal{C}\ell(V)\otimes\mathbf{C}$, the complexified Clifford algebra of $(V, \langle\ \cdot\ ,\ \cdot\ \rangle)$ is filtered, with A_m being the span of Clifford products of m or fewer elements. Note that A is not $\mathbf{Z}$-graded, because for α, β the product of m, n elements, respectively, $\alpha\cdot\beta$ may not be the product of $m+n$ elements. However, A is $\mathbf{Z}/2\mathbf{Z}$-graded.

The space $A = \mathcal{D}(M)$, the *algebra of linear differential operators* on $C^\infty_{\mathbf{C}}(M)$, is filtered. We call $(\mathcal{D}(M))_m$ the space of *differential operators of order* $\leqslant m$. This is not a graded algebra.

Our main goal now is to find a way to compute $\mathrm{str}(\Theta_{n/2}(p,p))$ without actually computing $\Theta_{n/2}(p,p)$.

Remark 4.5.5. Note that any graded algebra is filtered, by letting $A_m = \bigoplus_{n\in\mathbf{Z},n\leqslant m} A^n$. Note also for $f : A \to B$ a homomorphism of algebras, A filtered implies $f(A)$ is filtered. That is,

$$f(A_m)f(A_n) = f(A_mA_n) \subset f(A_{m+n})$$

as f is a homomorphism and A is filtered. Let's use these facts to construct a *canonical filtration.*

Definition 4.5.6. Let A be an algebra. A *filtration* of A is a collection of subspaces $A_m \subset A$ for all $m \in \mathbf{Z}$, such that $\bigcup_{m\in\mathbf{Z}} A_m = A$, $A_m \subset A_{m+1}$, and $A_mA_n \subset A_{m+n}$.

So suppose A is an algebra, B is a subalgebra of A, and V is a subspace of A such that A is generated by $B\cup V$. We will construct a filtration on A by assigning an order to elements of B and V: an element of B has order 0 and an element of V has order 1. Define

$$\bigotimes_B^* V = B \oplus (B\otimes V\otimes B) \oplus (B\otimes V\otimes B\otimes V\otimes B)\oplus\cdots.$$

There exists a surjective homomorphism $f : \bigotimes_B^* V \to A$. If $a \in A$ is $a = b_1v_1b_2v_2\cdots b_kv_kb_{k+1}$, then $f(b_1\otimes v_1\otimes\cdots\otimes b_x\otimes v_k\otimes b_{k+1}) = a$. Moreover, the space $\bigotimes_B^* V$ has a natural grading (where the degree is the number of vs). So A has a canonical filtration. An element $a \in A$ is of order k if it is a linear combination of products of elements of B, V with at most k elements of V in each term.

Example 4.5.7. Consider $\mathcal{C}\ell(V)\otimes\mathbf{C})$, the space of complexified differentials of $(V, \langle\ \cdot\ ,\ \cdot\ \rangle)$. Take $B = \mathbf{C}$, $V = V$. Then $\mathcal{C}\ell$ is generated by $B\cup V$. That canonical filtration described above is the usual filtration of $\mathcal{C}\ell$.

Definition 4.5.8. Let A be a filtered algebra. Define $(G(A))^m = A_m/A_{m-1}$, and $G(A) = \bigoplus_{m\in\mathbf{Z}}(G(A))^m$. It is left as an exercise to show that $G(A)$ is graded. Next, let G be graded, both A, B complex-valued algebras. A *symbol map* $\sigma_\bullet : A \to G$ is a collection of linear maps $\sigma_m : A_m \to G^m$ for all $m \in \mathbf{Z}$, such that

1. $\sigma_m(a) = 0$ if $a \in A_{m-1}$, and

2. if $a \in A_m$ and $b \in A_n$, then $\sigma_m(a)\cdot\sigma_n(b) = \sigma_{m+n}(ab)$.

Example 4.5.9. The map $\pi_k : A_k \to A_k/A_{k-1} = (G(A))^k$ is a symbol map. Also, if we let $A = \mathcal{C}\ell = \mathcal{C}\ell(V) \otimes \mathbf{C}$ and $G = G(A)$, then $G = \bigwedge^\bullet(V) \otimes \mathbf{C}$. We can also compute $\pi_n : (\mathcal{C}\ell)_n \to (G(A))^n$, by showing that for all $0 \leqslant k \leqslant n$,

$$\pi_k \left(\sum_{\substack{\text{multi-indices } I \\ |I| \leqslant k}} \alpha_I e_I \right) = \sum_{|I| \leqslant k} \alpha_I e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

Here the symbol map "picks out" the top degree part. This is what we need to compute $\operatorname{str}(\Theta_{n/2}(p,p))$.

Remark 4.5.10. Next we are going to consider a very important example. Let $A = \mathcal{D}(M)$, the algebra of linear differential operators on $C^\infty_{\mathbf{C}}(M)$. If $p \in A$, locally

$$p = \sum_{|\alpha| \leqslant m} f_\alpha \left(x^1, \ldots, x^n\right) \frac{\partial^{|\alpha|}}{\partial x^\alpha},$$

and p is of order m. We write x^α for $(x^1)^{\alpha_1} \cdots (x^n)^{\alpha_n}$. Note that A is filtered, with A_m the set of linear differential operators of order $\leqslant m$. We leave it as an exercise to show that this definition is independent of coocdinates. We will now construct a symbol map σ from $A = \mathcal{D}(M)$ to a particular graded algebra. So let V be a finite-dimensional vector space and let $\mathcal{C}(V)$ be the $\mathbf{C}$-algebra of constant coefficient differential operators acting on $C^\infty(V)$. If $\{e_1, \ldots, e_n\}$ is a basis of V, then $v \in V$ is $v = \sum_{k=1}^n x^x e_k$, so $x^1, \ldots, x^n$ are global coordinates on V. So

$$T \in \mathcal{C}(V) \qquad \text{coresponds to} \qquad T = \sum_{|\alpha| \leqslant k} c_\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}.$$

Then $\mathcal{C}(V)$ is a graded algebra, where $(\mathcal{C}(V))^m = \operatorname{span}\{\frac{\partial^m}{\partial x^\alpha} \ : \ |\alpha| = m\}$.

