Compact course notes

Pure Math 955, Fall 2013

Advanced Algebraic Geometry

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Contents

Note: Sections marked with a vertical line on the left side contain background information not presented in class.

1 Projective space and morphisms

1.1 Divisors

✠ *Lecture 1 - 09.09.2013*

Question: What are the transformations from one projective space into another projective space?

Begin with transformations from one-dimensional projective space into itself.

$$\mathbb{P}^1 \to \mathbb{P}^1 : \quad \text{Möbius transformation}$$
$$\text{Trivial morphism}$$

Definition 1.1.1. Let V, W be algebraic varieties. A *morphism* $V \to W$ is a function defined everywhere locally by rational functions. A morphism may also be viewed as a polynomial map.

Definition 1.1.2. An *algebraic variety* V over an n-dimensional field F is the set of points satisfying $f_i(x_1, \ldots, x_n) = 0$ for F-valued polynomials $f_i, i \in I$.

Definition 1.1.3. A *Möbius transformation* is a morphism $\mathbb{P}^1 \to \mathbb{P}^1$ of degree 1, that is, given by an invertible 2×2 matrix. It may be presented as:

$$\frac{az+b}{cz+d}, \; ad - bc \neq 0 \qquad \text{or} \qquad [az+b : cz+d], \; ad - bc \neq 0$$

Example 1.1.4. This is a morphism $\mathbb{P}^1 \to \mathbb{P}^2$ that parametrizes a circle:

$$[x : y] \mapsto [x^2 : xy : y^2] \in V(Y^2 = XZ)$$

Combining this morphism with a projection from $[0 : 1 : 0]$ gives a new morphism $\mathbb{P}^1 \to \mathbb{P}^1 : [x : y] \mapsto [x^2 : y^2]$ that still parametrizes a circle.

What are some other morphisms form $\mathbb{P}^1$ to itself? From the above example we have the following:

$$\begin{array}{rcl} [x : y] & \mapsto & [x^n : y^n] \\ [x : y] & \mapsto & [\varphi_0 : \varphi_1] \end{array}$$

Here $\deg(\varphi_0) = \deg(\varphi_1)$ and both are homogeneous polynomials and have no common factors. Generalizing to morphisms $\mathbb{P}^n \to \mathbb{P}^m$, we simply need an $(m+1)$-tuple of homogeneous polynomials of the same degree d that do not have any common zeros in $\mathbb{P}^n$:

$$[x_1 : \cdots : x_n] \mapsto [\varphi_0 : \cdots : \varphi_m]$$

Definition 1.1.5. A *homogeneous polynomial* is a polynomial with every term having the same degree combining degrees from different variables. For example, $x^6 + 2x^3y^3 - 5xy^5 + y^6$ is a homogeneous polynomial of degree 6.

Proposition 1.1.6. The set of homogeneous polynomials of degree d in $(n+1)$ variables in finite dimensions has size $\binom{d+n}{n}$.

Proof: The space of homogeneous polynomials in $d+1$ variables has a basis given by monomials. To each degree, from 1 to d, assign an object of type a. Amongst the objects of type a, place n objects of type b, that each represent a break between groups of objects a. Arranging all the objects in line, we have an analogy - by assigning to the first variable the degree that corresponds to the number of objects a in the first group (before the first object b), and to the kth variable the degree that corresponds to the number of objects a in the kth group (between the $(k-1)$th and kth objects b), we have created a way of designating all possible homogeneous monomials of degree d in $n+1$ variables. Hence the desired amount is equivalent to the number of ways of choosing n items from a list of $d+n$ items, or $\binom{d+n}{n}$. ■

Definition 1.1.7. Let $m_0, \dots, m_N$ be the set of monomials of degree d in $x_0, \dots, x_n$. The corresponding morphism $vZ : [x_0 : \cdots : x_n] \mapsto [m_0 : \cdots : m_N]$ is termed the *dth Veronese embedding* or d-uple embedding. Note that $N = \binom{n+d}{n} - 2$, so the projective space of the range has size $\binom{n+d}{n}$.

Remark 1.1.8. In general, given some moprhism $\varphi : \mathbb{P}^n \to \mathbb{P}^m$ with $\varphi : [x_0 : \cdots : x_n] \mapsto [\varphi_0 : \cdots : \varphi_m]$ (where $\deg(\varphi_i) = d$), the map φ may be presented as $\varphi = L \circ v$, where L is the linear map induced by the coefficients of φ_i.

Moreover, a *linear map* is a composition of a projection away form a linear subspace and a linear embedding of a projective space in another space.

✠ *Lecture 2 - 11.09.2013*

Definition 1.1.9. A *local ring* is a ring with a single maximal ideal.

Definition 1.1.10. A *discrete valuation ring* (or *DVR*) is a principal ideal domain (i.e. an integral domain such that for every ideal J that it contains, it also contains some a such that $J = \langle a \rangle$) with only one non-zero maximal ideal.

Definition 1.1.11. Given a DVR, a *uniformizer* of the DVR is any irreducible element in it. A uniformizer will be a generator of the only maximal ideal.

Definition 1.1.12. A *degree d morphism* from $\mathbb{P}^n$ into any projective space can be obtained by choosing a subset of the space spanned by monomials of degree d in $n+1$ variables.

Example 1.1.13. Let X be the zero set of $y^2z = x^3 - xz^2$ in $\mathbb{P}^2$. Consider a map $\varphi : X \to \mathbb{P}^1$ given by $\varphi(x : y : z) = [x : z]$. The map φ is not defined at $(0 : 1 : 0)$, even though X is. So we need another map that agrees with φ everywhere else, but is defined at $(0 : 1 : 0)$. Note that

$$y^2z = x^3 - xz^2 = x(x^2 - z^2) \implies \frac{y^2}{x^2 - z^2} = \frac{x}{z}$$

Hence $\varphi'(x : y : z) = [y^2 : x - z^2]$ is a map that agrees with φ everywhere, and is defined at $(0 : 1 : 0)$. This shows that sometimes a single polynomial may not describe a curve completely, whereas more than one will.

Definition 1.1.14. An *algebraic vairety*, or simply *variety*, is the solution set of a system of real- or complex-valued polynomial equations.

Definition 1.1.15. Suppose that $\varphi : X \to \mathbb{P}^m$ is a morphism, given by $\varphi = [\varphi_0 : \cdots : \varphi_m]$ for rational functions φ_i. The *linear system* associated to φ is defined as follows:

- Choose any irreducible subvariety $Y \subset X$ of codimension 1 (i.e. $\dim(Y) = \dim(X) - 1$)
- Choose a representation $[\varphi_0 : \cdots : \varphi_m]$ of rational functions for φ that is defined on a nonempty Zariski open subset of Y
- For any linear form $L(x_0, \dots, x_m)$, define

$$n_{y,L} = \mathrm{ord}_y L(\varphi_0, \dots, \varphi_m)$$
$$D_L = \sum_{y \in Y} n_{y,L}\, y \subseteq \mathrm{Div}(X), \qquad \mathrm{Div}(X) = \bigoplus_{\mathrm{codim}(Y)=1} \mathbb{Z}Y$$

Then the linear system associated to φ is $\{D_L\}_L$.

Definition 1.1.16. Given a space Y, a *divisor* of Y is a formal sum $\sum_{y \in Y} n_y y$ such that $n_y \in \mathbb{Z}$ and all but finitely many of the n_y are zero. Moreover, $\operatorname{codim}(y) = 1$ in Y. The group of divisors is denoted $\operatorname{Div}(Y)$. A divisor associated to a linear form L is denoted by D_L.

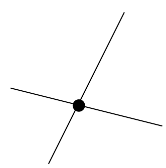
Definition 1.1.17. Note that divisors of rational functions form a subgroup of $\operatorname{Div}(X)$. Hence we may define the quotient group $\operatorname{Pic}(X) = \operatorname{Div}(X)/(\text{rat. func. divs})$ of divisor classes.

Remark 1.1.18. Note several things about divisors. First:

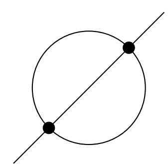
$$D_{L_1} - D_{L_2} = \operatorname{Div}\left(\frac{L_1}{L_2}\right) = \sum_{y \in Y} \operatorname{ord}_y \left(\frac{L_1}{L_2}\right) y$$

Next, if D_1 and D_2 are divisors of the same morphism, then $D_1 - D_2$ is a divisor of a rational function. This relationship is written $D_1 \sim D_2$, and said that "D_1 is linearly equivalent to D_2".

Example 1.1.19. Consider the map $\mathbb{P}^1 \to \mathbb{P}^2$ that takes everything to the z-axis. Then the values D_L associated to this map will be linear combinations of points on $\mathbb{P}^1$. Similarly, a map that takes everything to the circle will have a linear system composed of pairs of points on the circle.



linear system = {points in $\mathbb{P}^1$}



linear system = $\{P + Q : P, Q \in \mathbb{P}^1\}$

✠ *Lecture 3 - 13.09.2013*

Recall the previous lecture. If we have a morphism $\varphi : X \to \mathbb{P}^n$, then the linear system associated to φ is a collection of inverse images of linear slices of $\varphi(X)$. Moreover, it is a collection of divisors. If $X = \mathbb{P}^n$, then the linear systems are always subsets of "zero sets of polynomials of degree d" for some d (they are also all effective in this case).

Example 1.1.20. The linear system associated to $\varphi(x : y) = [x^2 : y^2]$ does not contain all of the subsets of degree 2, only some. The correct way to understand this morphism is that it is a small modification of the Veronese embedding of degree 2.

The question then arises, are there vector spaces that will work like this for arbitrary X?

Definition 1.1.21. A divisor $\sum n_Y Y$ is *effective* iff $n_Y \geqslant 0$ for all Y.

Definition 1.1.22. Let D be a divisor on a smooth variety X. Then another way to describe the *linear system* of D is as

$$L(D) = \{f \in k(X)^* \ : \ \operatorname{Div}(f) + D \text{ is effective}\} \cup \{0\}.$$

The set $k(X)^*$ contains all the non-zero rational functions on X.

Example 1.1.23. Let $X = \mathbb{P}^1$ and $D = P$, a single point $[0 : 1] \in \mathbb{P}^1$. Then the linear system is

$$\begin{aligned} L(D) &= \{f \in k(X)^* \ : \ f \text{ has at most one pole, necessarily at } P\text{, of order } \leqslant 1\} \\ &= \left\{ \frac{at + b}{t} \ : \ a, b \in k \right\} \end{aligned}$$

Note that if D is a divisor in a linear system attached to a morphism φ, then that linear system is contained in $\{D + \mathrm{Div}(f) \;:\; f \in L(D) - \{0\}\}$.

Example 1.1.24. Find a morphism from $X = V(y^2z - x^3 - xz^2)$ to projective space whose linear systom contaions $2[0:0:1]$.

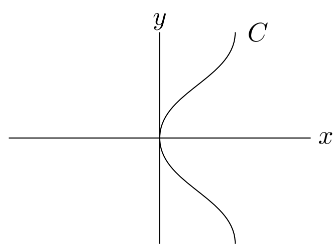
First note that this will be a map to $\mathbb{P}^1$. We are now looking for any rational function whose divisor plus $2[0:0:1]$ is effective.

✠ *Lecture 4 - 16.09.2013*

We continue with the problem from the previous class. Let the curve C be defined by $y^2z = x^3 + xz^2$, and consider the map

$$\begin{array}{rccc} \varphi : & C & \to & \mathbb{P}^1 \\ & [x:y:z] & \mapsto & [\frac{x^2}{y^2} : \frac{z^2}{y^2}] \end{array}.$$

However, this map doesn't work at $[0:1:0]$, so we use $\varphi_1 : [x:y:z] \to [y^4 : (x^2+z^2)^2]$. Now let $x = 0$, and away from $[0:1:0]$, $x = 0$ corresponds to $x^2 = 0$. This means that the affine piece at $z = 1$ looks like



So the divisor associated to $x = 0$ is $4[0:0:1]$. And so $\{[x:z] \text{ or } [y^2 : x^2+z^2]\}$ gives $2[0:0:1]$ as desired. It is natural now to ask why x vanishes twice at the origin. There is a theorem that answers this:

Theorem 1.1.25. Let C be a curve in $\mathbb{P}^n$ that is smooth at p, and L any linear form in $x_0, \ldots, x_n$ such that $L(p) = 0$. If the zero set of L does not contain the tangent line to C at p, then for any linear form L' that does not vanish at p, L/L' restricted to C is a uniformizer at p.

So let us find a uniformizer for our question. Note that anything that does not contain the tangent line will be a uniformizer. We see that y/z is a uniformizer at $p = [0:0:1]$, and so

$$\frac{x}{z} = \left(\frac{z^2}{x^2+z^2}\right)\left(\frac{y}{z}\right)^2$$

order of vanishing

non-vanishing at p

uniformizer at p

unit in the local ring at p

We note that

$$\mathrm{div}\left(\frac{x}{z}\right) = 2[0:0:1] + [0:1:0] - 3[0:1:0] = 2[0:0:1] - 2[0:1:0]$$
$$\mathrm{div}\left(\frac{x}{z}\right) + 2[0:0:1] = 2[0:1:0]$$

which is effective.

Remark 1.1.26. A curve of degree 2 is always isomorphic to $\mathbb{P}^1$. Simply pick a point on the curve and project away from it.

Now consider $L(D)$, which contains $\{1, \frac{z}{x}\}$. This is a basis of $L(D)$. So say $f_1 = 1, f_2 = z/x, f_3 \in L(D)$. Write $\varphi = [1 : \frac{z}{x} : f_3]$.

What is the degree of the image of φ? We claim that it is either 1 or 2. If it were bigger, Bezout's theorem would say that it has more points. If φ is injective, then 2 points. If not injective, then less, so 1 point. In this case, we note that $\deg(\varphi) \neq 2$, as φ is not injective, because degree 3 curves are isomorphic to $\mathbb{P}^1$, and $C \ncong \mathbb{P}^1$. Finally, if $\deg(\varphi) = 1$,then $\text{Im}(\varphi)$ is contained in a line, so $f_3 \in \text{span}\{1, \frac{z}{x}\}$.

Theorem 1.1.27. [Bezout]
Let k be a feld with $P, Q \in k[x, y]$ non-zero with no common factors. Then the algebraic curves

$$\{(x, y) \ : \ P(x, y) = 0\} \qquad \text{and} \qquad \{(x, y) \ : \ Q(x, y) = 0\}$$

have no common components, and intersect in at most $\deg(P)\deg(Q)$ points.

Remark 1.1.28. What is the diference between the "degree of a map" and the "degree of a curve"?

· The degree of a dominant rational map $F : X \to Y$ is the degree of the extension of the function field:

$$\deg(F) = [k(X) : F^*k(Y)]$$

· The degree of a curve $C \subset \mathbb{P}^n$ is the maximum number, with appropriate multiplicity, of the degrees of the points of the curve that intersect any one line.