Definition 4.5.11. Let M be a smooth manifold. Define $\mathcal{C}(TM)$ to be the bundle of algebras over M whose fiber over $p \in M$ is $\mathcal{C}(T_pM)$. The space of sections $\Gamma(\mathcal{C}(TM))$ is graded, by $T \in (\Gamma(\mathcal{C}(TM)))^m$ if $T_p \in (\mathcal{C}(T_pM))^m$.

We would next like to define $\sigma_\bullet : \mathcal{D}(M) \to \Gamma(\mathcal{C}(TM))$, map from a filtered algebra to a graded algebra. Let $p \in (M, g)$, and choose normal coordinates centered at p. Let

$$T = \sum_{|\alpha| \leqslant m} c_\alpha(x^1, \ldots, x^n) \frac{\partial^{|\alpha|}}{\partial x^\alpha}$$

in this chart, for $T \in (\mathcal{D}(M))_m$. Define $\sigma_m(T) \in (\Gamma(\mathcal{C}(TM)))^m = \Gamma((\mathcal{C}(TM))^m)$, so $(\sigma_m(T))_p = \sigma_{m,p}(T) \in (\mathcal{C}(T_pM))^m$, and set

$$\sigma_{m,p}(T) = \sum_{|\alpha| = m} c_\alpha(0) \frac{\partial^{|\alpha|}}{\partial x^\alpha},$$

which is an mth order constant coefficient differential operator on $C^\infty_{\mathbf{C}}(T_pM)$. This is well-defined, and $\sigma_{m,p}(T) = 0$ if $T \in (\mathcal{D}(M))_{m-1}$ (you will see this is Assginment 6). Also, if $T \in (\mathcal{D}(M))_m$ and $U \in (\mathcal{D}(M))_n$, then

$$\sigma_{m+n,p}(TU) = \sigma_{m,p}(T) \cdot \sigma_{n,p}(U).$$

Remark 4.5.12. Note that $\mathcal{D}(M)$ is generated as an algebra by $B = C^\infty_{\mathbf{C}}(M)$, which is of order 0, and by the vector space $W = \Gamma(TM)$, which is a smooth vector field on M. Then $\mathcal{D}(M)$ is the homomorphic image of $\bigotimes_B^* W$, hence it has a canonical filtration, which coresponds to the usual filtration. So to specify a symbol map σ on $\mathcal{D}(M)$, it is enough to specify the effect of σ on generators of $\mathcal{D}(M)$. For this σ, we then have $\sigma_0(f) = f$, i.e. $\sigma_{0,p}(f) = f(p)$ is multiplication by the constant $f(p)$ on $C^\infty_{\mathbf{C}}(T_pM)$. Then

$$\sigma_1(X) = \sigma_1 \left(a_i(x^1, \ldots, x^n) \frac{\partial}{\partial x^i} \right) = X \quad \text{and} \quad \sigma_{1,p} = X_p = \sum_{i=1}^n a_i(p) \frac{\partial}{\partial x^i} \in (\mathcal{C}(T_pM))^1.$$

Remark 4.5.13. On $\mathrm{End}_{\mathbf{C}}(S)$, the bundle of $\mathbf{C}$-linear endomorphisms of a Clifford bundle S, we want to construct a natural filtration. Recall that if $n = 2m$, any finite-dimensional representation of $\mathcal{C}\ell = \mathcal{C}\ell(\mathbf{R}^{2m}) \otimes \mathbf{C}$ is a direct sum of finitely many copies of Δ, the unique irreducible representation of $\mathcal{C}\ell$, with $\dim(\Delta) = 2^m$. Also, $\mathcal{C}\ell \cong \mathrm{End}_{\mathbf{C}}(\Delta)$ as we saw some time ago. Hence if S is a finite-dimensional representation of $\mathcal{C}\ell$, then

$$S = \underbrace{\Delta \oplus \cdots \oplus \Delta}_{k \text{ times}} \cong \Delta \otimes V$$

for some finite-dimensional V with $\dim(V) = k$, such that $- \otimes v \in \Delta \otimes v$, and $a(s \otimes v) = (as) \otimes v$. Note that is $S = \Delta \otimes_{\mathbf{C}} V$, then

$$\mathrm{Hom}_{\mathcal{C}\ell}(\Delta, S) \cong \Delta^* \otimes_{\mathcal{C}\ell} S \cong \left(\Delta^* \otimes_{\mathcal{C}\ell} \Delta\right) \otimes_{\mathbf{C}} V \cong V.$$

Moreover, observe that

$$\begin{aligned} \mathrm{End}_{\mathbf{C}}(S) &\cong S^* \otimes_{\mathbf{C}} S \\ &\cong (V^* \otimes_{\mathbf{C}} \Delta^*) \otimes_{\mathbf{C}} (\Delta \otimes_{\mathbf{C}} V) \\ &\cong (\Delta^* \otimes_{\mathbf{C}} \Delta) \otimes_{\mathbf{C}} (V^* \otimes_{\mathbf{C}} V) \\ &\cong \mathrm{End}_{\mathbf{C}}(\Delta) \otimes_{\mathbf{C}} \mathrm{End}_{\mathbf{C}}(V) \\ &\cong \mathcal{C}\ell \otimes_{\mathbf{C}} \mathrm{End}_{\mathbf{C}}(V), \end{aligned}$$

and

$$\begin{aligned} \mathrm{End}_{\mathbf{C}}(V) &\cong V^* \otimes_{\mathbf{C}} V \\ &\cong \left(S^* \otimes_{\mathcal{C}\ell} \Delta\right) \otimes_{\mathbf{C}} \left(\Delta^* \otimes_{\mathcal{C}\ell} S\right) \\ &\cong S^* \otimes_{\mathcal{C}\ell} (\Delta \otimes_{\mathbf{C}} \Delta^*) \otimes_{\mathcal{C}\ell} S \\ &\cong S^* \otimes_{\mathcal{C}\ell} \mathrm{End}_{\mathbf{C}}(\Delta) \otimes_{\mathcal{C}\ell} S \\ &\cong S^* \otimes_{\mathcal{C}\ell} \mathcal{C}\ell \otimes_{\mathcal{C}\ell} S \\ &\cong S^* \otimes_{\mathcal{C}\ell} S \\ &\cong \mathrm{End}_{\mathcal{C}\ell}(S). \end{aligned}$$

Hence $\mathrm{End}_{\mathbf{C}}(S) \cong \mathcal{C}\ell \otimes_{\mathbf{C}} \mathrm{End}_{\mathcal{C}\ell}(S)$, so any $\mathbf{C}$-linear endomorphism of S may be written as $T = \sum_i \alpha_i \otimes T_i$ for $\alpha_i \in \mathcal{C}\ell$ and T_i a $\mathcal{C}\ell$-linear endomorphism of S.

Definition 4.5.14. Let $F \in \mathrm{End}_{\mathcal{C}\ell}(S)$, and define $\mathrm{Tr}^{S/\Delta}(F) = \mathrm{Tr}(T)$ to be the *relative trace* of F. Hence $F \leftrightarrow T$ under the isomorphism $\mathrm{End}_{\mathcal{C}\ell}(S) \cong \mathrm{End}_{\mathbf{C}}(V)$. It helps to think of S/Δ as V.

We now use the isomorphism $\mathrm{End}_{\mathbf{C}}(S) \cong \mathcal{C}\ell \otimes_{\mathbf{C}} \mathrm{End}_{\mathcal{C}\ell}(S)$ to make $\mathrm{End}_{\mathbf{C}}(S)$ a bundle of filtered algebras, by using the standard filtration on $\mathcal{C}\ell$ and assigning order 0 to the elements of $\mathrm{End}_{\mathcal{C}\ell}(S)$. Also note that if A, b are filtered, then $A \otimes B$ is filtered, i.e. $(\mathcal{C}\ell \otimes_{\mathbf{C}} \mathrm{End}_{\mathcal{C}\ell}(S))_m \cong (\mathcal{C}\ell)_m \otimes_{\mathbf{C}} \mathrm{End}_{\mathcal{C}\ell}(S)$.

4.6 Getzler's method

Let $\mathcal{D}(S)$ be the algebra of linear differential operators acting on $\Gamma(S)$. This is generated by Clifford multiplication, covariant derivatives, and sections of $\mathrm{End}_{\mathcal{C}\ell}(S)$.

Definition 4.6.1. The *Getzler filtration* on $\mathcal{D}(S)$ is that determined by the following assignment of orders to generators of $\mathcal{D}(S)$:

1. a $\mathcal{C}\ell$-module endomorphism $T \in \mathrm{End}_{\mathcal{C}\ell}(S)$ has order 0,

2. $c(X)$, for $X \in \Gamma(TM)$, has order 0, and

3. ∇_X, for $X \in \Gamma(TM)$, has order 1,

where ∇ is the connection on S that makes it a Clifford bundle. With this, we would like to get a symbol map $\sigma_\bullet : \mathcal{D}(S) \to G$ for G some graded algebra.

Definition 4.6.2. As before, V is a finite-dimensional vector space. Let $\mathcal{P}(V)$ be the algebra of polynomial coefficient linear differential operators acting on $C^\infty_{\mathbf{C}}(V)$. Note that

$$\mathcal{P}(V) = \mathrm{span}\left\{ x^\alpha \frac{\partial^{|\beta|}}{\partial x^\beta} \ : \ \alpha, \beta \text{ are multi-indices} \right\}.$$

We get that $\mathcal{P}(V)$ is a graded algebra, if we define $x^\alpha \frac{\partial^{|\beta|}}{\partial x^\beta}$ to have degree $|\beta| - |\alpha|$. Then $(x^\alpha \frac{\partial^{|\beta|}}{\partial x^\beta})(x^\gamma \frac{\partial^{|\delta|}}{\partial x^\delta})$ has degree $|\beta| + |\delta| - |\gamma| - |\alpha|$.

Definition 4.6.3. Let (M, g) be a manifold with a metric, with $p \in M$, and $(x^1, \ldots, x^n)$ normal coordinates centered at p. Then $\mathcal{P}(TM)$ is a bundle of algebras over M with fiber $(\mathcal{P}(TM))_p = \mathcal{P}(T_pM)$, and $\Gamma(\mathcal{P}(TM))$ is a graded algebra. We write

$$U \in (\Gamma(\mathcal{P}(TM)))^m \ \leftrightarrow \ U_p \in (\mathcal{P}(T_pM))^m = \mathrm{span}\left\{ x^\alpha \frac{\partial^{|\beta|}}{\partial x^\beta} \right\}$$

for all $p \in M$.

Recall that we want a symbol map on $\mathcal{D}(S)$ with respect to the filtration we defined. The map will be $\sigma_\bullet : \mathcal{D}(S) \to \Gamma(\mathcal{P}(TM) \otimes \bigwedge^\bullet(T^*M) \otimes \mathrm{End}_{\mathcal{C}\ell}(S))$.

Example 4.6.4. Let (M, g) be a compact, oriented, Riemann surface. Then the Riemann curvature operator $R \in \Omega^2(\mathrm{End}(TM))$. Let $Y \in \Gamma(TM)$ and consider the map $T_pM \to \bigwedge^2(T_p^*M)$, given by $V \mapsto (R_p(\ \cdot\ ,\ \cdot\ )Y_p, V)$. Explicitly, if $(x^1, \ldots, x^n)$ are normal coordinates centered at p and $V = V^i \frac{\partial}{\partial x^i}$, then

$$V \mapsto \frac{1}{2} V^k Y^\ell R_{ijk\ell} e^i \wedge e^j$$

for $e^i = dx^i|_p$. Identify T_pM with T_p^*M using the metric g. Then this map is a degree 1 polynomial function on T_pM with vectors in $\bigwedge^2(T_p^*M)$. Denote it by $\langle RY,\ \cdot\ \rangle$, the function

$$\frac{1}{2} X^k Y^\ell R_{ijk\ell} e^i \wedge e^j \in \mathcal{P}(T_pM) \otimes {\textstyle\bigwedge}^\bullet(T_p^*M) \otimes \mathrm{End}_{\mathcal{C}\ell}(S).$$