✠ *Lecture 5 - 18.09.2013*

Recall our curve $C \subset \mathbb{P}^2$, defined by $y^2z = x^3 + xz^2$, and that we were looking for a morphism $\varphi : C \to \mathbb{P}^m$ for some m such that the divisor $2[0 : 0 : 1]$ would be associated to φ. We begin with the map

$$\varphi(x : y : z) = \begin{cases} [x : z] & \text{if } [x : y : z] \neq [0 : 1 : 0] \\ [y^2 : x^2 + z^2] & \text{else} \end{cases}$$

On the destination space, take a linear form. For $m = 1$, take a linear form in $[X : Y]$. Taking $x = 0$ corresponds to the first coordinate of the representation being zero. And $x = 0$ corresponds to

$$\begin{aligned} x = 0 \quad &\text{if} \quad [x : y : z] \neq [0 : 1 : 0] \\ y^2 = 0 \quad &\text{if} \quad [x : y : z] = [0 : 1 : 0] \end{aligned}$$

Now we have to deduce the order of vanishing of $x = 0$. Well, it vanishes at $[0 : 0 : 1] = p$, so we should find the multiplicity of $x = 0$ at $[0 : 0 : 1]$, so consider

$$\frac{x}{z} = \frac{y^2}{x^2 + z^2} = \frac{y^2}{x^2 + z^2}\left(\frac{z^2}{z^2}\right) = \frac{z^2}{x^2 + z^2} \cdot \frac{y^2}{z^2}$$

To get the second equality we multiplied by the expression because it was the simplest that would change the main expression in any meaningful way. Further, the right-most factor in the last expression is a uniformzier, and the left-most term in that same expression is a unit because it doesn't vanish at p. Hence the divisor corresponding to $x = 0$ is $2[0 : 0 : 1]$.

The linear system corresponding to φ is the collection of these divisors corresponding to the collection of linear forms in $\{x, y\}$, or

$$L(D) = \{f \in k(C) \ : \ D + \text{div}(f) \text{ is effective}\}.$$

Given an $L(D)$, one may construct a rational map to projective space $\mathbb{P}^m$ by choosing $m + 1$ elements $f_0 : f_1 : \cdots : f_m \in L(D)$ and writing $\varphi(p) = [f_0(p), f_1(p), \ldots, f_m(p)]$. However, deciding whether or not $\varphi : X \to \mathbb{P}^m$ is a morphism is hard in general.

If φ is a morphism, then if $\text{span}\{f_i\} = L(D)$, then $\{D + \text{div}(f) \ : \ f \in L(D) - \{0\}\}$ is the linear system associated to φ.

Example 1.1.29. Consider $[1 : \frac{p_1}{q_1} : \cdots : \frac{p_m}{q_m}]$. As $D+\mathrm{div}(\frac{p_i}{q_i})$ is effective for all i, the zeros of q_i are cancelled by the zeros of $D = \sum n_Y Y$. Clearing the denominators gives a simpler expression.

We would like a single "thing" that will correspond to $L(D)$ and any of its associated morphisms.

✠ *Lecture 6 - 20.09.2013*

Recall from the last lecture that for a map $K \to \mathbb{P}^m$, we associated a linear system

$$L(D) = \{f \in k(X)^* \ : \ D + \mathrm{div}(f) \geqslant 0\} \cup \{0\}$$

Now let $U \subset X$ be a non-empty open subset. Define

$$L(D)(U) = \{f \subset k(X)^* \ : \ (D + \mathrm{div}(f))|_U \geqslant 0\} \cup \{0\}$$

Note that $D \geqslant 0$ for divisors iff $D = \sum n_Y Y$ for $n_Y \geqslant 0$, and $D|_U = \sum_{Y \cap U \neq \emptyset} n_Y Y$, and $L(D)(\emptyset) = 0$. Notice that there are "restriction maps" $\mathrm{res}_{U \to V} : L(D)(U) \to L(D)(V)$ for any $V \subset U$ given by $\mathrm{res}_{U \to V} f = f|_V$. This leads us into the next topic.

1.2 Sheaves and presheaves

Definition 1.2.1. Let X be a topological space. A *sheaf* $\mathcal{F}$ of things on X is a thing $\mathcal{F}(U)$ for every open subset $U \subset X$ together with morphisms $\mathrm{res}_{U \to V} : \mathcal{F}(U) \to \mathcal{F}(V)$ for every $V \subset U$, satisfying:

1. $\mathcal{F}(\emptyset) = 0$
2. $\mathrm{res}_{U \to U} = \mathrm{id}$
3. $\mathrm{res}_{U \to V} \circ \mathrm{res}_{W \to U} = \mathrm{res}_{W \to V}$
4. if $U = \bigcup_i U_i$ and $\mathrm{res}_{U \to U_i}(f) = 0$ for all i, then $f = 0$
5. if $U = \bigcup_i U_i$ and the $f_i \in \mathcal{F}(U_i)$ satisfy $\mathrm{res}_{U_i \to U_i \wedge U_j}(f_i) = \mathrm{res}_{U_j \to U_i \wedge U_j}(f_j)$ for all i, j, then there exists $f \in \mathcal{F}(U)$ satisfying $f_i = \mathrm{res}_{U \to U_i}$ for all i.

The elements of $\mathcal{F}(U)$ are termed *sections*. If the thing $\mathcal{F}(U)$ satisfies the first three axioms, then it is termed a *presheaf*.

Example 1.2.2.
· The object $L(D)$ as we have defined it above is a sheaf
· The set of functions on a topological space that satisfy any local property is a sheaf

> **Definition 1.2.3.** Let $\mathcal{F} \subset k[x_1, \ldots, x_n]$ and $V(\mathcal{F})$ denote the set of common zeros of the elements of $\mathcal{F}$ in affine n-space. Then a subset X of affine n-space that has the form $V(\mathcal{F})$ for some $\mathcal{F}$ is termed *Zariski closed* in the space. Such sets define a topology on the affine n-space, termed the *Zariski topology.*
>
> Put another way, a closed set in the Zariski topology, for $\mathbb{A}^n$ affine n-space, is
>
> $$\{x \in \mathbb{A}^n \ : \ f(x) = 0 \ \forall \ f \in S\}$$
>
> for S any set of polynomials in n variables over the base field k.

Definition 1.2.4. Let $X \ni p$ be a topological space. Then the *skyscraper sheaf* at p is defined by

$$\mathcal{F}(U) = \begin{cases} \mathbb{C} & \text{if } p \in U \\ 0 & \text{if } p \notin U \end{cases}$$

The maps $\mathrm{res}_{U \to V}$ are 0 or id as appropriate.

Example 1.2.5. Let $X = \mathbb{P}^1$ with homogeneous coordinates $[x : y]$ and $U \neq \emptyset$. Let $\mathcal{F}(U) = \{f_U(ax+by) : a, b \in k, f_U \in \mathcal{O}(U)\}$, where $\mathcal{O}(U)$ is the ring of regular (defined everywhere) functions on U. The restriction maps are defined by

$$\text{res}_{U\to V}[f_u(ax+by)] = (f_u|_V)(ax+by)$$

Then $\mathcal{O}(1)$ is termed *Serre's twisting sheaf*, which is a sheaf of $\mathcal{O}$-modules.

Example 1.2.6. The objects $\mathcal{O}(U) = \{\text{regular functions on } U\}$ is a sheaf of rings on X. For $W \subset V \subset U$, we have that $\mathcal{O}(V)$ is bigger than $\mathcal{O}(U)$, and $\mathcal{O}(W)$ is bigger than $\mathcal{O}(V)$. Further, the object $\mathcal{O}_p(X)$ is the set of functions that are regular at p, the local ring at p. It also may be viewed as the union of all $\mathcal{O}(U)$ over all U containing p.

In general, we have a poset of open sets containing $p \in X$, by

$$\begin{array}{ccccc} & \supset & V & \supset & \\ U & & & & V \cap W \\ & \supset & W & \supset & \end{array}$$

with each corresponding poset using $\text{res}_{U\to V}$. We would like for $\mathcal{F}_p = \bigcup_{U \ni p} \mathcal{F}(U)$, but that is impossible, so we settle for

$$\mathcal{F}_p = \varinjlim_{U \ni p} [\mathcal{F}(U)]$$

Here we have used the direct limit to define $\mathcal{F}_p$, the *stalk* of $\mathcal{F}$ at p.

✠ *Lecture 7 - 23.09.2013*

Definition 1.2.7. Let $\{M_i\}$ be a directed system (a collection of groups with maps between them) of Abelian groups. That is, there are maps $M_{ij} : M_i \to M_j$ for some i, j such that for all i, j there exists some k such that m_{ik} and m_{jk} both exist.

For example, we may have $M_i = \mathcal{F}(U_i)$ for some sheaf $\mathcal{F}$, with $M = \bigoplus M_i$. Let R be the submodule of M generated by all elements of the form

$$(\dots, 0, \underset{M_i \text{ factor}}{x}, 0, \dots, 0, \underset{M_j \text{ factor}}{-m_{ij}(x)}, 0, \dots) \quad .$$

Then define the *direct limit* of M to be $\varinjlim [M_i] = M/R$. This essentially tries to mimic a union of things.

Example 1.2.8. With the structure above, consider R defined as

$$R = \text{span}\Big\{(\dots, 0, \underset{\mathcal{F}(U)}{f_U}, 0, \dots, 0, \underset{\mathcal{F}(V)}{-\text{res}_{U\to V}(f_U)}, 0, \dots)\Big\} \quad .$$

Then we may express $\mathcal{F}_p$ as

$$\mathcal{F}_P = \varinjlim_{U \ni P} [\mathcal{F}(U)].$$

Definition 1.2.9. Lot $\mathcal{F}, \mathcal{G}$ be sheaves of abelian groups on a topological space X. A *morphism of sheaves* $\psi : \mathcal{F} \to \mathcal{G}$ is a collection of homomorphisms

$$\Big\{f_U \;:\; \mathcal{F}(U) \to \mathcal{G}(U) \;:\; \text{res}^{\mathcal{G}}_{U\to V} \circ f_U = f_V \circ \text{res}^{\mathcal{F}}_{U\to V},\Big\}$$

or equivalently, maps f_U for all $U \subset X$ and $V \subset U$ such that the diagram below commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f_U} & \mathcal{G}(U) \\ \downarrow{\scriptstyle \operatorname{res}^{\mathcal{F}}_{U \to V}} & & \downarrow{\scriptstyle \operatorname{res}^{\mathcal{G}}_{U \to V}} \\ \mathcal{F}(V) & \xrightarrow[f_V]{} & \mathcal{G}(V) \end{array}$$

Note that once we have a morphism of sheaves, we may turn it into a morphism of stalks.

A natural question to ask next is what is $f_P : \mathcal{F}_P \to \mathcal{G}_P$? So let $s \in \mathcal{F}_P$. Then $f_P(s) = \overline{f_U(\bar{s})}$, where $\bar{s} \in \mathcal{F}(U)$ represents the corresponding equivalence class in $\mathcal{G}_P$. This observation leads us to the next theorem.

Theorem 1.2.10. Let $\psi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of an abelian group. Then ψ is an isomorphism if and only if ψ_P is an isomorphism for all P.

Proof: The forward direction is immediate. The reverse direction requires some work. We begin b noting that we wish to prove that $\psi|_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism.

For injectivity, we consider $f \in \ker(\psi|_U)$, for which we would like to show $f = 0$. Then $\psi(f) = 0$ in $\mathcal{G}_P$ for all $P \in U$. So $f_P = 0$ for all $P \in U$, as ψ_P is an isomorphism (that is, f_P is the equivalence class of f in $|mF_P$). So for all P, there exists a V_P such that $f|_{V_P} = 0$. By the zero axiom, $f = 0$ on U.

For surjectivity, choose $g \in \mathcal{G}(U)$, for which we would like to find $f \in \mathcal{F}(U)$ such that $\psi|_U(f) = g$. For all P, we can find $f_P \in \mathcal{F}_P$ such that $\psi_P(f_P) = g_P$ for all P. For all P, choose V_P such that f_P is represented by f_{V_P} in $\mathcal{F}(V_P)$. Note that $\psi_{V_P}(f_{V_P}) = g_P$, so $\psi_{V_P}(f_{V_P})$ and $\psi_{V_P^*}(f_{V_P^*})$ agree on some small neighborhood of P. So we may glue the f_{V_P} together to make f as desired. ■

✠ *Lecture 8 - 25.09.2013*

We now explore the standard way to take a presheaf and make it into a sheaf, a process called *sheafification*.

Theorem 1.2.11. Let $\mathcal{F}$ be a presheaf. There is a sheaf $\mathcal{F}^+$ and a morphism $\theta : \mathcal{F} \to \mathcal{F}^+$ such that every morphism $\mathcal{F} \to \mathcal{G}$ for $\mathcal{G}$ a sheaf factors through θ. That is, there exists some unique $\tilde{f}$ making the diagram below commute.

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{f} & \mathcal{G} \\ & {\scriptstyle\theta}\searrow \quad \nearrow{\scriptstyle\tilde{f}} & \\ & \mathcal{F}^+ & \end{array}$$

Proof: This theorem follows by considering the action on the stalks. Note that

$$\mathcal{F}^+(U) = \left\{ f : U \to \bigsqcup_{P \in V} \mathcal{F}_P \ : \ \begin{array}{l} f(P) \in \mathcal{F}_P \text{ and for all } P, \text{ there is some open } V \ni P \text{ and} \\ t \in \mathcal{F}(V) \text{ such that } f = t \text{ as functions from } V \text{ to } \bigsqcup_{R \in V} \mathcal{F}_R \end{array} \right\}.$$

Then θ is defined as

$$\begin{array}{rccc} \theta : & \mathcal{F}(U) & \to & \mathcal{F}^+(U) \\ & f & \mapsto & \{P \mapsto f_P \in \mathcal{F}_P\} \end{array}.$$

It is now trivial to see that $\mathcal{F}^+$ is a sheaf. Now say that $f : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves. Define $\tilde{f} : \mathcal{F}^+(U) \to \mathcal{G}(U)$ by noting that there is some open cover $\{U_i\}$ of U such that $t|_{U_i} \in \mathcal{F}(U_i)$ for all i, so $f(t|_{U_i})$ is well-defined. And

$$f(t|_{U_i})|_{U_i \cap U_j} = f(t|_{U_i \cap U_j}) = f(t|_{U_j})|_{U_i \cap U_j},$$

so we can glue $\{f(t|_{U_i})\}$ to $\tilde{f}(t)$. This is well-defined and satisfies $f = \tilde{f} \circ \theta$, as desired. ■

Theorem 1.2.12. For sheaves $\mathcal{F}, \mathcal{G}$ and $f : \mathcal{F} \to \mathcal{G}$ a morphism that is an isomorphism on stalks, the map f is an isomorphism of sheaves.

✠ *Lecture 9 - 27.09.2013*

What is the corresponding sheaf on an atlas of a manifold? This sheaf contains continuous differentiable maps from the manifold to the disk in $\mathbb{R}^n$ that are everywhere locally a homeomorphism.

For example, if the manifold is $X = \mathbb{S}^2$, then $\mathcal{F}(X)$ is empty. However, if $X = \mathbb{S}^2 \setminus \{\text{pt}\}$, then $\mathcal{F}(X) = X$.

In general, if X is not homeomorphic to a disk, then we can always find in $\mathcal{F}(X)$ the covering map. In fact, this will note be a diffeomorphism from X to the disk.

Definition 1.2.13. Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Define the *kernel presheaf* $\mathcal{K}$ of f by $\mathcal{K}(U) = \ker(f|_U) \subset \mathcal{F}(U)$. Note that $\mathcal{K}$ is a sheaf.

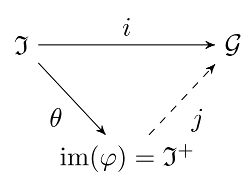
Define the *image presheaf* $\mathfrak{I}$ by $\mathfrak{I}(U) = \operatorname{Im}(f|_U) \subset \mathcal{G}(U)$. This is not a sheaf, however, as the gluing axiom fails.