Proposition 4.6.5. There exists a unique symbol map $\sigma_\bullet : \mathcal{D}(S) \to \Gamma(\mathcal{P}(TM) \otimes \bigwedge^\bullet(T^*M) \otimes \mathrm{End}_{\mathcal{C}\ell}(S))$ that has the following effect on generators:

1. if $F \in \mathrm{End}_{\mathcal{C}\ell}(S)$, then $\sigma_0(F) = F$

2. if $X \in \Gamma(TM)$, then $\sigma_1(c(X)) = X \in \bigwedge^1(TM)$

3. if $Y \in \Gamma(TM)$, then $\sigma_1(\nabla_Y) = \partial_Y + \frac{1}{4}\langle RY,\ \cdot\ \rangle$

The proof is postponed until we can generalize this further. Also, note that

$$\sigma_1(\nabla_{e_i}) = \frac{\partial}{\partial x^i} - \frac{1}{8} X^m R_{ijm\ell} e^\ell \wedge e^j.$$

Remark 4.6.6. A smybol map is uniquely determined by its effect on generators. The conditions **1.**, **2.**, **3.** above uniquely determine a symbol map on $\bigotimes_B^* V$ for $B = \mathrm{End}_{\mathcal{C}\ell}(S)$ and $V = \Gamma(TM) \oplus \Gamma(TM)$. So we only need to show that $\sigma_\bullet(T)$ is independent of choice of representative of $T \in \mathcal{D}(S)$ using these generators (i.e. that it is compatible with the relations). We will accept it as fact for now.

Example 4.6.7. Consider the expression

$$\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} = F^{\nabla}(X,Y) = R^S(X,Y) + \underbrace{F^S(X,Y)}_{\in \text{ End}_{\mathcal{C}\ell}(S)}.$$

Both sides are in $\mathcal{D}(S)$, so we take the symbol σ of both sides. We take σ_2 since the order is $\leqslant 2$. It is enough to show this for $X = e_i$, $Y = e_j$. We would like to show that

$$\underbrace{\sigma_1(\nabla_{e_i}\nabla_{e_j}}_{(*)} - \nabla_{e_j}\nabla_{e_i}) = \sigma_2(R^s(e_i, e_j)).$$

The left side expands as

$$\begin{aligned}
(*) &= \sigma_1(\nabla_{e_i})\sigma_1(\nabla_{e_j}) - \sigma_1(\nabla_{e_j})\sigma_1(\nabla_{e_i}) \\
&= \left(\frac{\partial}{\partial x^i} - \frac{1}{8} R_{isk\ell} x^s e^k \wedge e^\ell\right)\left(\frac{\partial}{\partial x^j} - \frac{1}{8} R_{jkab} x^x e^a \wedge e^b\right) - (i \leftrightarrow j) \\
&= \frac{\partial^2}{\partial x^i \partial x^j} - R_{isk\ell} e^k \wedge e^\ell x^s \frac{\partial}{\partial x^s} - \cdots \\
&= \frac{1}{4} R_{ijab} e^a \wedge a^b.
\end{aligned}$$

Some calculations are ommitted because hey, who the fuck wants to do this shit anyways. The right side expands to the same expression, yielding the desired result and justifying the definition of $\sigma_1(\nabla_X)$.

Example 4.6.8. Let D be the Dirac operator on S, so $D = \sum_{i=1}^n c(e_i)$ in an orthonormal frame. This has *Getzler order* 2, i.e. the order of the element in the filtered algebra is 2. So

$$\begin{aligned}
\sigma_2(D) &= \sum_{i=1}^n \sigma_1(c(e_i))\sigma_1(\nabla_{e_i}) \\
&= \sum_{i=1}^n e_i \left(\frac{\partial}{\partial x^i} - \frac{1}{8}\sum_{j,k,\ell} R_{ijk\ell} x^j e^k \wedge e^\ell\right) \\
&= \sum_{i=1}^n e_i \frac{\partial}{\partial x^i} - \frac{1}{8}\sum_{i,j,k\ell} R_{ijk\ell} x^j e^i \wedge e^k \wedge e^\ell \\
&= \sum_{i=1}^n e_i \frac{\partial}{\partial x^i}
\end{aligned}$$

by the 1st Bianchi identity. Also

$$\sigma_2(D) = \sum_{i=1}^n e^i \frac{\partial}{\partial x^i} = dT_pM,$$

which is the exterior derivative on the smooth manifold T_pM.

Corollary 4.6.9.

$$\sigma_4(D^2) = \sigma_2(D) \cdot \sigma_2(D) = d_{TM} \cdot d_{TM} = 0,$$

so D^2 has actually Getzler order < 4. We will see that D^2 has Getzler order 2.

Example 4.6.10. D^2 has Getzler order 2 and

$$\sigma_2(D^2) = -\sum_{i=1}^n \underbrace{\left(\frac{\partial}{\partial x^i} - \frac{1}{4}\sum_{j=1}^n R_{ij} x^j\right)^2}_{\sigma_1(\nabla_{e_i})} + F^S,$$

where $R_{ij} \in \bigwedge^2(T_p^*M)$.

Proof: We showed that $D^2 = \nabla^*\nabla + \frac{1}{4}K + c(F^S)$. In local coordinates,

$$\begin{aligned}
\nabla^*\nabla &= \sum_{j,k} \left(-g^{jk}\nabla_j\nabla_k - \Gamma^i_{jk}\nabla_i\right), \\
\sigma_2(\nabla^*\nabla) &= -\sum_{i=1}^n \sigma_2(\nabla_i\nabla_i) \\
&= -\sum_{i=1}^n \sigma_1(\nabla_i)\sigma_1(\nabla_i) \\
&= \cdots .
\end{aligned}$$

This apparently completes the proof. ∎

Remark 4.6.11. We would now like to apply Getzler symbol calculus to the asymptotic expansion of the heat kernel. Recall that

$$h_t(p,q) \sim \frac{1}{(4\pi t)^{n/2}} \exp\left(\frac{-\mathrm{dist}(p,q)^2}{4t}\right)\left(\sum_{k=0}^\infty t^k\Theta_k(p,q)\right),$$

with $\Theta_0(p,p) = \mathrm{id}_{S_p}$. This is not in $\mathcal{D}(S)$; it is the kernel of a smoothing operator. We have $\sigma_\bullet : \mathcal{D}(S) \to \Gamma(\mathcal{P}(TM) \otimes \bigwedge^\bullet(T^*M) \otimes \mathrm{End}_{\mathcal{C}\ell}(S))$. The idea is to replace polynomials by formal power series. Note that $\mathcal{D}(S)$ acts on kernels of smoothing operators.

Definition 4.6.12. Let V be a finite-dimensional vector space. Let $\mathbf{C}[[V]]$ be the *ring of formal power series* on V. That is, if $(e_1, \ldots, e_n)$ is the basis of V, then for $v \in V$, $v = \sum_{i=1}^n x^i e_i$ an element of $\mathbf{C}[[V]]$ is a formal series

$$\sum_\alpha c_\alpha x^\alpha = \sum_\alpha c_\alpha (x^1)^{\alpha_1}\cdots(x^n)^{\alpha_n}.$$

Note that $\mathcal{P}(V)$ acts naturally on $\mathbf{C}[[V]]$:

$$\left(x^\gamma \frac{\partial^{|\delta|}}{\partial x^\delta}\right)\left(\sum_\alpha c_\alpha x^\alpha\right) = \sum_\alpha c_\alpha x^\alpha \left(\frac{\partial^{|\delta|}}{\partial x^\delta} x^\alpha\right).$$

The space $\mathbf{C}[[V]]$ is graded, where $\deg(x^\alpha) = -|\alpha|$. The gradings are compatible with the action. That is, if $p \in (\mathcal{P}(V))^m$ and $a \in (\mathbf{C}[[V]])^n$, then $pa \in (\mathbf{C}[[V]])^{m+n}$.

Remark 4.6.13. We will now define a filtration on $\Gamma(S \boxtimes S^*)$ and an induced map

$$\sigma_\bullet : \Gamma(S \boxtimes S^*) \to \Gamma\left(\mathbf{C}[[TM]] \otimes \textstyle\bigwedge^\bullet(T^*M) \otimes \mathrm{End}_{\mathcal{C}\ell}(S)\right).$$

Let $s \in \Gamma(S \boxtimes S^*)$. Fix $q \in M$, fix normal coordinates $(x^1, \ldots, x^n)$ centered at q. Let $t_1|_q, \ldots, t_r|_q$ be an orthonormal frame on S_q. Define $t_1, \ldots, t_r$ in the domain U of normal coordinates by parallel transport [..] Each t_i has $\nabla_{\frac{\partial}{\partial r}} t_i = 0$ on U and $\nabla_{X_q} t_i = 0$ for all $X_q \in T_qM$. Consider the map

$$p \mapsto S(p,q) \in \mathrm{End}(S_q, S_p).$$

We can write this map with respect to the coordinates $(x^1, \ldots, x^n)$ in this frame as

$$(x^1, \ldots, x^n) \mapsto \sum_{i=1}^r \sum_{j=1}^r S_{ij}(x^1, \ldots, x^n) t_i(x) \otimes t_j^*(0) = S_q(x^1, \ldots, x^n).$$

Expand $S_q(x^1,\dots,x^n)$ in a Taylor series at the origin. $S_q(x^1,\dots,x^n) \sim \sum_\alpha s_\alpha x^\alpha$, where s_α are sections of $S \otimes S_q^*$ that are parallel along radial geodesics emanating from q, i.e.

$$s_\alpha = \sum_{i,j=1}^{r} c_\alpha \left(\frac{\partial^{|\alpha|} S_{ij}(0)}{\partial x^\alpha} \right) t_i(x) \otimes t_j^*(0).$$

Note that since s_α is determined by it value $s_\alpha(0)$ at q. We may think of this as an element of $\mathbf{C}[[T_qM]] \otimes \mathrm{End}(S_q)$. Hence, as q varies over M, we get a soction of the bundle $\mathbf{C}[[TM]] \otimes \mathrm{End}(S)$, which is filtered. So since $\mathbf{C}[[T_qM]]$ is graded and $\mathrm{End}(S_q)$ has the canonical filtration, $\mathrm{End}(S_q) \cong \mathcal{C}\ell \otimes \mathrm{End}_{\mathcal{C}\ell}(S_q)$.

Definition 4.6.14. Define a filtration on $\Gamma(S \boxtimes S^*$ as follows: $s \in \Gamma(S \boxtimes S^*)$ has order $\leqslant m$ if its Taylor series at q has order $\leqslant m$, at each point $q \in M$. We then get a symbol map

$$\sigma_\bullet : \Gamma(S \boxtimes S^*) \to \Gamma\left(\mathbf{C}[[TM]] \otimes \textstyle\bigwedge^\bullet (T^*M) \otimes \mathrm{End}_{\mathcal{C}\ell}(S) \right),$$

where $\sigma_m(s)$ is a section of the image. We define $\sigma_m^0(s)$ to be the constant term in this power series.

Theorem 4.6.15. [MAIN THEOREM]
Let $T \in \mathcal{D}(S)$ be one of the operators described before. Let $m \in \{0,1\}$ be the Getzler order of T. Let $Q \in \Gamma(S \boxtimes S^*)$ be of Getzler order $\leqslant k$. Then $TQ \in \Gamma(S \boxtimes S^*)$ has Getzler order $\leqslant m+k$, and

$$\sigma_{m+k}(TQ) = \sigma_m(T) \cdot \sigma_k(Q), \tag{21}$$

where the left side and the second factor on the right are symbols on $\Gamma(S \boxtimes S^*)$, and the first factor on the right is a symbol on $\mathcal{D}(S)$. This works for:
· $T = F \in \Gamma(\mathrm{End}_{\mathcal{C}\ell}(S))$,
· $T = c(X)$ for $X \in \Gamma(TM)$,
· $T = \nabla_X$ for $X \in \Gamma(TM)$.