Example 1.2.14. Consider the following spaces and maps:

$$\begin{array}{rcl} X & = & \mathbb{R}^2 \setminus \{0\} \\ \mathcal{F} & = & \{f : X \to \mathbb{R}\} \subset C^\infty(X) \\ \mathcal{G} & = & \{(f,g) \ : \ f_y = g_x\} \end{array} \qquad \begin{array}{rrcl} \varphi : & \mathcal{F} & \to & \mathcal{G} \\ & f & \mapsto & (f_x, f_y) \end{array}$$

Since X is not simply connected, φ is not globally surjective. For example, $(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$ is not globally the image of anything in $\mathcal{F}$, although it is in $\mathcal{G}(X)$ and everywhere locally in $\operatorname{im}(\varphi)$. Hence the image presheaf is not a sheaf.

So we define the image sheaf to be the sheafification of the image presheaf:



The map j is injective because i is injective and θ is an isomorphism of stalks. So $\mathfrak{I}^+$ may be naturally considered as a subsheaf of $\mathcal{G}$.

Remark 1.2.15. Given a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$, we say that φ is surjective if and only if $\operatorname{im}(vp) = \mathcal{G}$. Note this is not equivalent to saying "$\varphi|_U$ is onto for all U". It is equivalent to "$\varphi|_U$ is surjective for small enough U". It is also equivalent to "φ_P is surjective for all $P \in X$".

Definition 1.2.16. Let $\mathcal{F}$ be a sheaf of abelian groups, and $\mathcal{G} \subset \mathcal{F}$ a subsheaf. Define $(\mathcal{F}/\mathcal{G})(U) = \mathcal{F}(U) = \mathcal{G}(U)$. This is a presheaf but is not a sheaf, as it may be represented in terms of the image presheaf. So we define the *quotient sheaf* to be the sheafification of $\mathcal{F}/\mathcal{G}$.

1.3 Modules

✠ *Lecture 10 - 30.09.2013*

A homogeneous coordinate ring is like an affine variety ring, but less useful. We may define it as follows:

Definition 1.3.1. Let $X = V(f_1, \dots, f_n) \subset \mathbb{P}^m$. For $f_1, \dots, f_n$ homogeneous, define the *homogeneous coordinate ring* of $X \hookrightarrow \mathbb{P}^m$ by

$$S = k[x_0, \dots, x_m] \Big/ (f_1, \dots, f_n)$$

The ring does not contain functions on X, but the elements do have well-defined zero sets on X. So S is a graded ring, with $S = \bigoplus_{-\infty}^{\infty} S_i$, where the S_i are subgroups of the additive groups of S with $S_i \cap S_j = \{0\}$ if $i \neq j$, and $S_i S_j \subset S_{i+j}$. So in this construction, $S_i = \{0\}$ for $i < 0$.

> The idea of a module over a ring generalizes the idea of a vector space over a field.
>
> **Definition 1.3.2.** Let R be a ring. A *left R-module* consists of an abelian group $(M, +)$ and an operation $\cdot : R \times M \to M$ with $\cdot$-identity $1_R \in R$ such that for all $r, s \in R$ and $x, y \in M$,
>
> **1.** $r(x + y) = rx + ry$
> **2.** $(r + s)x = rx + sx$
> **3.** $(rs)x = r(sx)$
> **4.** $1_R x = x$
>
> A *right R-module* is defined the same way, except the associated operation is $M \times R \to R$, and the axioms are expressed accordingly.

Definition 1.3.3. A *graded S-module* is an S-module M with $M = \cdots \oplus M_{-1} \oplus M_0 \oplus M_1 \oplus \cdots$ for additive subgroups M_i such that $M_i \cap M_j = \{0\}$ for $i \neq j$, and $M_i M_j \subset M_{i+j}$.

Example 1.3.4. Let $S[1] = M$ be given by $M = S$, but $M_i = S_{i+1}$. $S[n]$ is defined similarly for any $n \in \mathbb{Z}$, and is called the *Serre twist*, or just *twist*.

Example 1.3.5. Another example of a graded S-module is the free graded S-module M generated by ✹ , symbolizing absolutely anything. Then:

$$\begin{aligned} S &= k[x_0, \dots, x_m] \\ M_0 &= \{0\} \\ M_1 &= (✹)k \\ M_2 &= k\text{-span of } x_0(✹), \dots, x_m(✹) \\ &= ✹(\text{linear polynomials}) \end{aligned}$$

How would a sheaf be made of this? The sheaf will be termed $\tilde{M}$, and will be defined by

$$\tilde{M}(U) = \left\{ \frac{m}{f} \ : \ \deg(m) = \deg(f), f \in S, m \in M, f(p) \neq 0 \ \forall \ p \in U \right\}$$

The restriction maps in this sheaf are inclusion maps. Note that $\tilde{M}$ is actually a presheaf, but sheafification will give a sheaf.

Example 1.3.6.
· If $M = S$, then $\tilde{M} = \mathcal{O}_X$.
· If $M = S[1]$, then $\tilde{M} = \mathcal{O}(1)$, with

$$\begin{aligned} \tilde{M}_p &= \text{ free rank-1 module over the local ring } \mathcal{O}_p \\ &= \left\{ \frac{f}{g} \ : \ f, g \text{ homogeneous} \right\} \end{aligned}$$

In fact, there is always an open cover $\{U_0\}$ of X such that for all i, $\tilde{M}(U_i)$ is a free rank-1 $\mathcal{O}(U)$-module. Then $\tilde{M}$ is called locally free of rank 1, or an invertible sheaf. In general, an *invertible sheaf* is exactly a free module over the local ring of rank 1. In this case, we would like to relate $\mathcal{O}(1)$ with $\mathcal{L}(D)$ and with $\mathcal{O}(1)$ on $\mathbb{P}^m$. As the definition of $\mathcal{O}(1)$ is the same as that of $\mathcal{L}(D)$, things are slightly easier.

✠ *Lecture 11 - 2.10.2013*

Remark 1.3.7. Consider the following situation:

$$\begin{aligned} &\mathcal{O}(1) \text{ on } \mathbb{P}^n \\ S &= k[x_1, \dots, x_n] \\ M &= S[1] \end{aligned}$$

What is $[\mathcal{O}(1)](\mathbb{P}^n)$? One way to describe it is by

$$\begin{aligned} [\mathcal{O}(1)](\mathbb{P}^n) &= \left\{ \frac{m}{f} \ : \ \deg(m) = \deg(f), m \in M, m, f \text{ homogeneous}, f \in \mathcal{O}(\mathbb{P}^n), f(p) \neq 0 \ \forall\, p \in \mathbb{P}^n \right\} \\ &= \left\{ \frac{m}{f} \ : \ \deg(m) = 0, f \in k \right\} \\ &= M_0 \\ &= \{\text{linear functions}\} \end{aligned}$$

Suppose that $\frac{m_1}{f_1} = \frac{m_2}{f_2}$, for which we assume $m_i, f_i \in k[x_1, \dots, x_n]$ and are homogeneous. Since S is a UFD, we get that $\gcd(m_1, f_1) = 1$ implies $m_1 | m_2$, so $m_1 = cm_2$ for some $c \in k$. Therefore $f_1 = c' f_2$ for some $c' \in k$.

Remark 1.3.8. Suppose that $\varphi : X \to \mathbb{P}^n$ is a morphism. How can we use φ to construct a sheaf on X out of $\mathcal{O}(1)$ on $\mathbb{P}^n$?

Let us call the putative new sheaf $\mathcal{F}$. For $U \subset X$ open, what is $\mathcal{F}(U)$? Well, we would like $\mathcal{F}$ to be a locally free sheaf of $\mathcal{O}_X$-modules of rank one. So say that $f \in [\mathcal{O}(1)](V)$ for some space V. Then we plug in the coordinates of φ into f to get a new rational function, which we put in $\mathcal{F}(U)$. The space V is defined by the necessity that V contains $\varphi(U)$. Essentially, we would like

$$f \in \varinjlim_{V \supset \varphi(U)} \big[[\mathcal{O}(1)](V) \big]$$

So we define $\mathcal{F}(U) = f \in \varinjlim_{V \supset \varphi(U)} \big[[\mathcal{O}(1)](V) \big] = \varphi^{-1}(\mathcal{O}(1))$.

Example 1.3.9. Let $X = \mathbb{P}^1$, and $\varphi : X \to \mathbb{P}^1$ be defined by $\varphi(x, y) = [x^2 : y^2]$. The associated divisor class is $[2[0 : 1]] = \{\text{divisors of degree } 2\}$. And $\mathcal{F}(\mathbb{P}^1) = \text{span}\{x, y\} = \text{span}\{x^2, y^2\}$.

Further, we may let $U = \mathbb{P}^1 \setminus \{[1 : 0], [0 : 1]\}$, with

$$[\mathcal{O}(1)](U) = \left\{ \frac{m}{f} \ : \ \cdots (\text{as above}) \right\} = \left\{ \frac{m}{f} \ : \ \deg(m) = \deg(f), f = \left(\frac{x}{y}\right)^n \right\} = \frac{x}{y} \mathcal{O}_{\mathbb{P}^1}(U)$$

Then $\mathcal{F}(U) = \frac{x^2}{y^2} \cdot (\text{functions in } \mathcal{O}_{\mathbb{P}^1}(U))$, which is not an $\mathcal{O}_X(U)$-module.

Remark 1.3.10. In general, given a morphism $f : X \to Y$ and a sheaf $\mathcal{F}$ on Y, define

$$(f^{-1}\mathcal{F})(U) = \varinjlim_{V \supset f(U)} \big[\mathcal{F}(U) \big]$$

and sheafify to get the desired sheaf. That is, make $\mathcal{O}_X$ into a sheaf of $f\mathcal{O}_Y$-modules, and let $f^*\mathcal{F} = f^{-1}\mathcal{F} \bigoplus_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$. This takes the approach above by taking the $\mathcal{O}_X$-modules everywhere locally.

From the example above, we would have $[\varphi^*\mathcal{O}(1)](\mathbb{P}^1) = \text{span}\{x^2, y^2, xy\}$, where the xy comes from multiplying x^2 by $\frac{y}{x}$ on $x \neq 0$, and multiplying y^2 by $\frac{x}{y^3}$ on $y \neq 0$.

✠ *Lecture 12 - 4.10.2013*

Given a divisor (or a sheaf), we want to make a sheaf (or divisor). We want to do this because divisors make a group, and invertible sheaves make a group as well.

Definition 1.3.11. A *Weil divisor* is an integer linear combination of subvarieties of codimension 1.

A *Cartier divisor* is an equivalence class of

$$\{(f_i, U_i) \ : \ U_i \subset X \text{ open}, \bigcup U_i = X, f_i \text{ rational}, f_i/f_j \in \mathcal{O}^*(U_i \cap U_j)\}$$

where

$$\{(f_i, U_i)\} \sim \{(g_j, V_j)\} \text{ iff } f_i/g_j \in \mathcal{O}^* \text{ on } U_i \cap V_j \ \forall\ i, j$$

The last statement says that f_i/g_j is a unit on $U_i \cap V_j$.

Proposition 1.3.12. There exists a 1-1 correspondence between Weil and Cartier divisors.

Proof: Say $D = \sum n_Y Y$ is a Weil divisor. For each subvariety Y of the variety X, there is some U_Y open and t_y rational such that $\text{div}(t_Y|_{U_Y}) = Y$. Let $\{(t_Y^{n_Y}, U_Y)\}$ be the Cartier divisor corresponding to D. We choose U_y small enough so that it does not contain any point of Y' if $n_{Y'} \neq 0$ and $Y' \neq Y$.

Conversely, if $\{(f_i, U_i)\}$ is a Cartier divisor, let

$$D = \sum_{\substack{i \text{ such that} \\ U_i \cap Y \neq \emptyset}} (\text{ord}_Y(f_i))Y$$

This is well defined because if $Y \subset U_i \cap U_j$, then f_i/f_j is a unit on $U_i \cap U_j$, so $\text{ord}_Y(f_i) = \text{ord}_Y(f_j)$. ■

Definition 1.3.13. A *principal Weil divisor* is a Weil divisor such that $\text{div}(f) = \sum \text{ord}_Y(f)Y$. A *principal Cartier divisor* is a Cartier divisor $\{(f, X)\}$.

Note that the principal Weil (or Cartier) divisor induces a corresponding Cartier (or Weil) divisor, by the construction described above.

Definition 1.3.14. The *divisor class group* is defined to be

$$\text{Cl}(X) = \text{Div}(X)/\text{Rat}(X)$$

or the group of divisors modulo the group of rational functions. Note that the same group results if the Weil or Cartier divisors are chosen.

Remark 1.3.15. Adding Cartier divisors $\{(f_i, U_i)\}$ and $\{(g_j, U_j)\}$ is done in the following manner:

$$\{(f_i, U_i)\} + \{(g_j, U_j)\} = \{(f_i g_i, U_i \cap V_j)\}$$

Example 1.3.16. Let $D = 2[0 : 1]$ on $\mathbb{P}^1 = \{[x : y]\}$. What is the Cartier divisor corresponding to this Weil divisor? By trial and error, we find that it will be

$$\begin{aligned} D' &= \left\{(f_i, U_i) \ : \ f_i \text{ rational}, U_i \text{ open}, f_i/f_j \text{ unit on } U_i \cap U_j, \bigcup U_i = \mathbb{P}^1\right\} \\ &= \left\{\left(\frac{x^2}{y^2}, \mathbb{P}^1 - \{[1:0]\}\right), \left(1, \mathbb{P}^1 - \{[0:1]\}\right)\right\} \end{aligned}$$

✠ *Lecture 13 - 7.10.2013*

We were working on a corespondence between divisors and sheaves. Recall how the correspondence between Weil and Cartier divisors worked.

Example 1.3.17.
The Weil divisor in $\mathbb{P}^2$ is $\{x = 0\} - \{y = 0\}$. The corresponding Cartier divisor is $\{(\frac{x}{y}, \mathbb{P}^2)\}$.

The Weil divisor in $\mathbb{P}^2$ is $\{x = 0\} - 2\{y = 0\}$. The corresponding Cartier divisor is

$$\left\{ \left(\frac{xz}{y^2}, \mathbb{P}^2 - \{z = 0\} \right), \left(1, \mathbb{P}^2 - \{xy = 0\}\right), \left(\frac{x(x + y + z)}{y^2}, \mathbb{P}^2 - \{x + y + z = 0\} \right) \right\}$$

Remark 1.3.18. The above shows a general characteristic of $\mathbb{P}^n$. If a Cartier divisor cannot be constructed with only one element, it must have at least $n + 1$ elements.

Definition 1.3.19. Let R be a commutative ring, with A and B R-modules. Define

$$H(A, B) = \bigoplus_{a\in A, b\in B} R(a \otimes b)$$

$$Z(A, B) = \Big(\text{span of all elements of } H(A, B) \text{ of the forms, for all } a, a_1, a_2 \in A,\ b, b_1, b_2 \in B \text{ and } r \in R: \\ a_1 \otimes b + a_2 \otimes b - (a_1 + a_2) \otimes b, a \otimes b_1 + a \otimes b_2 - a \otimes (b_1 + b_2), \\ (ra) \otimes b - r(a \otimes b), a \otimes (rb) - r(a \otimes b)\Big)$$

Note that every ideal of R is an R-module. Indeed, even R is an R-module (it is a free R-module of rank 1). Finally, define

$$A \otimes_R B = H(A, B)/Z(A, B)$$

Example 1.3.20. Consider the following objects.

$$\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C} \text{ via } \varphi(z) = z(1 \otimes 1)$$
$$X \otimes_{\mathbb{C}} \mathbb{C} \cong X \text{ via } \varphi(x) = x \otimes 1 \quad \text{for every } \mathbb{C}\text{-module } X$$
$$X \otimes_R R \cong X \text{ via } \varphi(x) = x \otimes 1 \quad \text{for every } R\text{-module } X$$
$$\mathbb{C} \otimes_{\mathbb{Z}} \mathbb{C} \text{ is very bad, as it is huge}$$

What does $(\mathbb{Z}/3\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/4\mathbb{Z})$ look like?

✠ *Lecture 14 - 9.10.2013*

Definition 1.3.21. Let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. Then $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$ is the *tensor product* of sheaves.

Remark 1.3.22. Is the tensor product of sheaves still a sheaf? The restriction map we employ is $\text{res}_{\mathcal{F}\otimes\mathcal{G}} = \text{res}_{\mathcal{F}} \otimes \text{res}_{\mathcal{G}}$ The presheaf axioms are satisfied, but it is not a sheaf in general. So we must sheafify it to get a sheaf.

The reason for introducing the tensor product arises from the desire to make a group of invertible sheaves. This group will have the tensor product as the group operation.

Theorem 1.3.23. Let $\mathcal{F}, \mathcal{G}$ be invertible sheaves on a smooth variety X. Then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is invertible.

<u>*Proof:*</u> This involves the following algebra fact:

$$\varinjlim [M_i \otimes_{R_i} N_i] \cong \left(\varinjlim[M_i]\right) \otimes_{\varinjlim[R_i]} \left(\varinjlim[N_i]\right)$$

Using this, we get that

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_p \cong \mathcal{F}_p \otimes_{\mathcal{O}_p(X)} \mathcal{G}_p$$

And as we have that

$$(\mathcal{F} \otimes \mathcal{G})_p \cong \mathcal{F}_p \otimes G_p \cong \mathcal{O}_p \otimes_{\mathcal{O}_p} \mathcal{O}_p \cong \mathcal{O}_p$$

And as $\mathcal{O}_X$ is an invertible sheaf, $\mathcal{F} \otimes \mathcal{G}$ is invertible. ■

So now we have that $\otimes$ is an associative binary operation on isomorphism classes of invertible sheaves. As $\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{F} \cong \mathcal{F}$ for all $\mathcal{F}$, it follows that $\mathcal{O}_X$ is the identity element in the group we will soon have. Next we seek to define inverses.

Definition 1.3.24. Let $\mathcal{F}$ be sheaf of an $\mathcal{O}_X$-module. Define the *dual sheaf* of $\mathcal{F}$ to be the sheaf

$$\mathcal{F}^\vee = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$$

Definition 1.3.25. Let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. Define $\mathcal{H} = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ by

$$\mathcal{H}(U) = \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U) \qquad \mathrm{res}_{U \to V}(h) = h|_V$$

Theorem 1.3.26. The sheaf $\mathcal{F}^\vee$ is an invertible sheaf.

Proof: This follows from considering the following algebraic fact:

$$\mathcal{F}_p^\vee \cong \mathrm{Hom}_{\mathcal{O}_p}(\mathcal{F}_p, \mathcal{G}_p) \cong \mathrm{Hom}_{\mathcal{O}_p}(\mathcal{O}_p, \mathcal{O}_p) \cong \mathcal{O}_p$$

■

✠ *Lecture 15 - 11.10.2013*

Remark 1.3.27. For $\mathcal{H} = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, is $\mathcal{H}_p \cong \mathrm{Hom}_{\mathcal{O}_p}(\mathcal{F}_p, \mathcal{G}_p)$?

In general, no. But if $\mathcal{F}, \mathcal{G}$ are invertible (more generally, if $\mathcal{F}, \mathcal{G}$ are coherent), then yes.

Proposition 1.3.28. Let $\mathcal{F}$ be an invertible sheaf. Then $\mathcal{F} \otimes \mathcal{F}^\vee \cong \mathcal{O}_X$.

Proof: To show this, we need to censtruct an isomorphism from $\mathcal{F} \otimes \mathcal{F}^\vee$ to $\mathcal{O}_X$. The isomorphism will be constructed on the presheaf $\mathcal{F} \otimes \mathcal{F}^\vee$, but will be such that if we choose an open set U small enough, then all the desired properties will hold for stalks, and so will extend through sheafification to the sheaf $\mathcal{F} \otimes \mathcal{F}^\vee$. So begin by defining a map

$$\begin{array}{rrcl} \varphi: & (\mathcal{F} \otimes \mathcal{F}^\vee)(U) & \to & \mathcal{O}_X(U) \\ & f \otimes \mu & \mapsto & \mu(f) \end{array}$$

We have constructed this map on the pure elements of $(\mathcal{F} \otimes \mathcal{F}^\vee)(U)$, but it can be extended linearly to all the elements. We now check that it satisfies all the necessary conditions from the big set H.

$$\varphi(\lambda f \otimes \mu - \lambda(f \otimes \mu)) = \varphi(\lambda f \otimes \mu) - \varphi(\lambda(f \otimes \mu)) = \mu(\lambda f) - \lambda\mu(f) = 0$$

Check the other condition too:

$$\varphi((f_1 + f_2) \otimes \mu - f_1 \otimes \mu - f_2 \otimes \mu) = \mu(f_1 + f_2) - \mu(f_1) - \mu(f_2) = 0$$

So we conclude that it is well-defined, but only on the presheaf $\mathcal{F} \otimes \mathcal{F}^\vee$. Now choose U small enough so that

$$\mathcal{F}(U) \cong \mathcal{F}^\vee(U) \cong \mathcal{O}_X(U) \qquad \text{and} \qquad \mathcal{F}|_U \cong \mathcal{F}^\vee|_U \cong \mathcal{O}_X|_U$$

Then we write $\mathcal{F}|_U = x\mathcal{O}_X|_U$ and $\mathcal{F}^\vee|_U = y\mathcal{O}_X|_U$, which implies that

$$\mathcal{F}(U) = f\mathcal{O}_X(U) \qquad \text{and} \qquad \mathcal{F}^\vee(U) = \mu\mathcal{O}_X(U)$$

Injectivity is then proved by noting that $\varphi(\lambda(f \otimes \mu)) = \lambda\mu(f)$, where $\lambda(f \otimes \mu)$ is an arbitrary element of $(\mathcal{F} \otimes \mathcal{F}^\vee)(U)$. If that expression is zero, then either $\lambda = 0$ ar $\mu(f) = 0$. However, since μ generates $\mathcal{F}^\vee(U)$, μ cannot be zero, so $\mu(f) \neq 0$. Therefore $\lambda = 0$, and so φ is injective.

Surjectivity is given by taking some $\alpha \in \mathcal{F}^\vee(U)$ such that $\alpha(f) = 1$. Then $\varphi(\lambda(f \otimes \alpha)) = \lambda$ for any $\lambda \in \mathcal{O}_X$. So φ is surjective, completing the proof that φ is an isomorphism. ■

Remark 1.3.29. Now we see that invertible sheaves form a group with inverses. Now we want to show that this group is isomorphic to the other two groups that we already have. We do this by finding a correspondence between invertible sheaves (up to isomorphism) and Cartier divisors modulo principal divisors (f, X).

Let $D = \{(f_i, U_i)\}$ be a Cartier divisor. Let $\mathcal{K}$ be the constant sheaf associated to the function field $k(X)$, so that $\mathcal{K}(U) = k(X)$ if $u \neq \emptyset$.

Definition 1.3.30. Define a sheaf $\mathcal{L}$ by $\mathcal{L}|_{U_i} = f_i^{-1}\mathcal{O}_X|_{U_i}$ and sheafify. To see that the presheaf is well defined, notice that if $V \subset U_i \cap U_j$, then $f_i^{-1}\mathcal{O}_X|_V = f_j^{-1}\mathcal{O}_X|_V$ because $f_i^{-1}/f_j^{-1} = f_i/f_j$ is a unit on $\mathcal{O}_X|_U$ by definition of Cartier divisors.

✠ *Lecture 16 - 16.10.2013*

Our strategy so far has been to make the following relations clear:

$$\Big(\text{morphisms}\Big) \longleftrightarrow \Big(\text{Weil divisors}\Big) \longleftrightarrow \Big(\text{Cartier divisors}\Big) \longleftrightarrow \Big(\text{invertible sheaves}\Big)$$

Recall what we were working on in the last lecture. We had $\{(f_i, U_i)\}$ as a Cartier divisor, from which we made an invertible sheaf by setting

$$\mathcal{L}|_{U_i} = f_i^{-1}\mathcal{O}_X|_{U_i} \qquad \text{i.e.} \qquad \mathcal{L}(V) = f_i^{-1}\mathcal{O}_X(V) \ \forall\ V \subset U_i,$$

where $f_i^{-1}\mathcal{O}_X(V) \subset \mathcal{K}(V)$, and $\mathcal{K}(V)$ is the constant sheaf associated to $k(X)$. The associated restriction maps are given by $\text{res}(f_i^{-1}\alpha) = f_i^{-1}\text{res}(\alpha)$, and we use the sheaf gluing axiom to define $\mathcal{L}(U)$ for the open set U.

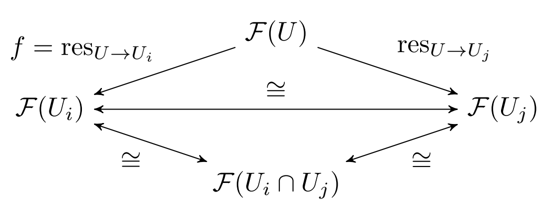
> **Definition 1.3.31.** Let X be a topological space. Define the *constant sheaf* associated to X to be the sheaf $\mathcal{K}(X)$ whose stalks are all equal to X.

So now we have constructed an invertible sheaf from a Cartier divisor. Is the reverse possible?

Let $\mathcal{L}$ be an invertible sheaf, and let's try to embed $\mathcal{L}$ in the sheaf $\mathcal{K}$, and then run the previous process backward. We will show that $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K} \cong \mathcal{K}$. Begin by picking any open set U such that $\mathcal{L}|_U \cong \mathcal{O}_X|_U$. Then

$$\mathcal{L}|_U \otimes_{\mathcal{O}_X} \mathcal{K}|_U \cong \mathcal{O}_X|_U \otimes_{\mathcal{O}_X|_U} \mathcal{K}|_U \cong \mathcal{K}|_U.$$

This is a local fact, not yet a global fact. Note also that it holds true for any $\mathcal{K}$. Next, the restriction maps of $\mathcal{F} = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}$ are all isomorphisms for small enough open sets. Let U be any non-empty subset, and let $\{U_i\}$ be an open cover of X so that $\mathcal{L}|_{U_i} \cong \mathcal{O}_X|_{U_i}$. This gives the following commutative diagram:



The map $f = \text{res}_{U \to U_i}$ is an isomorphism by the gluing and zero sheaf axioms, as well as the commutativity of the diagram. Hence $\mathcal{F} \cong \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}$ is a constant sheaf, and is thus isomorphic to $\mathcal{K}$.

We now embed $\mathcal{L}$ in $\mathcal{K}$ via $\mathcal{L} \mapsto \mathcal{L} \otimes 1 \hookrightarrow \mathcal{L} \otimes \mathcal{K} \cong \mathcal{K}$. So assume that $\mathcal{L}$ is a subsheaf of $\mathcal{K}$, and choose an open cover $\{U_i\}$ such that $\mathcal{L}|_{U_i} \cong f_i^{-1}\mathcal{O}_x|_{U_i}$ for all i and some $f_i \| k(X)$. We now make a Cartier divisor $\{(f_i, U_i)\}$. We know the f_i are rational functions and the U_i are open sets that agree on the sections (this remains to be checked). Since $\mathcal{L}$ is well-defined,

$$f_i^{-1}\mathcal{O}_X|_{U_i \cap U_j} = f_j^{-1}\mathcal{O}_X|_{U_i \cap U_j} \qquad \text{so} \qquad f_i/f_j \in \mathcal{O}_X|_{U_i \cap U_j}$$

as desired. So our Cartier divisor is well-defined. Note that the principal Cartier divisors correspond to $\mathcal{O}_X$, so linearly equivalent divisors correspond to isomorphic invertible sheaves. Furthermore, the correspondence is a group isomorphism. In other words, $\mathrm{Cl}(X) \cong \mathrm{Pic}(X)$.

Example 1.3.32. Let X be the zero set of $y^2z = x^3 - xz^2 \subset \mathbb{P}^2$. Let $S = k[x, y, z]/(y^z - x^3 + xz^2)$ and $M = S[1]$. Let L be $\tilde{M}$, the sheaf associated to M. Find a Weil divisor associated to $\tilde{M}$.

Recall that, as a presheaf,

$$L(U) = \left\{ \frac{m}{f} \ : \ \deg(m) = \deg(f), m \in M, f \in S, f \text{ is non-vanishing on } U \right\}$$

✠ *Lecture 17 - 18.10.2013*

Recall the problem from last time, where we were trying to find a Weil divisor associated to $\mathcal{L}$, given

$$\begin{aligned} X &= \{y^2z = x^3 - xz^2\} \subset \mathbb{P}^2 \\ S &= k[x, y, z]/(y^2z - x^3 + xz^2) \\ M &= S[1] \\ \mathcal{L} &= \tilde{M} \end{aligned}$$

We then made a presheaf $\tilde{M}(U) = \{\frac{m}{f} \ : \ m \in M, f \in S, \deg(m) = \deg(f), f \text{ non-vanishing on } U\}$. Then we embed $\mathcal{L}$ into $\mathcal{K} =$ the constant sheaf of $k(X)$ by

$$\begin{array}{rcl} \varphi(U): \ \ \mathcal{L}(U) & \to & \mathcal{K}(U) \\ {[\varphi(U)]} \left(\frac{m}{f}\right) & = & \frac{1}{z}\left(\frac{m}{f}\right) \end{array}$$

This clearly works (i.e. nothing gets killed), so $\varphi(L)$ is an invertible subsheaf of K. We now need an open cover $\{U_i\}$ of X such that $L|_{U_i} \cong \mathcal{O}_X|_{U_i}$, that is, $L|_{U_i} = f_i^{-1}\mathcal{O}_X|_{U_i}$ for some $f_i \in k(X)$. First we consider $U_1 = \{z \neq 0\}$. Then we have that

$$\begin{aligned} \tilde{M}(U_i) &= \left\{ \tfrac{m}{z^n} \ : \ \deg(m) = n \text{ in } M, m \in M \right\} \\ \mathcal{O}_X(U_i) &= \left\{ \tfrac{f}{z^n} \ : \ \deg(f) = n \text{ in } S, f \in S \right\} \end{aligned}$$

Now we check that $\tilde{M}(U_1) = z\mathcal{O}_X(U_1)$. This indeed is true, by changing indeces. Hence the associated Cartier divisor to $\tilde{M}$ is

$$\left\{ (1, U_1), \left(\frac{z}{x}, U_2\right), \left(\frac{z}{y}, U_3\right) \right\}$$

because $\varphi(z\mathcal{O}_X(U_1)) = \frac{z}{z}\mathcal{O}_X(U_1) = \mathcal{O}_X(U_1)$, and $\varphi(x\mathcal{O}_X(U_2)) = \frac{x}{z}\mathcal{O}_X(U_2)$, and $\varphi(y\mathcal{O}_X(U_3)) = \frac{y}{z}\mathcal{O}_X(U_3)$. Now we need a Weil divisor. To make it, we check with each element in the Cartier divisor. On the first, it is 0 on U_1. For the second, it is 0 on U_2. From the third, we get $3[0:1:0]$, as a line intersects a plane cubic curve 3 times. This also follow from the fact that x/y, $(x - z)/y$, and $(x + z)/y$ are all uniformizers at $[0:1:0]$ on X, as

$$y^2z = x(x - z)(x + z) \quad \Longrightarrow \quad \frac{z}{y} = \frac{x}{y} \cdot \frac{x - z}{y} \cdot \frac{x + z}{y}.$$

Therefore $\text{ord}_p(z/y) = 1 + 1 + 1 = 3$, and hence from the third element we get $3[0:1:0]$, for $p = [0:1:0]$. Note that this result depends on the choice of $1/z$ initially. The result could have been

$$\left\{ \left(\frac{ax+by+cz}{z}, U_1 \right), \left(\frac{ax+by+cz}{x}, U_2 \right), \left(\frac{ax+by+cz}{y}, U_3 \right) \right\}.$$

Also, if we would have had $S[2]$ initially, then we would have a degree 2 function in z, instead of $ax+by+cz$. Also note that $\tilde{M} \cong \psi^* \mathcal{O}(1)$, for $\psi : X \hookrightarrow \mathbb{P}^2$.