Proof: Fix $q \in M$, normal coordinates $(x^1,\dots,x^n)$ centered at q. Let $s_q(x) \sim \sum_\alpha s_\alpha x^\alpha$ be the Taylor series of $s_q(x)$ at q. We would like to verify (21) for the three Ts described above. First, if $T = F \in \mathrm{End}_{\mathcal{C}\ell}(S)$ and $\nabla F = 0$ at q, then the Taylor coefficients of Fs are Fs_α. So, when $m = 0$,

$$\sigma_k(Fs) = F\sigma_k(s) = \sigma_0(F)\sigma_k(s).$$

In general, let F_0 be the parallel transport of $F|_q$ along radial geodesics emanating from q. Then $\nabla F_0 = 0$ at q and $F - F_0$ tas vanishing constant terms in its Taylor expansion at q. Hence $\sigma_0(F - F_0) = 0$, since σ_0 picks out the constant term. Hence

$$\sigma_k(Fs) = \sigma_k(F_0 s)\sigma_k(s) = \sigma_0(F)\sigma_k(s),$$

by the above. Case 2, $T = c(X)$, is identical to the above case. For the third case, let $T = \nabla_X$, which is linear in X, so it is enough to prove it for $X = \frac{\partial}{\partial x^i}$. Let $Y = r\frac{\partial}{\partial r} = x^i \frac{\partial}{\partial x^i}$. First we assume that s is parallel along radial geodesics emanating from q (this is a shot special case). Then $\nabla_{\frac{\partial}{\partial r}} s = 0$ everywhere, so $\nabla_Y = 0$ everywhere. Then for $\nabla_X s \sim \sum_\alpha t_\alpha x^\alpha$,

$$\nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X,Y]} s = F^\nabla(X,Y)s.$$

Also note that

$$\left[\frac{\partial}{\partial x^i}, x^j \frac{\partial}{\partial x^j} \right] = \frac{\partial}{\partial x^i} = X \qquad , \qquad Y(x^\alpha) = x^j \frac{\partial}{\partial x^j}(x^\alpha) = |\alpha| x^\alpha,$$

so the equation above becomes $0 - \nabla_Y(\nabla_X s) - \nabla_X s = F^\nabla(X,Y)s$. Replace by the Taylor expansion of $\nabla_X s$ to get

$$\begin{aligned} -\nabla_Y(t_\alpha x^\alpha) - t_\alpha x^\alpha &\sim F^\nabla(X,Y)s, \\ -(|\alpha|+1)t_\alpha x^\alpha &\sim F^\nabla(X,Y)s = \sum_j F_{ij} x^j s, \end{aligned}$$

where the last term has order $\leqslant k+1$, F_{ij} having order 2, x^j having order -1, and s having order k. So the Taylor coefficients of $\nabla_X s$ are determined by the Taylor coefficients of F^∇. Next, equate powers of x and keep terms of order $\leqslant k+1$ on both sides to get

$$-\sum_{j=1}^{n} 2t_j x^j = \sum_j x^j R^S\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) s + \text{(linear order)}$$
$$\implies \quad t_j = -\frac{1}{2} R^S\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) s,$$
$$\sigma_2\left(R^S\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\right) = \sigma_2\left(\frac{1}{4} R_{ijk\ell} c(e^k) \wedge c(e^\ell)\right) = \frac{1}{4} R_{ijk\ell} e^k \wedge e^\ell = \frac{1}{2} R_{ij}.$$

Now take σ_{k+1} of both sides, so $\sigma_{k+1}(\nabla_X s) = \sigma_{k+1}(t_\alpha x^\alpha) = -\frac{1}{4} R_{ij} x^j \wedge \sigma_k(s)$. That concludes the special case. Now for the general case, where $s \sim \sum_\alpha s_\alpha x^\alpha$. Then

$$\nabla_{\frac{\partial}{\partial x^i}} \sim \sum_j (\nabla_i s_\alpha) x^\alpha + \sum_i s_\alpha \left(\frac{\partial}{\partial x^i} s^\alpha\right),$$

and

$$\sigma_{k+1}\left(\nabla_{\frac{\partial}{\partial x^i}} s\right) = -\frac{1}{4} R_{ij} x^j \wedge \sigma_{k+|\alpha|}(s_\alpha) x^\alpha + \frac{\partial}{\partial x^i}(\sigma_k(s)) = \left(\frac{\partial}{\partial x^i} - \frac{1}{4} R_{ij} x^j\right) \wedge \sigma_k(s) = \sigma_1(\nabla_X)\sigma_k(s).$$

■

Corollary 4.6.16. The Getzler symbol on $\mathcal{D}(S)$ is well-defined.

Proof: Let $T \in \mathcal{D}(S)$. Let $\widetilde{T}$ be a partial representation of T in a basis of generators $T = T_1, \ldots, T_k$ of order $\leqslant k$. Then by the main theorem, $\sigma_{\ell+k}(Ts) = \sigma_\ell(T)\sigma_k(s)$. Since this holds for all $s \in \Gamma(S \boxtimes S^*)$, $\sigma_\ell(\widetilde{T})$ is uniquely determined by T. Hence $\sigma_\bullet$ is well-defined on $\mathcal{D}(S)$. ■

Recall that we wanted $\operatorname{str}(\Theta_{n/2}(p,p))$. Let's now apply the Getzler formalism to the heat kernel h_t.

Theorem 4.6.17. The terms $\Theta_k(p,q)$ have Getzler order $\leqslant 2k$ and the heat symbol, defined as

$$W_t = W = \rho_t\left(\sigma_0(\Theta_0) + t\sigma_2(\Theta_1) + t^2\sigma_4(\Theta_2) + \cdots + t^{m/2}\sigma_n(\Theta_{n/2})\right),$$

satisfies the equation

$$\frac{\partial W}{\partial t} + \sigma_2(D^2) W = 0 \tag{22}$$

and is the unique solution of this equation of the form $\rho_t(v_0 + tv_1 + \cdots + t^{m/2} v_{m/2})$, where v_j is a symbol of Getzler order $\leqslant 2j$ and $v_0 = 1$.