✠ *Lecture 18 - 21.10.2013*

Proposition 1.3.33. A morphism $\varphi : X \to \mathbb{P}^n$ corresponds to the invertible sheaf $\varphi^* \mathcal{O}(1) = \mathcal{L}$ with

$$\mathcal{L}(U) = \underbrace{\varinjlim_{V \supset \varphi(U)} \Big[[\mathcal{O}(1)](V) \Big]}_{\text{not a sheaf of } \mathcal{O}_x\text{-modules}} \otimes_{\#} \mathcal{O}_X(U)$$

Here $\# = [\varphi^{-1}\mathcal{O}_{\mathbb{P}^n}](U) = \varinjlim_{v \supset \varphi(U)} [\mathcal{O}_{\mathbb{P}^n}(V)]$, which consists of regular functions on $\mathbb{P}^n$ pulled back to X by φ. The factor on the left approximates $\mathcal{O}(1)(\varphi(U))$ as close as it can for U open. The main idea is that $\varphi^* \mathcal{O}(1)$ is just linear forms, pre-composed with φ to get linear forms on X.

Recall that:

$$\begin{aligned} \text{Pic}(X) = &\text{ invertible sheaves wih } \otimes \text{ up to isomorphism} \\ \text{Cl}(X) = &\text{ Weil divisors modulo linear equivalence} \end{aligned}$$

We already showed that $\text{Pic}(X) \cong \text{Cl}(X)$. The question arises: what is $\text{Pic}(\mathbb{P}^n)$?

Example 1.3.34. Consider the two following divisors:

$$\begin{aligned} [x^2 + yz = 0] - 2[x = 0] + [x^3 + y^3 + z^3 = 0] &= D \\ 3[x = 0] - 2[z = 0] &= D' \end{aligned}$$

By observation, we see that D has degree $2 - 2 + 3 = 3$, and D' has degree $3 - 2 = 1$, so they can not be linearly equivalent. This may be formalized by stating

$$D - D' = [x^2 + yz = 0] - 5[x = 0] + [x^3 + y^3 + z^3 = 0] + 2[z = 0] \overset{?}{=} \text{div}\left(\frac{(x^2 + yz)z^2(x^3 + y^3 + z^3)}{x^5} \right)$$

As the agument of Div on the right side is not a rational function (the degrees do not match up), the equality does not hold. However, such an argument does not hold in $\mathbb{P}^n$.

Remark 1.3.35. We know that $\text{Pic}(\mathbb{P}^n) \cong \text{Cl}(\mathbb{P}^n) \cong \mathbb{Z}$, but we would like an explicit isomorphism. So define $\varphi : \text{Div}(\mathbb{P}^n) \to \mathbb{Z}$ by $\varphi(f = 0) = \deg(f)$ and extend linearly. Then

$$\varphi\left(\sum n_i [f_i = 0] \right) = 0 \quad \Longrightarrow \quad \sum n_i \deg(f_i) = 0$$

So let $f = \prod f_i^{n_i}$, which is a well-defined rational function with divisor $\text{div}(f) = \sum n_i [f_i = 0]$. Conversely, if $\varphi(\text{div}(f)) = 0$ for all rational functions f, then $\ker(\varphi) = $ (principal divisors). Since φ is injective, φ induces an isomorphism $\text{Cl}(X) \cong \mathbb{Z}$ since $\text{Cl}(X) \cong \text{Div}(X)/\text{Rat}(X)$.

Every invertible sheaf on $\mathbb{P}^n$ is of the form $\mathcal{O}(n)$ for some n, so we now know what all the morphisms from $\mathbb{P}^n$ to $\mathbb{P}^n$ are. Although, we already know from earlier that they are a dth Veronese embedding.

Remark 1.3.36. A sheaf of ideals on X is a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ with $\mathcal{F}(U) \subset \mathcal{O}_X(U)$ for all U. So for every subvariety $Y \subset X$, there is a corresponding sheaf of ideals

$$\mathcal{I}_Y(U) = \{f \in \mathcal{O}_Y(U) \ : \ f(Y) = U\}$$

It turns out that $\mathcal{I}_Y$ is invertible iff $\text{codim}(Y) = 1$, in which case $\mathcal{I}_Y \cong \mathcal{L}(-Y)$. Here, $\mathcal{I}_Y$ is generated locally by f, and $\mathcal{L}(-Y)$ is generated locally by $1/f^{-1}$.

Definition 1.3.37. An invertible sheaf $\mathcal{F}$ on X is *generated by global sections* iff $\mathcal{F} \cong \varphi^*\mathcal{O}(1)$ for some morphism $\varphi : X \to \mathbb{P}^n$.

We say that $\mathcal{F}$ is *very ample* iff φ may be chosen to be an embedding.

Remark 1.3.38. Note that if $\mathcal{F}, \mathcal{G}$ are generated by global sections, then so is $\mathcal{F} \otimes \mathcal{G}$. The same thing goes for very ample sheaves. Further, $\mathcal{F}$ is generated by global sections iff for all $f \in X$, $\mathcal{F}_p$ is generated as an $\mathcal{O}_p$-module by $[\mathcal{F}(X)]_p$.

The next big question that we will ask, is how do we tell if an invertible sheaf is very ample?

1.4 Differentials

✠ *Lecture 19 - 23.10.2013*

Consider: given an invertible sheaf, for which, if any, does it define a morphism to $\mathbb{P}^n$?

Definition 1.4.1. Let A be a k-algebra (i.e. a ring and a k-vector space, with compatible structure). Define $\Omega_A = H/R$, where $H = \bigoplus_{a \in A} A(da)$ and

$$R = \text{submodule of } H \text{ spanned by } \begin{matrix} d\lambda \\ d(a_1 + a_2) - da_1 - da_2 \\ d(a_1 a_2) - a_1 da_2 - a_2 da_1 \end{matrix} \quad \forall\ \lambda \in k, a_1 a_2 \in A.$$

Example 1.4.2. Let $A = k[x_1, \dots, x_n]$. Then $\Omega_A = Adx_1 \oplus \cdots \oplus Adx_n$, because

$$df(x_1, \dots, x_n) = f_1 dx_1 + \cdots + f_n dx_n \quad \text{for} \quad f_i = \frac{\partial f}{\partial x_i}.$$

So Ω_A is the module of differentials on A relative to k.

Remark 1.4.3. Let X be a smooth projective variety. Define a presheaf $\Omega_X(U) = \Omega_{\mathcal{O}_X(U)}$ with restriction maps

$$\text{res}_{U \to V}\left(\sum a_i db_i\right) = \sum \left(\text{res}_{U \to V} a_i\right) d \left(\text{res}_{U \to V} db_i\right).$$

Unfortunately, Ω_X is not a sheaf, so we sheafify. Then Ω_X is called the *sheaf of differentials* on X.

Example 1.4.4. Consider the following example, which demonstrates why the original Ω_X fails the second sheaf axiom. Let $X = \{y^2 z = x^3 - xz^2\}$, and consider the regular differential dx/y on $zy \neq 0$.

· If $y = 0$, we can be at one of three points: $[1 : 0 : 1]$, $[-1 : 0 : 1]$, $[0 : 0 : 1]$

· If $z = 0$, we can be at $[0 : 1 : 0]$

On $z = 1$,

$$2ydy = 3x^2 dx - dx \implies 2ydy = (3x^2 - 1)dx \implies \frac{dx}{y} = \frac{2}{3x^2 - 1} dy$$

which is regular if $3x^2 \neq 1$ and $z \neq 0$. So the points where $y = 0$ are all regular points for this differential. At $[0 : 1 : 0]$,

$$\frac{dx}{y} = \frac{d\left(\frac{x}{z}\right)}{\frac{y}{z}} = zd\left(\frac{x}{z}\right) = z\left(\frac{1}{z}dx - \frac{x}{z^2}dz\right) = dx - \frac{x}{z}dz.$$

But

$$dz = 3x^2dx - z^2dx - 2xzdz \quad \Longrightarrow \quad dz = \left(\frac{3x^2 - z^2}{2xz + 1}\right)dx,$$

hence

$$dx - \frac{x}{z}dz = dx - \frac{x}{z}\left(\frac{3x^2 - z^2}{2xz+1}\right)dx = \left(\frac{2xz^2 + z - 3x^3 + xz^2}{(2xz+1)z}\right)dx = \frac{3(xz^2 - x^3) + z}{(2xz+1)z}dx = \frac{-2}{2xz+1}dx,$$

which is regular at $[0:1:0]$. So there exist functions that are regular on points of X, but are not regular everywhere. Hence Ω_X can not be a sheaf.

Remark 1.4.5. It turns out that the stalk $(\Omega_X)_P \cong (\mathcal{O}_P)^{\dim(X)}$ is generated by $dz_1, \ldots, dz_{\dim(X)}$, where the z_i are local coordinates for X at P. So Ω_X is locally free of rank $\dim(X)$. If $\dim(X) = 1$, then Ω_X is invertible.

Note that in order to make Ω_X, no choice was made. Hence any invariant calculated of Ω_X is an invariant of the variety X.

✠ *Lecture 20 - 25.10.2013*

What is $\Omega_{\mathbb{P}^1}$? It is a locally free sheaf of rank 1, so it is invertible. So $\Omega_{\mathbb{P}^1} = \mathcal{O}(n)$ for some n, but which n?

To find the solution, we construct a Cartier divisor. Let $\omega = d(\frac{x}{y})$. Then if $xy \neq 0$, ω will generate the stalk of $\Omega_{\mathbb{P}^1}$ as an $\mathcal{O}_{\mathbb{P}^1}$-module because $\frac{x}{y} - \frac{x_0}{y_0}$ is a uniformizer at $[x_0 : y_0]$ if $x_0y_0 \neq 0$.

- for $x = 0$: $d(x/y) = d(x/1) = dx$, which generates the stalk of $\Omega_{\mathbb{P}^1}$
- for $y = 0$: $d(x/y) = d(1/y) = -dy/y^2$, which has order of vanishing -2 at $[1:0]$

So the morphism $\varphi : \Omega_{\mathbb{P}^1} \to K$ could be $\varphi(\alpha) = \alpha/\omega$, which is the unique rational function f such that $\alpha = f\omega$. This morphism is also an embedding.

Local generators for the image are 1 for $p \in \mathbb{P}^1 - \{[1:0]\}$, and $-x^2/y^2$ for $p \in \mathbb{P}^1 - \{[0:1]\}$. So the associated Cartier divisor is

$$\left\{\left(1, \mathbb{P}^1 - \{[1:0]\}\right), \left(-\frac{x^2}{y^2}, \mathbb{P}^1 - \{[0:1]\}\right)\right\}$$

Therefore $\Omega_{\mathbb{P}^1} \cong \mathcal{O}(-2)$.

Remark 1.4.6. The sequence $0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\pi} C \longrightarrow C$ is a short exact sequence iff $B/\mathrm{Im}(A) \cong C$, which simplifies to $B/A \cong C$ if ι is an injection. The isomorphism is induced by π.

Theorem 1.4.7. The following sequence is exact:

$$0 \longrightarrow \Omega_{\mathbb{P}^n} \xrightarrow{\ \varphi\ } \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \xrightarrow{\ \pi\ } \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0$$

where $\varphi(d(x_j/x_i)) = (x_i\vec{e}_j - x_j\vec{e}_i)/x_i^2$ and $\vec{e}_i$ is the ith standard basis vector in $S[-1]^{n+1}$ of degree 1, and $\pi(\vec{e}_i) = x_i$ (i.e. $\vec{e}_i = 0 \oplus \cdots \oplus 0 \oplus 1 \oplus 0 \oplus \cdots \oplus 0$, and the 1 is in the ith position).

Proof: First note that $\mathrm{Im}(\varphi) \subset \ker(\pi)$, as $x_ix_j - x_jx_i = 0$. Next observe that this sequence is induced by the sequence $0 \longrightarrow \Omega_S \xrightarrow{\varphi} S[-1]^{n+1} \xrightarrow{\pi} S \longrightarrow 0$, where $S = k[x_0, \ldots, x_n]$. The map π is onto except in degree 0, so the corresponding sheaf map π is surjective in stalks, and thus onto. As $\pi \circ \varphi = 0$ clearly, it remains to show that φ is injective and $\ker(\pi) \subset \mathrm{Im}(\varphi)$.

We do this calculation on the stalks. Choose $P \in \mathbb{P}^n$, with (say) $x_0(P) \neq 0$. Then $(\Omega_{\mathbb{P}^n})_P \cong d(x_1/x_0)\mathcal{O}_P \oplus \cdots \oplus d(x_n/x_0)\mathcal{O}_P$. Now plug in an arbitrary linear combination of this into φ and see what happens. Suppose that $\varphi(f_1d(x_1/x_0) + \cdots + f_nd(x_n/x_0)) = 0$. We would like all the f_is to be zero. This expression simplifies:

$$f_1\varphi(d(x_1/x_0)) + \cdots + f_n\varphi(d(x_n/x_0)) = 0 \quad \Longrightarrow \quad f_1\frac{1}{x_0^2}(x_0\vec{e}_i - x_1\vec{e}_0) + \cdots + f_n\frac{1}{x_0^2}(x_0\vec{e}_n - x_n\vec{e}_0) = 0.$$

Rearranging this expression gives

$$(f_0x_0)\vec{e}_0 + (f_1x_0)\vec{e}_1 + \cdots + (f_nx_n)\vec{e}_n = 0,$$

implying that $f_i = 0$ for all i, as $x_0 \neq 0$. So φ is injective.