Proof: Recall that in normal coordinates centered at $q \in M$, $h_t(x) \sim \rho_t(x)(v_0(x) + tv_1(x) + \cdots)$ such that

$$\nabla_{\frac{\partial}{\partial r}}\left(r^k g^{1/4} u_k\right) = -r^{k-1} g^{1/4} D^2 u_{k-1} \quad , \quad u_{-1} = 0. \tag{23}$$

These equations determine the u_k uniquely, given $v_0(q) = \mathrm{id}$. We will take the Taylor series of both sides. Our aim is to prove u_k has Getzler order $\leqslant 2k$, by induction. First, $v_0 = \mathrm{id} \in \operatorname{End}_{\mathcal{C}\ell}(S)$, so $\operatorname{order}(v_0) = 0$. Let $k \geqslant 1$ and assume that u_{k-1} has Getzler order $\leqslant 2(k-1)$. Take σ of both sides and use the fact that D^2 has Getzler order 2, to get

$$r^{k-1} g^{1/4} u_k + r^k\left(\nabla_{\frac{\partial}{\partial r}} g^{1/4}\right) u_k + r^k g^{1/4} \nabla_{\frac{\partial}{\partial r}} u_k = -r^{k-1} g^{1/4} D^2 u_{k-1}.$$

Let a be the Getzler order of u_k and b the Getzler order of $r^k g^{1/4}$. Then everything on the left has Getzler order $b+1+a$. And on the right, everything has order $b+1+2+2(k-1)$, so equating, we get that $a=2k$. So u_k has Getzler order $\leqslant 2k$. Then σ_{2k+1-k} of both sides of (23) implies that

$$\frac{\partial}{\partial r}\left(r^k \sigma_{2k}(u_k)\right) = -r^{k-1}\sigma_2(D^2)\sigma_{2k-2}(u_{k-1}).$$

But this is exactly the recurrence relation satisfied by the heat symbol W. We don't have to find the heat kernel h_t for all terms in the asymptotic expansion or even the first $n/2$ terms to get the sum of their symbols. Let a_{ij} be a skew-symmetric real $n\times n$ matrix, for $n=2m$, and $b\in\mathbf{R}$. Consider the differential equation on $\mathbf{R}^n$ given by

$$\frac{\partial W}{\partial t} - \sum_{i=1}^{n}\left(\frac{\partial}{\partial x^i} - \frac{1}{4}\sum_{j=1}^{n} a_{ij}x^j\right)^2 W + bW = 0. \tag{24}$$

We want to show that this has a solution for small t. It is an analytic function of b and the $a_{ij}s$, and is asymptotic to $\exp(-|X|/(4t))/(4\pi t)^{n/2}$ as $t\to 0$. The first step is to let $W = e^{-tb}W_0(x^1,\dots,x^n,t)$ such that

$$\frac{\partial W}{\partial t} = -bW + e^{-tb}\frac{\partial W_0}{\partial t} \qquad \text{and} \qquad \frac{\partial W_0}{\partial t} - \sum_{i=1}^{n}\left(\frac{\partial}{\partial x^i} - \frac{1}{4}a_{ij}x^j\right)^2 W_0 = 0$$

are satisfied. Since a_{ij} is skew-symmetric, there exists an orthonormal basis $e_1, f_1, \dots, e_m, f_m$ of $\mathbf{R}^{2m}$ and corresponding coordinates $u^1, v^1, \dots, u^m, v^m$ where the bilinear form becomes

$$a = \begin{pmatrix} 0 & \lambda_1 & & & & & & \\ -\lambda_1 & 0 & & & & & & \\ & & 0 & \lambda_2 & & & & \\ & & -\lambda_2 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & \lambda_k \\ & & & & & & -\lambda_k & 0 \end{pmatrix},$$

with zeroes in the empty spots. In this new coordinate system (check this),

$$\frac{\partial W}{\partial t} - \sum_{i=1}^{m}\left(\frac{\partial}{\partial x^i} - \frac{1}{4}\lambda_i v^i\right)^2 W - \sum_{i=1}^{m}\left(\frac{\partial}{\partial x^i} + \frac{1}{4}\lambda_i u^i\right)^2 W = 0.$$

Now use the separation of variables. Assume that it is a function of several variables and check it reduces to

$$\frac{\partial W_i}{\partial t} - \left(\frac{\partial}{\partial x^i} - \frac{1}{4}\lambda_i v^i\right)^2 W_i - \left(\frac{\partial}{\partial x^i} + \frac{1}{4}\lambda_i u^i\right)^2 W_i = 0,$$

so we are now reduced to solving

$$\frac{\partial W}{\partial t} - \left(\frac{\partial}{\partial u} - \frac{\lambda v}{4}\right)^2 W - \left(\frac{\partial}{\partial v} + \frac{\lambda u}{4}\right)^2 W = 0.$$

For $W = W(u,v,t)$, we leave it as an exercise to show that

$$W(u,v,t) = \frac{1}{\sqrt{4\pi t}}\left(\frac{it\lambda/2}{\sinh(it\lambda/2)}\right)^{1/2} \exp\left(\frac{-i\lambda}{8}(u^2+v^2)\coth(it\lambda/2)\right).$$

Hence

$$
\begin{aligned}
W &= e^{-tb} W_1 W_2 \cdots W_m \\
&= \frac{1}{(4\pi t)^{n/2}} \left(\det \left(\frac{ta/2}{\sinh(ta/2)} \right) \right)^{1/2} \exp\left(\frac{-1}{4t} \left\langle \frac{ta}{2} \coth(ta/2) X, X \right\rangle \right) e^{-tb} \\
&\xrightarrow{t \to 0} \exp\left(\frac{-|X|^2}{4t} \right).
\end{aligned}
$$

So this solves (24). Next, recall we saw $\sigma_2(D^2) = -\sum_{i=1}^n (\frac{\partial}{\partial x^i} - \frac{1}{4} R_{ij} x^j)^2 + F^S$, which is a differential operator on $C^\infty(T_q M)$. Hence $R_{ij} = \frac{1}{2} R_{ijk\ell} e^k \wedge e^\ell$ is a skew-symmetric matrix whose entries are 2-forms. So F^S is a 2-form with sections in $\mathrm{End}_{\mathcal{C}\ell}(S)$, so the R_{ij} and F^S terms all commute. Since 2-forms are nilpotent elements of the exterior algebra,

$$
W = \frac{1}{(4\pi t)^{n/2}} \left(\det \left(\frac{tR/2}{\sinh(tR/2)} \right) \right)^{1/2} \exp\left(\frac{-1}{4t} \left\langle \frac{tR}{2} \coth(tR/2) X, X \right\rangle \right) \exp(-tF^S)
$$

is a formal power series and solves (22). ■

So W is of the form $W = \frac{1}{(4\pi t)^{n/2}}(v_0 + t v_1 + \cdots + t^{n/2} v_{n/2})$, where v_k has Getzler order $\leqslant 2k$ and $v_0(0) = 1$ (by explicit calculation). Hence we have shown the following:

Proposition 4.6.18. With notation as above,

$$
\sum_{k=0}^{n/2} \sigma_{2k}^0(\Theta_k) = \left(\det \left(\frac{R/2}{\sinh(R/2)} \right) \right)^{1/2} e^{-F^S} \in \Gamma\left(\bigwedge\nolimits^\bullet (T^*M) \otimes \mathrm{End}_{\mathcal{C}\ell}(S) \right).
$$

Theorem 4.6.19. [Atiyah, Singer (1960s-1980s)]
With notation as above,

$$
\mathrm{ind}(D^+) = \int_M \left(\det \left(\frac{R/4\pi i}{\sinh(R/4\pi i)} \right) \right)^{1/2} \mathrm{Tr}^{S/\Delta} e^{-F^S/2\pi i}.
$$

Proof: We have shown, by McKean–Singer, that $\mathrm{ind}(D^+) = \frac{1}{(4\pi)^{n/2}} \int_M \mathrm{str}(\Theta_{n/2})$, but $\Theta_{n/2} \in \Gamma(\mathrm{End}_{\mathbf{C}}(S)) = \Gamma(\mathcal{C}\ell \otimes_{\mathbf{C}} \mathrm{End}_{\mathcal{C}\ell}(S))$, and by a previous result,

$$
\mathrm{str}(\Theta_{n/2}) = (-2i)^m \underbrace{(\Theta_{n/2})_\Gamma}_{\substack{\text{top degree} \\ \text{part of } \Theta_{n/2}}},
$$

so $\mathrm{str}(\Theta_{n/2}) = (-2i)^{n/2} \mathrm{Tr}^{S/\Delta} \sigma_n^0(\Theta_{n/2})$, so we get

$$
\mathrm{ind}(D^+) = \frac{1}{(4\pi)^{n/2}} (-2i)^{n/2} \int_M \left(\det \left(\frac{R/2}{\sinh(R/2)} \right) \right)^{1/2} \mathrm{Tr}^{S/\Delta} e^{-F^S}.
$$

Replace R and F^S by $\frac{1}{2\pi i} R$ and $\frac{1}{2\pi i} F^S$, respectively, to get the result. ■

Remark 4.6.20. If F is the curvature of a connection E, then $(\det(\frac{F/4\pi i}{\sinh(F/4\pi i)}))^{1/2}$ is the $\hat{A}$-genus of E. This is a closed mixed degree form whose cohomology class is independent of ∇. Also, $\mathrm{Tr}^{S/\Delta} e^{-F^S/2\pi i} = ch(S/\Delta)$ is called the *relative Chern character* of S. So the index theorem may be written as

$$
\mathrm{ind}(D^+) = \int_M \hat{A}(TM) ch(S/\Delta) = \left(\hat{A}(TM) \smile ch(S/\Delta) \right)[M].
$$

Finally, consider some special cases of the theorem:

Theorem 4.6.21. [Chern, Gauss, Bonnet]

$$\chi(M^{2m}) = \int_M e(TM),$$

where e is the Euler class, $S = \bigwedge^\bullet(T^*M) \otimes \mathbf{C}$, and $D = d + d^*$.

Theorem 4.6.22. [Signature theorem]
Using the same S and D as above, but with a different splitting $S = S^+ \oplus S^-$,

$$\text{sign}(M^{4k}) = \int_M \alpha(TM) \qquad , \qquad \alpha(TM) = \left(\det\left(\frac{R/2\pi i}{\tanh(R/2\pi i)}\right)\right)^{1/2},$$

where $n = 4k$.

Theorem 4.6.23. [Hirzebruch, Riemann, Roch]

$$\chi_{\mathbf{C}}(M^{2m}) = \sum_{k=0}^{m} (-1)^k \dim(H^{0,k}(M)) = \int td(T^{1,0}M)ch(TM),$$

where $\chi_{\mathbf{C}}$ is the holomorphic genus, $S = \bigwedge^\bullet(T^{0,1}M)$ and $D = \delta + \delta^*$.

Index of notation

$\mathcal{E}$	grading operator	51
$\mathrm{str}(T)$	supertrace of an operator T	52
$S_2 \boxtimes S_2$	box tensor of two bundles S_1 and S_2	53
$(h_t)_{t>0}$	heat kernel	55
$\widehat{S \boxtimes S^*}$	hat box tensor of a bundle S	55
$f(t) \sim \sum_{k=0}^{\infty} a_k(t)$	the formal series $\sum_{k=0}^{\infty} a_k(t)$ is an asymptotic expansion for f near $t = 0$	59
$\rho_t(p,q)$, $\Theta_k(p,q)$	auxiliary functions in asymptotic expansion of heat kernel	60
c	Clifford multiplication	66
R^S	Riemann endomorphism	66
F^S	twisting curvature	67
$[\,\cdot\,,\,\cdot\,]_S$	supercommutator of elements of $\mathrm{End}(S)$ for S supersymmetric	68
$\mathcal{D}(M)$	algebra of linear differential operators on $C^\infty_{\mathbf{C}}(M)$	70
$G(A)$	graded algebra of a filtered algebra A	70
$\sigma_\bullet$	symbol map	70
$\mathcal{C}(V)$	algebra of constant coeff. diff. operators acting on $C^\infty(V)$	71
$\mathrm{Tr}^{S/\Delta}(F)$	relative trace of F	72
$\mathcal{P}(V)$	algebra of poly. coeff. lin. diff. operators	73
$\mathbf{C}[[V]]$	ring of formal power series on V	75
$ch(S/\Delta)$	relative Chern character of S	**??**

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