It remains to show that $\ker(\pi) \subset \mathrm{Im}(\varphi)$. So suppose that $\pi(f_0\vec{e}_0 + \cdots + f_n\vec{e}_n) = 0$. Then $\sum f_ix_i = 0$. Since $\ker(\pi)$ and $\mathrm{Im}(\varphi)$ are both rank n locally free $\mathcal{O}_X$-modules with $\mathrm{Im}(\varphi) \subset \ker(\pi)$, it follows that $\mathrm{Im}(\varphi) = \ker(\pi)$. In particular, $\varphi(\sum f_i d(x_i/x_0)) = \sum f_i\vec{e}_i$. ■

✠ *Lecture 21 - 28.10.2013*

Remark 1.4.8. How do we compute Ω_X, the sheaf of differentials on X, for $X \neq \mathbb{P}^n$? So let $X \neq \mathbb{P}^n$ be a smooth variety with coordinate ring A, which is a k-algebra, and a subvariety Y, with coordinate ring A/I, where I is an ideal of A. Consider

$$\begin{array}{ccccccc} I/I^2 & \xrightarrow{j} & \Omega_A \otimes_A A/I & \xrightarrow{q} & \Omega_{A/I} & \longrightarrow & 0 \\ \uparrow & & & & \uparrow & & \\ \text{local coordinates} & & & & \text{local coordinates} & & \\ \text{normal to variety} & & & & \text{on variety} & & \end{array}$$

Here, $j(x) = dx \otimes 1$, and $q(da \otimes b) = bda$, and the above is an exact sequence of (A/I)-modules. Note that I/I^2 is an (A/I)-module as $(x+I^2)(a+I) = ax + aI^2 + xI + I^3$. The version of the sequence with sheaves is

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_X \otimes \mathcal{O}_Y \longrightarrow \Omega_Y \longrightarrow 0$$

Example 1.4.9. Consider a bicuspid cubic with $X = \mathbb{A}^2$ and $Y = \{x^2 = y^3\}$, and just in case, we say $\mathrm{char}(k) \neq 2, 3$. Here, $A(X) = k[x,y]$ and $I = (y^2 - x^3)$ and $A/I = k[x,y]/(y^2 - x^3)$. Then $1, x, y$ are all k-linearly independent in $A/\mathcal{I}$. Note that $\mathcal{I}^2 = (y^2 - x^3)^2 = ((y^2 - x^3)^2)$, so $y^2 - x^3, xy^2 - x^3y, y^3 - x^3y$ are all k-linearly independent in $\mathcal{I}/\mathcal{I}^2$. Let us now try the map:

$$\begin{aligned} d(y^2 - x^3) \otimes 1 &= (2ydy - 3x^2dx) \otimes 1 \\ d(xy^2 - x^4) \otimes 1 &= (y^2dx - 2ykdy - 4x^3dx) \otimes 1 \\ d(y^3 - yx^3) \otimes 1 &= (3y^2 - x^3dy - 3x^2dx) \otimes 1 \end{aligned}$$

This example will be left unfinished. The moral is that the two sequences presented are exact on the left iff Y is smooth (X is assumed to be smooth throughout).

Remark 1.4.10. The canonical sheaf of a smooth variety X is $\omega_X = \bigwedge^{\dim(X)} \Omega_X$. To see this, let V be a vector space, for which

$$\begin{aligned} &\textstyle\bigwedge^n V = (V \otimes \cdots \otimes V)/R, \text{ and} \\ &\quad R = \mathrm{span}_{i,j}\{v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n + v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n\} \subset V \otimes \cdots \otimes V. \end{aligned}$$

Elements of $\bigwedge^n V$ are expressed as $v_1 \wedge \cdots \wedge v_n$. Note that $v_1 \wedge \cdots \wedge v_n = 0$ iff $\{v_1, \ldots, v_n\}$ is linearly dependent. Further, the dimension of $\bigwedge^n V$ is $\binom{\dim(V)}{n}$ for $n \in \{0, \ldots, \dim(V)\}$, and 0 otherwise. So if $\dim(V) = n$, thon $\dim(\bigwedge^n V) = 1$.

The above analysis also works for free modules over a ring, but requires slightly more work. For example, let $\Omega_X|_U$ be free, i.e. $\Omega_X|_U \cong (\mathcal{O}_X|_U)^{n=\dim(X)}$. Then

$$(\textstyle\bigwedge^n \Omega_X)|_U \cong \bigwedge^n \mathcal{O}_X|_U \qquad \text{where} \qquad (\bigwedge^n \mathcal{O}_X|_U)(V) = \bigwedge^n \mathcal{O}_X(V)$$

with restriction maps induced by $\mathrm{res}(v_1 \wedge \cdots \wedge v_n) = \mathrm{res}(v_1) \wedge \cdots \wedge \mathrm{res}(v_n)$.

1.5 Canonical sheaves

✠ *Lecture 22 - 01.11.2013*

What is $\omega_{\mathbb{P}^n} = \bigwedge^n \Omega_{\mathbb{P}^n}$? To find the solution, we stick $\mathbb{P}^n$ in the constant sheaf, find generators, find Cartier divisors, find Weil divisors. As this is an invertible sheaf, to construct an invertible embedding into $\mathcal{K}$, let's first find a non-zero element in it.

Let $\alpha = dX_1 \wedge \cdots \wedge dX_n$ with coordinates $x_0, \ldots, x_n$, where $X_i = \frac{x_i}{x_0}$ and $Y_i = \frac{x_i}{x_1}$. Note that α has no poles if $x_0 \neq 0$. If $x_0 = 0$ then (say) $x_1 \neq 0$, so

$$\begin{aligned}
\alpha &= d\frac{1}{Y_0} \wedge d\frac{Y_2}{Y_0} \wedge \cdots \wedge d\frac{Y_n}{Y_0} \\
&= \frac{-1}{Y_0^2} dY_0 \wedge \left(\frac{-Y_2}{Y_0^2} dY_0 + \frac{1}{Y_0} dY_2\right) \wedge \cdots \wedge \left(\frac{-Y_n}{Y_0^2} dY_0 + \frac{1}{Y_0} dY_n\right) \\
&= \frac{-1}{Y_0^2} dY_0 \wedge \frac{1}{Y_0} dY_2 \wedge \cdots \wedge \frac{1}{Y_0} dY_0 \\
&= \frac{-1}{Y_0^{n+1}} dY_0 \wedge dY_2 \wedge \cdots \wedge dY_n
\end{aligned}$$

which has a pole of order $n+1$ along $Y_0 = 0$, which is $x_0 = 0$. So α is our generator. Hence the Cartier divisor associated to $\omega_{\mathbb{P}^n}$ is

$$\left\{(1, \mathbb{P}^n, \{x_0 = 0\}), \left(\frac{-1}{Y_0^{n+1}}, \mathbb{P}^n - \{x_1 = 0\}\right), \cdots\right\}.$$

Every point in $\mathbb{P}^n$ contains a point that is in at least one of those sets, so we are done. This corresponds to the Weil divisor $(-n-1)[x_0 = 0]$, because $\mathrm{div}(Y_0) = [x_0 = 0]$. This Weil divisor cerrosponds to $\mathcal{O}(-1)$, so $\omega_{\mathbb{P}^n} \cong \mathcal{O}(-n-1)$.

Note that at every step of the process above we preserved isomorphism classes.

What if $X \ncong \mathbb{P}^n$? How would we compute ω_X?

Definition 1.5.1. Let $Y \subset X$ be a subvariety for X smooth. The *conormal sheaf* of Y in X is $\mathcal{I}_Y/\mathcal{I}_Y^2$. The *normal sheaf* of Y in X is the dual, $(\mathcal{I}_Y/\mathcal{I}_Y^2)^\vee = \mathrm{Hom}(\mathcal{I}_Y/\mathcal{I}_Y^2, \mathcal{O}_X)$.

Theorem 1.5.2. Let $Y \subset X$ be a smooth variety. Then $\omega_Y = \omega_X \otimes_{\mathcal{O}_X} \bigwedge^{\mathrm{codim}(Y)} \mathcal{N}_{Y/X}$, where $\mathcal{N}_{Y/X}$ is the normal sheaf of Y in X. If $\mathrm{codim}(Y) = 1$, then $\omega_Y \cong \omega_X \otimes_{\mathcal{O}_X} (\mathcal{I}_Y/\mathcal{I}_Y^2)^\vee$.

Proof: The sequence $0 \to \mathcal{I}_Y/\mathcal{I}_Y^2 \to \Omega_X \otimes \mathcal{O}_X \to \Omega_X \to 0$ is exact, so $\bigwedge^{\dim(X)} \Omega_X \otimes \mathcal{O}_Y \cong \left(\bigwedge^{\mathrm{codim}(Y)} (\mathcal{I}_Y/\mathcal{I}_Y^2)\right) \otimes \left(\bigwedge^{\dim(Y)} \Omega_Y\right)$. This is a general fact about exterior powers and short exact sequences. The proof follows by choosing bases for the two free modules on each end, building a basis for the free module in the middle and computing straightforwardly. So

$$\begin{aligned}
\omega_X \otimes \mathcal{O}_Y &\cong \left(\bigwedge^{\mathrm{codim}(Y)} \mathcal{I}_Y/\mathcal{I}_Y^2\right) \otimes \omega_Y \\
\implies \quad \omega_Y &\cong \omega_X \otimes \mathcal{O}_Y \otimes \left(\bigwedge^{\mathrm{codim}(Y)} \mathcal{I}_Y/\mathcal{I}_Y^2\right)^\vee \\
&\cong \omega_X \otimes \bigwedge^{\mathrm{codim}(Y)} \mathcal{N}_{Y/X}.
\end{aligned}$$

This completes the proof. ∎

Remark 1.5.3. Note the corresponding identifications:

$\bigwedge^n(V^\vee)$	$(\bigwedge^n V)^\vee$
$\bigwedge^n \mathrm{Hom}(V,k)$	$\mathrm{Hom}(\bigwedge^n V, k)$
$\sum_j f_{j_1}\wedge\cdots\wedge f_{j_n}$	$\sum_i v_{i_1}\wedge\cdots\wedge v_{i_n}\mapsto \sum_{i,j} f_{j_1}(v_{i_1})\wedge\cdots\wedge f_{j_n}(v_{i_n})$

This works with sheaves and free modules as well, even though this only shows the isomorphisms for vector spaces V.

Example 1.5.4. What is ω_Y if $Y=\{y^2z=x^3-xz^2\}\subset\mathbb{P}^2=X$? Then $\omega_Y\cong\omega_X\otimes_{\mathcal{O}_X}\mathcal{N}_{Y/X}$, where $\omega_X\cong\mathcal{O}(-3)$, and $\mathcal{N}_{Y/X}$ is an invertible sheaf on Y. What will be the Weil divisor?

✠ *Lecture 23 - 04.11.2013*

Let us continue where the previous lecture left off.

We had that $\omega_Y\cong\omega_X\otimes\mathcal{O}_Y\otimes\bigwedge^{\mathrm{codim}(Y)}\mathcal{N}_{Y/X}$, with $\mathcal{N}_{Y/X}=(\mathcal{I}_Y/\mathcal{I}_Y^2)^\vee$. Note that $\omega_X\otimes\mathcal{O}_Y$ is an invertible sheaf, and equals $i^*\omega_X$, for $i:Y\hookrightarrow X$ the inclusion map. Let us calculate what ω_Y is, for $Y=\{x=0\}\subset\mathbb{P}^2$.

First of all, $\omega_Y\cong i^*\mathcal{O}(-3)\otimes\mathcal{N}_{Y/X}$, where $i^*\mathcal{O}(-3)$ is the set of all rational functions with degree 3 more in the numerator than in the denominator. Restricting to the line in $\mathbb{P}^n$ (i.e. not $x=0$) leaves it at $\mathcal{O}(-3)$. Since we know the answer is $\omega_Y\cong\mathcal{O}(2)$, showing $\mathcal{N}_{Y/X}\cong\mathcal{O}(1)$ will give us the result. That is because taking the dual of a Weil divisor is multiplying by -1. So let's find a Weil divisor corresponding to $\mathcal{N}_{Y/X}=(\mathcal{I}_Y/\mathcal{I}_Y^2)^\vee$. Note that it is enough to get a Weil divisor of $\mathcal{I}_Y/\mathcal{I}_Y^2$.

We want to take something that is non-zero in $\mathcal{I}_Y/\mathcal{I}_Y^2$. As $\mathcal{O}_Y\cong\mathcal{O}_X/\mathcal{I}_Y$, consider $a+\mathcal{I}_Y^2$, for $a\in\mathcal{I}_Y$. We want to multiply this by $x+\mathcal{I}_Y\in\mathcal{O}_X/\mathcal{I}_Y$: $(x+\mathcal{I}_Y)(a+I_Y^2)=ax+\mathcal{I}_Y^2$.

Remark 1.5.5. Before we continue this example, consider $R=k[x,y]$ and $\mathcal{I}=\langle x\rangle$, and compute a non-zero element of $\mathcal{I}/\mathcal{I}^2$. One such element is $x+\mathcal{I}^2$.

Now observe that $\frac{x}{z}+\mathcal{I}_Y^2(z\neq0)\in(\mathcal{I}_Y/\mathcal{I}_Y^2)(z\neq0)$ is non-zero, as is $\frac{xy}{z^2}+\mathcal{I}_Y^2(z\neq0)$. Define a map

$$\begin{array}{rrcl}\varphi: & \mathcal{I}_Y/\mathcal{I}_Y^2 & \to & \mathcal{K}(Y)\\ & a+\mathcal{I}_Y^2(U) & \mapsto & \frac{az}{x}\in\mathcal{K}(Y)\end{array}.$$

If $\alpha\in\mathcal{I}_Y^2(U)$, then $\frac{(a+\alpha)z}{x}=\frac{az}{x}$ as elements of $\mathcal{K}(Y)$, the fraction field of Y, because $\frac{\alpha z}{x}\in\mathcal{I}_Y(U)$, so it is 0 in $k(Y)=\mathcal{K}(\mathcal{O}(U)/\mathcal{I}_Y(U))$.

> **Definition 1.5.6.** Let X be an integral domain. Then the *field of fractions* of X is the smallest field into which X can be embedded. It is denoted by $\mathcal{K}(X)$.

Note that if Y is smooth of codimension 1 in X, then $\mathcal{I}_Y/\mathcal{I}_Y^2$ is a locally free module of rank r.

✠ *Lecture 24 - 06.11.2013*

Recall the previous lecture's question - what is ω_Y if $Y=\{x=0\}\subset\mathbb{P}^2$? In general, we know that $\omega_Y=i^*\mathcal{O}_{\mathbb{P}^2}(-3)\otimes(\mathcal{I}_Y/\mathcal{I}_Y^2)^\vee$, but we would like to simplify this expression. We begin by embedding $\mathcal{I}_Y/\mathcal{I}_Y^2$ into the constant sheaf $\mathcal{K}$ by "dividing" by $\frac{x}{z}+\mathcal{I}_Y^2$. The local generators are then given by

$$\frac{x}{z}+\mathcal{I}_Y^2\text{ on }\{z\neq0\}\qquad\text{and}\qquad\frac{x}{y}+\mathcal{I}_Y^2\text{ on }\{y\neq0\}$$

The associated Cartier divisor is then $\{(1,\{z\neq0\}),(\frac{y}{z},\{y\neq0\})\}$. This is well-defined, as the divisor of 1 and of $\frac{y}{z}$ is 0 on $\{y\neq0\}\cap\{z\neq0\}$. This corresponds to the Weil divisor $-1\cdot[0:1:0]$, which corresponds to $\mathcal{O}(-1)$. Multiplying by -1, we then get that

$$\omega_Y=i^*\mathcal{O}_{\mathbb{P}^2}(-3)\otimes(\mathcal{I}_Y/\mathcal{I}_Y^2)^\vee\cong\mathcal{O}_{\mathbb{P}^1}(-3)\otimes\mathcal{O}_{\mathbb{P}^1}(1)\cong\mathcal{O}_{\mathbb{P}^1}(-2)$$

Example 1.5.7. Let us consider a more difficult example: $Y = \{y^2z = x^3 - xz^2\}$. First, we find a non-zero element in $\mathcal{I}_Y/\mathcal{I}_Y^2$, say

$$\frac{y^2z + xz^2 - x^3}{z^3} + \mathcal{I}_Y^2 \text{ on } \{z \neq 0\}$$

The local generators then are

$$\frac{y^2z + xz^2 - x^3}{z^3} + \mathcal{I}_Y^2 \text{ on } \{z \neq 0\} \;\leftrightarrow\; (1, \{z \neq 0\}) \qquad \text{and} \qquad \frac{y^2z + xz^2 - x^3}{y^3} + \mathcal{I}_Y^2 \text{ on } \{y \neq 0\} \;\leftrightarrow\; (\tfrac{y^3}{z^3}, \{y \neq 0\})$$

As Weil divisors, we have that

$$\omega_Y \cong -i^*3L + 9[0:1:0] \cong -3(z=0) + 9[0:1:0] \cong -9[0:1:0] + 9[0:1:0] = 0$$

So as sheaves, $\omega_Y \cong \mathcal{O}_Y$. Therefore $\omega_Y(1) \cong \mathcal{O}_Y(Y) \cong k$, meaning that the global differential we found on Y before is, up to scalars, the only one.

Example 1.5.8. Let $Y = \{x^4 + y^4 = z^4 + w^4\} \subset \mathbb{P}^2 = X$. Then we have

$$\begin{aligned} \text{non-zero element on } \mathcal{I}_Y/\mathcal{I}_Y^2\text{:} \quad & \frac{x^4 + y^4 - z^4 - w^4}{z^4} + \mathcal{I}_Y^2 \quad \text{on } \{z \neq 0\} \\ \text{local generator:} \quad & \frac{x^4 + y^4 - z^4 - w^4}{z^4} + \mathcal{I}_Y^2 \quad \text{on } \{z \neq 0\} \\ \text{local generator:} \quad & \frac{x^4 + y^4 - z^4 - w^4}{y^4} + \mathcal{I}_Y^2 \quad \text{on } \{y \neq 0\} \end{aligned}$$

The allociated Cartier divisor is $\{(1, \{z \neq 0\}), (\frac{y^4}{z^4}, \{y \neq 0\})\}$. The associated Weil divisor is $-4(z=0)$. The Weil divisor for $\mathcal{N}_{Y/X}$ is $4(z=0)$ and the Weil divisor for $i^*\mathcal{O}_Y(-4)$ is $-4(z=0)$, so the Weil divisor for ω_Y is 0. Therefore $\omega_Y \cong \mathcal{O}_Y$.

This gives a general rule:

Proposition 1.5.9. If $Y = \{f = 0\}$ for $\deg(f) = d$ in $\mathbb{P}^n$, then $\mathcal{N}_{Y/X} \cong i^*\mathcal{O}(d)$ and

$$\omega_Y \cong i^*\mathcal{O}(-n-1) \otimes i^*\mathcal{O}_Y(d) \cong i^*\mathcal{O}(d-n-1)$$

Now we know what happens on hypersurfaces. Let's consider something that is not a hypersurface.

Example 1.5.10. Let $Y = \left\{ \substack{xy = zw \\ 2x^2 + y^2 = z^2 + w^2} \right\} \subset \mathbb{P}^2$. We first need to know if Y is smooth. So we consider the Jacobian and the determinants of the minors of the Jacobian:

$$J = \begin{pmatrix} x & y & -w & -z \\ 4x & 3y & -2z & -2w \end{pmatrix} \qquad [\det(J^{ij})]_{ij} = \begin{pmatrix} y^2 - 2x^2 & 2xw - yz & 2xz - yw \\ yw - xz & yz - xw & w^2 - z^2 \end{pmatrix}_{ij}$$

Since the rank is maximal, the variety is smooth.

Theorem 1.5.11. If $V = z(f_1, \ldots, f_r) \subset \mathbb{P}^n$, then V is smooth if and only if $\text{rank}(J_V) = \dim(V)$ for all points of V, where

$$J_V = \begin{pmatrix} \frac{\partial f_1}{\partial x^0} & \cdots & \frac{\partial f_1}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_r}{\partial x^0} & \cdots & \frac{\partial f_r}{\partial x^n} \end{pmatrix}$$

✠ *Lecture 25 - 08.11.2013*

Example 1.5.12. Let's consider an even more difficult example than last time:

$$Y = \left\{ \begin{array}{c} xy = zw \\ 2x^2 + y^2 = z^2 + w^2 \end{array} \right\} \subset \mathbb{P}^3 = X \qquad\qquad \omega_Y = i^*\omega_X \otimes \bigwedge^2 \left(\mathcal{I}_Y/\mathcal{I}_Y^2\right)^\vee$$

We get a 2-form because $2 = \dim(X) - \dim(Y \text{ in } X) = 3 - (3-2)$. But now, instead of finding a non-zero element of $\mathcal{I}_Y/\mathcal{I}_Y^2$, we define a new variety $Z = \{xy = zw\}$, with the maps

$$\begin{array}{c} Y \xhookrightarrow{\;j\;} Z \xhookrightarrow{\;h\;} X \\ \underbrace{\qquad\qquad\qquad\qquad}_{i} \end{array}$$

Then for $\mathcal{J}_Y$ the ideal sheaf of Y in Z and $\mathcal{I}_Z$ the ideal sheaf of Z, we have

$$\begin{aligned} \omega_Z &= h^*\omega_X \otimes \left(\mathcal{I}_Z/\mathcal{I}_Z^2\right)^\vee = h^*\mathcal{O}(-2) \\ \omega_Y &= j^*\omega_Z \otimes \left(\mathcal{J}_Y/\mathcal{J}_Y^2\right)^\vee = j^*h^*\mathcal{O}(-2) \otimes \left(\mathcal{J}_Y/\mathcal{J}_Y^2\right)^\vee \end{aligned}$$

The Weil divisor for $j^*h^*\mathcal{O}(-2)$ has degree -8, which comes from intersecting the variety Y with the curve $z = 0$, and getting $4 = 2 + 2$ points. Then as we are in $\mathcal{O}(-2)$, we multiply 4 by -2 to get -8. Further, a non-zero element in $\mathcal{J}_Y/\mathcal{J}_Y^2$ is

$$\frac{2x^2 + y^2 - z^2 - w^2}{x^2} + \mathcal{J}_Y^2$$

Then local generators for $\mathcal{J}_Y/\mathcal{J}_Y^2$ and associated Cartier divisors are

$$\begin{aligned} \frac{2x^2 + y^2 - z^2 - w^2}{x^2} + \mathcal{J}_Y^2 \;\text{ on }\; \{x \neq 0\} &\leftrightarrow (1, \{x \neq 0\}) \\ \frac{2x^2 + y^2 - z^2 - w^2}{y^2} + \mathcal{J}_Y^2 \;\text{ on }\; \{y \neq 0\} &\leftrightarrow (\tfrac{y^2}{x^2}, \{y \neq 0\}) \end{aligned}$$

Hence the associated Weil divisor is $-2\mathrm{div}(X)$. We can now calculate ω_Y, as

$$\omega_Y \leftrightarrow -2\mathrm{div}(x) + 2\mathrm{div}(y) = 0 \implies \omega_Y \cong \mathcal{O}_Y$$

Remark 1.5.13. A local generator is a non-zero element of the invertible sheaf. That is, it is a generator at p when every element of the stalk of the sheaf is every element of the stalk times it.

Remark 1.5.14. Note that in the previous example, we did not use the first equation in defining the local generators. We didn't use either for the divisors. In general, if $Y \subset \mathbb{P}^n$ is the smooth intersection of exactly $\mathrm{codim}(Y) = r$ equations of degrees $d_1, \ldots, d_r$, then

$$\omega_Y = i^*\mathcal{O}(d_1 + \cdots + d_r - n - 1)$$

where $i : Y \hookrightarrow \mathbb{P}^n$ is the inclusion.

Proposition 1.5.15. Whenever we embed a variety in $\mathbb{P}^n$ as an intersection, then the canonical sheaf of that variety is the restriction of $\mathcal{O}(d)$ for some d. Moreover, any divisor of degree 0 on an elliptic curve is linearly equivalent to some point minus some other point.

Example 1.5.16. Consider $X = \mathbb{P}^3$ and $Y = \{xw = yz, y^2 = xz, z^2 = yw\} \subset \mathbb{P}^3$. Let's take a similar two-step approach that we did earlier. Define a new variety $Z = \{xw = yz\}$, for which

$$\begin{aligned} \omega_Z &\cong h^*\mathcal{O}(-2) \\ \omega_Y &\cong j^*h^*\mathcal{O}(-2) \otimes (\mathcal{J}_Y/\mathcal{J}_Y^2)^\vee \end{aligned}$$

A non-zero element in $\mathcal{J}_Y/\mathcal{J}_Y^2$ is $\frac{y^2-xz}{x^2}+\mathcal{J}_Y^2$ on $\{x\neq 0\}$. When are $xw=yz$ and $y^2=xz$ enough to define Y near p? Note that if $w\neq 0$, then $x=yz/w$, so $y^2=yz^2/w$. Hence $y=0$ or $yw=z^2$.

So long as $y\neq 0$ or $w\neq 0$, the expression $\frac{y^2-xz}{x^2}+\mathcal{J}_Y^2$ is a local generator (it works away from $[1:0:0:0]$) for $\mathcal{J}_Y$, and thus also for $\mathcal{J}_Y/\mathcal{J}_Y^2$.

✠ *Lecture 26 - 11.11.2013*

Continue with the example from the last lecture. Recall that we had

$$\begin{aligned}\omega_Z &\cong h^*\mathcal{N}_{Z/X}\cong h^*\mathcal{O}(-2)\\ \omega_Y &\cong j^*\omega_Z\otimes\mathcal{N}_{Y/X}\cong j^*h^*\mathcal{O}(-2)\otimes(\mathcal{J}_Y/\mathcal{J}_Y^2)^\vee\cong i^*\mathcal{O}(-2)\otimes(\mathcal{J}_Y/\mathcal{J}_Y^2)^\vee\end{aligned}$$

We also had a non-zero element $\frac{y^2-xz}{x^2}+\mathcal{J}_Y^2$ in $\mathcal{J}_Y/\mathcal{J}_Y^2$. The next question is, where is this a local generator for $\mathcal{J}_Y/\mathcal{J}_Y^2$? This will happen anwhere that $\frac{z^2-yw}{x^2}$ is in the ring $\mathcal{O}_{Z,p}\left[\frac{y^2-xz}{x^2}\right]$. This gives

$$xw=yz\implies w=yz/x\implies \frac{z^2-yw}{x^2}=\frac{z^2-y(yz/x)}{x^2}=\frac{z^2x-y^2z}{x}=\frac{-z(y^2-xz)}{x^3}=\frac{-z}{x}\left(\frac{y^2-xz}{x^2}\right)$$

So now we have the function that multiplies this non-zero element to get $xw=yz$. So the local generator is $\frac{y^2-xz}{x^2}+\mathcal{J}_Y^2$ on $\{x\neq 0\}$. Note that the only point on Y where $x=0$ is $[0:0:0:1]$. As $w\neq 0$ here, we consider the element $\frac{z^2-yw}{w^2}+\mathcal{J}_Y^2$. Then

$$x=yz/w\implies\frac{y^2-xz}{w^2}=\frac{y^2-(yz/w)z}{w^2}=\frac{y^2w-yz^2}{w^3}=\frac{-y}{w}\left(\frac{z^2-yw}{w^2}\right)$$

Here also we have the function that multiplies this non-zero element to get $xw=yz$, so the local generator is $\frac{z^2-yw}{w^2}+\mathcal{J}_Y^2$ on $\{w\neq 0\}$. Now we want a Cartier divisor out of $\mathcal{J}_Y/\mathcal{J}_Y^2$. The first element will be $(1,\{z\neq 0\})$, and the second is calculated below:

$$\frac{\frac{z^2-yw}{w^2}}{\frac{y^2-xz}{x^2}}=\frac{x^2z^2-ywx^2}{w^2y^2-w^2xz}=\frac{x^2z^2-y\left(\frac{yz}{x}\right)x^2}{\left(\frac{yz}{x}\right)^2y^2-\left(\frac{yz}{x}\right)^2xz}=\frac{x^4z^2-y^2zx^3}{y^4z^2-y^2z^3x}=\frac{-x^3}{y^2z}\left(\frac{y^2z-z^2x}{y^2z-z^2x}\right)=\frac{-x^3}{y^2z}$$

Hence the second element of the Cartier divisor is $(\frac{-y^2z}{x^3},\{w\neq 0\})=(\frac{-yw}{x^2},\{w\neq 0\})$. The associated Weil divisor is then

$$\mathrm{div}(y)-\mathrm{div}(x^2)=\mathrm{div}(y)-2\mathrm{div}(x),$$

but only away from $\{w=0\}$, or equivalently the point $[1:0:0:0]$. This divisor is a multiple of $[0:0:0:1]$, because $y=0$ and $z=0$, and $x=0$ implies $y=0$, and $[1:0:0:0]$ is excluded. Next, we are looking for the tangent line to Y at $[0:0:0:1]$, so we calculate the Jacobian:

$$J=\begin{pmatrix}-z & 2y & -x & 0\\ w & -z & -y & x\\ 0 & -w & 2z & -y\end{pmatrix}\qquad\text{and}\qquad J_{[0:0:0:1]}=\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&-1&0&0\end{pmatrix}.$$

Hence the tangent line is $x = y = 0$, or rather the solution set of linear systems of equations that correspond to it. So now we have that both $x = 0$ and $y = 0$ contain the tangent line at $p = [0 : 0 : 0 : 1]$, but $z = 0$ does not, so $\text{ord}_p(z) = 1$. Further,

$$y = \frac{z^2}{w} \implies \text{ord}_p(y) = 2 \qquad \text{and} \qquad x = \frac{y^2}{z} \implies \text{ord}_p(x) = 4 - 1 = 3$$

So our Weil divisor is $-4[0 : 0 : 0 : 1]$ for $\mathcal{J}_Y/\mathcal{J}_Y^2$. So $\omega_Y \cong i^*\mathcal{O}(-2) \otimes \mathcal{L}(4[0 : 0 : 0 : 1])$ and

$$-2(x = 0) + 4[0 : 0 : 0 : 1] = -6[0 : 0 : 0 : 1] + 4[0 : 0 : 0 : 1] = -2[0 : 0 : 0 : 1]$$

Therefore the Weil divisor for ω_Y is $-2[0 : 0 : 0 : 1]$. This completes the example.

Recall that $\mathbb{P}^1$ had a similar Weil divisor. So $Y \cong \mathbb{P}^1$.

2 Cohomology

2.1 Some important theorems

✠ *Lecture 27 - 13.11.2013*

Our main question this term has been: what are morphisms $X \to \mathbb{P}^n$? To any morphism is associated a linear system, which is associated to a linear equivalence class of divisors (Weil or Cartier), which is associated to an invertible sheaf.

There are lots of different morphisms corresponding to the same invertible sheaf. However, any morphism associated to a linear system $\mathcal{L}$ is, up to composition with linear maps, equal to the complete morphism associated to $\mathcal{L}$ coming from a basis of $L(\mathcal{L})$.

Remark 2.1.1. How do we make a morphism out of an invertible sheaf $\mathcal{F}$?

· Choose $f_0, \ldots, f_n \in \mathcal{F}(X)$ and define $\varphi_{\mathcal{F}}(p) = [f_0(p) : \cdots : f_n(p)]$

Here, $f_0(p)$ is f_0 of the stalk at p, i.e. $(f_0)_p$. Note that the stalk is a free rank-1 madule over the local ring, that is, $\mathcal{F}_p \cong \alpha\mathcal{O}_p$ for some $\alpha \in \mathcal{F}_p$. Hence $(\mathcal{F}_i)_p = \alpha r_i$ for some $r_i \in \mathcal{O}_p$, and so

$$\varphi(p) = [\alpha r_1(p) : \cdots : \alpha r_n(p)] = [r_0(p) : \cdots r_n(p)].$$

This completely defines φ. So for any $\mathcal{F}$, there is a best $\mathbb{P}^n$ that is the image for a rational map associated to $\mathcal{F}$. That "best" n is $n = \dim(\mathcal{F}(X)) - 1$.

Remark 2.1.2. $\mathcal{F}(X)$ is a finite-dimensional k-vector space.

The rest of the class will continue on the question of what are the morphisms $X \to \mathbb{P}^n$. We begin by considering $\mathcal{O}$ as an irreducible smooth Weil divisor on a smooth projective variety X. We can then construct a short exact sequence of sheaves:

$$0 \longrightarrow \mathcal{I}_D \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0$$

Note that the stalk of $\mathcal{O}_D$ is zero outside D. Further, $\mathcal{I}_D = \mathcal{L}(-D)$. Taking the tensor product with $\mathcal{L}(D)$ (which preserves exactness, as $\mathcal{L}(D)$ is an $\mathcal{O}_X$-sheaf), gives

$$0 \longrightarrow \mathcal{L}(-D) \otimes \mathcal{L}(D) \longrightarrow \mathcal{O}_X \otimes \mathcal{L}(D) \longrightarrow \mathcal{O}_D \otimes \mathcal{L}(D) \longrightarrow 0$$

$$\approx \quad 0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{L}(D) \longrightarrow \mathcal{O}_D \otimes \mathcal{L}(D) \longrightarrow 0$$

$$\approx \quad 0 \longrightarrow \mathcal{O}_X(X) \longrightarrow [\mathcal{L}(D)](X) \to [\mathcal{O}_D \otimes \mathcal{L}(D)](X) \to 0$$

In general, the last map in the last sequence is not surjective, because surjective morphisms of sheaves are not surjective for every open set. To solve this problem, we turn to *cohomology*.

Proposition 2.1.3. For any coherent sheaf $\mathcal{F}$ on X, for X a smooth projective variety defined over an algebraically closed field k, there are k-vector spaces $H^i(\mathcal{F})$ such that

· $M^0(\mathcal{F}(X)) \cong \mathcal{F}(X)$ naturally

· If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is exact, then there is an exact sequence:

$$0 \longrightarrow H^0(\mathcal{F}_1) \longrightarrow H^0(\mathcal{F}_2) \longrightarrow H^0(\mathcal{F}_3) \longrightarrow H^1(\mathcal{F}_1) \longrightarrow H^1(\mathcal{F}_2) \longrightarrow H^1(\mathcal{F}_3) \longrightarrow H^2(\mathcal{F}_1) \longrightarrow \cdots$$

Theorem 2.1.4. [Grothendieck]
$H^i(\mathcal{F}) = 0$ if $i > \dim(X)$

Theorem 2.1.5. [Kodaira vanishing]
If $\mathcal{F}$ is an ample sheaf, then $H^i(\mathcal{F} \otimes \omega_X) = 0$ if $i \geqslant 1$.

Note that $\mathcal{F}$ is ample iff $\mathcal{F}^{\otimes n}$ is very ample (i.e. it corresponds to some embedding) for some $n > 0$.

Theorem 2.1.6. [Serre duality]
If $\mathcal{F}$ is a locally free sheaf, then $H^i(\mathcal{F}) \cong H^{\dim(X)-i}(\mathcal{F}^\vee \otimes \omega_X)^*$.

In terms of the dimensions $h^i(X) = \dim(H^i(X))$, this may be expressed, for D a divisor on a curve X (so $\dim(X) = 1$), as $h^1(D) = h^0(K - D)$.

2.2 Riemann–Roch for curves

✠ *Lecture 28 - 15.11.2013*

Definition 2.2.1. A divisor D is called *effective* if all the coefficients in the sum are non-negative. This is expressed as $D \geqslant 0$.

Example 2.2.2. Suppose that X is a curve and D is a divisor on X. Let $\mathcal{L} = \mathcal{L}(D)$ be the corresponding invertible sheaf. What is $\dim_k(\mathcal{L}(X))$?

If $\deg(D) < 0$, then $\mathcal{L}(D) = 0$, because no non-zero rational function satisfies $\text{div}(f) + D \geqslant 0$.

If $\deg(D) = 0$, then $\mathcal{L}(D) = 1$ if $D = 0$, and 0 if $D \neq 0$. This follows as if $\text{div}(f) + D \geqslant 0$ for $\deg(D) = 0$, then $D = -\text{div}(f) = \text{div}(1/f)$.

If $\deg(D) = 1$, then (WLOG) $D = p$ for some point p. This gives a short exact sequence:

$$0 \longrightarrow \mathcal{I}_p \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_p \longrightarrow 0$$

$$\approx \qquad 0 \longrightarrow \mathcal{L}(-p) \longrightarrow \mathcal{O}_X \longrightarrow k(p) \longrightarrow 0$$

Above, $k(p)$ is the skyscraper sheaf on p with value k. Tensor with $\mathcal{L}(p)$ to get a new short exact sequence:

$$0 \longrightarrow \mathcal{L}(-p) \otimes \mathcal{L}(p) \longrightarrow \mathcal{O}_X \otimes \mathcal{L}(p) \longrightarrow k(p) \otimes \mathcal{L}(p) \longrightarrow 0$$

$$\approx \qquad 0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{L}(p) \longrightarrow k(p) \longrightarrow 0$$

On X, we get the following longer exact sequence:

$$0 \longrightarrow \mathcal{O}_X(X) \longrightarrow [\mathcal{L}(p)](X) \longrightarrow [k(p)](X) \longrightarrow H^1(\mathcal{O}_X) \longrightarrow H^1(\mathcal{L}(p)) \longrightarrow H^1(k(p)) \longrightarrow 0$$

Recall that the *Euler characteristic* of the sheaf $\mathcal{F}$ is $\chi(\mathcal{F}) = \sum(-1)^i h^i(\mathcal{F})$. It has to sum to zero, so

$$h^0(\mathcal{O}_X) - h^0(\mathcal{L}(p)) + h^0(k(p)) - h^0(\mathcal{O}_X \otimes \omega_X) + h^)(\mathcal{L}(p)^\vee \otimes \omega_X) - h^1(k(p)) = 0$$

We leave this example unfinished, because DM says so.

For a general divisor D, with $H^i(D) = H^i(\mathcal{L}(D))$, we have the exact sequence

$$0 \longrightarrow H^0(D) \longrightarrow H^0(D+p) \longrightarrow k(p) \longrightarrow H^1(D) \longrightarrow H^1(D+p) \longrightarrow H^1(k(p)) \longrightarrow 0$$

Note that $H^1(k(p)) = 0$, as $\dim(p) = 0$. Taking the alternating sum here, we find that

$$\begin{aligned} & h^0(D) - h^0(D+P) + 1 - h^0(K-D) + h^1(D+p) = 0 \\ \iff & \underbrace{h^0(D) - h^0(K-D)}_{\chi(\mathcal{L}(D))} + 1 = \underbrace{h^0(D+P) - h^0(K-(D+p))}_{\chi(\mathcal{L}(D+P))} \end{aligned}$$

for K the Weil divisor corresponding to ω_X. Plug in $D = 0$ for

$$1 - h^0(K) + 1 = h^0(p) - h^0(K-p) \quad \Longrightarrow \quad h^0(p) = 2 - h^0(K) + h^0(K-p).$$

Remark 2.2.3. Consider the following remarks:

- $\chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) = 1 - h^0(\omega_X)$
- the *genus* of X is $g = \dim(H^0(\omega_X))$

Hence $\chi(\mathcal{O}_X) = 1 - g$. If D is effective, then it may be obtained from 0 by ading points, which adds 1 to the Euler characteristic. So

$$\chi(D) = 1 - g + \deg(D) \quad \Longrightarrow \quad h^0(D) - h^0(K-D) = 1 - g + \deg(D)$$

This is known as the *Riemann–Roch* theorem for curves. It holds for general divisors D.

✠ *Lecture 29 - 18.11.2013*

Recall the Riemann–Roch theorem for curves: $\chi(D) = \deg(D) + 1 - g$, where $\chi(D) = h^0(D) - h^1(D) = h^0(D) - h^0(K-D)$, and $g = h^0(K)$.

2.3 Worked examples

Example 2.3.1. What are $h^0(D)$ and $h^1(D)$ for a divisor D on $\mathbb{P}^1$?

First, we know that $\omega_p \cong \mathcal{O}(-2)$, so $k = -2p$ for p some point. For any D, we get $h^0(D) - h^1(D) = \deg(D) + 1$, since $g = h^0(K) = h^0(\mathcal{O}(-2)) = 0$. Using Serre duality, we rewrite this as $h^0(D) - h^0(-2p-D) = \deg(D) + 1$. This splits into two cases.

- If $\deg(D) \geqslant -1$, then $h^0(-2p-D) = 0$ implies that $h^0(D) = \deg(D) + 1$
- If $\deg(D) \leqslant -2$, then $h^0(D) = 0$ implies that $h^0(-2p-D) = h^1(D) = -\deg(D) - 1$

Now we know what h^1 and h^0 are for an arbitrary divisor on $\mathbb{P}^1$.

Remark 2.3.2. A rational function is *exactly* a map to $\mathbb{P}^1$, no more, no less.

Remark 2.3.3. What are all the curves of genus $g = 0$? Let X have $g = 0$. Ley $f \in k(X)$ is a rational function on X, and $\varphi : X \to \mathbb{P}^1$ is the corresponding morphism. Consider $\varphi^{-1}([0:1])$. If φ is a morphism, then $\varphi^{-1}([0:1])$ is a single point P. Similarly. $\varphi^{-1}([1:0]) = Q$. Moreover, we must have $\varphi^*([0:1]) = P$ and $\varphi^*([1:0]) = Q$, where $\varphi^* : \mathrm{Cl}(\mathbb{P}^1) \to \mathrm{Cl}(X)$ is the pullback of Weil divisors.

Note that $\varphi = [g : h]$, or $f = g/h$. What is $\mathrm{div}(f)$? It is $P - Q$! So $h^0(P - Q) \geqslant 1$, as $1/f \in \mathcal{L}(P - Q)$. Further, if $g = 0$, then

$$h^0(P-Q) - h^1(P-Q) = \deg(P-Q) + 1 \quad \Longrightarrow \quad h^0(P-Q) = 1 + h^1(P-Q) \geqslant 1.$$

So there exists an $f \in k(X)$ such that $\mathrm{div}(f) = Q - P$, with $Q \neq P$. Let $\varphi : X \to \mathbb{P}^1$ be the morphism corresponding to f. Then $\varphi^*([0:1])$ is a Weil divisor of degree $\deg(vp)$. But $\varphi^*([0:1]) = Q$, so $\deg(\varphi) = 1$.

> **Proposition 2.3.4.** [Poincare duality]
> Let X be a topological space. Then for $k \in \mathbb{Z}$, $H^k(X) \cong H_{\dim(X)-k}(X)$.

✠ *Lecture 30 - 20.11.2013*

Example 2.3.5. Let X be the curve defined by $\{y^z = x^3 - xz^2\} \subset \mathbb{P}^2$. Compute $h^0(nP)$ for any fixed point P on the curve. Let's first write down what Riemann-Roch says about this curve:

$$h^0(nP) - h^1(nP) = \deg(nP) + 1 - g$$

Since $\deg(nP) = n$ and the genus $g = h^0(\omega_X) = h^0(\mathcal{O}_X) = 1$, the right side of the equation is just n. The equation then simplifies to $h^0(nP) - h^0(-nP) = n$. We can divide this into the following situations:
- $\cdot$ $n > 0$: $h^0(nP) = n$ and $h^1(nP) = 0$
- $\cdot$ $n < 0$: $h^0(nP) = 0$ and $h^1(nP) = -n$
- $\cdot$ $n = 0$: $h^0(nP) = 1$ and $h^1(nP) = 1$

If $P \neq Q$, what is $h^0(P+Q)$? Plug this into Riemann–Roch to get

$$h^0(P+Q) - h^1(P+Q) = 2 + 1 - 1 = 2 \quad \Longrightarrow \quad h^0(P+Q) - h^0(-P-Q) = 2 \quad \Longrightarrow \quad h^0(P+Q) = 2.$$

Remark 2.3.6. Let D be a divisor on X. Then $h^0(D) - h^0(-D) = \deg(D)$. Consider all possibilites:

- $\cdot$ if $\deg(D) > 0$, then $h^0(D) = \deg(D)$ and $h^1(D) = 0$
- $\cdot$ if $\deg(D) < 0$, then $h^0(D) = 0$ because $\deg(\mathrm{div}(f) + D) = \deg(D) < 0$, so $D + \mathrm{div}(f)$ is not effective
- $\cdot$ if $\deg(D) = 0$ and $h^0(D) > 0$, then there is some rational function f such that $\mathrm{div}(f) + D$ is effective. Then $D + \mathrm{div}(f) = 0$, so $D = -\mathrm{div}(f) = \mathrm{div}(1/f)$, so $D \equiv 0$. Hence if $\deg(D) = 0$, then either $h^0(D) = 0$ and $h^1(D) = 0$ or $D \equiv 0$ and $h^0(D) = h^1(D) = 1$.

What are the morphisms in each case? When $\deg(D) < 0$, there are no morphisms. When $\deg(D) = 0$, the only morphism is the trivial morphism. When $\deg(D) > 0$, the situation is more complicated.

Bofero we consider that, let $D = 2P$, for which $h^0(D) = 2$, implying $H^0(D)$ is 2-dimensional, so it has a basis $\{f_1, f_2\}$. This allows us to build a rational map $\varphi : X \to \mathbb{P}^1$ by $\varphi(P) = [f_1(P) : f_2(P)]$. Since X is smooth, φ is a morphism. We generalize from this example.

Take any divisor D with $\deg(D) > 0$. If $h^0(D) = 0$, then D does not correspond to a morphism. If $h^0(D) = \deg(D) > 0$, then let $\{f_1, \ldots, f_d\}$ be a basis of $H^0(D)$, and define $\varphi : X \to \mathbb{P}^{d-1}$ by $\varphi(P) = [f_1(P) : \cdots : f_d(P)]$. Since X is smooth, this is a morphism.

Remark 2.3.7.
· Any morphism of positive degree can be made by the described tactic. However, if the degree is 1, it might not correspond to D.

· Every divisor of degree 3 corresponds to a morphism of $\mathbb{P}^2$ that is an embedding.

· Every divisor of degree 3 or higher on an elliptic curve corresponds to an embedding of the curve in projective space.

· The above works for any smooth curve of genus 1 in any $\mathbb{P}^n$.

> **Definition 2.3.8.** An *elliptic curve* is a type of cubic curve constrainde by the equation $y^2 = x^3+ax+b$ for some a, b. Its solution set is topologically equivalent to a torus.

✠ *Lecture 31 - 22.11.2013*

Remark 2.3.9. Do genus 2 curves exist at all?

If the genus is 2, then $h^0(K) = 2$. Let $\varphi : X \to \mathbb{P}^1$ be the associated morphism. It is not constant, so it has degree $\deg(K) = 2$. This follows from Riemann-Roch, as $h^0(K) - h^0(K - K) = \deg(K) + 1 - g$ implies that $\deg(K) = 2g - 2$, where K is any Weil divisor corresponding to the constant sheaf. Such a 2-1 map is called a *hyperelliptic map*. Every 2-1 map from this X to $\mathbb{P}^1$ is this map up to automorphisms of $\mathbb{P}^1$. Let us analyse such a curve further.

- · if $\deg(D) < 0$, then $h^0(D) = 0$
- · if $\deg(D) = 0$, then $h^0(D) = 1$ if $D \equiv 0$ and 0 otherwise. This follows as $\deg(D) = h^0(D) - h^0(K - D) = 1 + 1 - 2 = 0$.
- · if $\deg(D) > 0$, the situation is more complicated

Suppose that $\deg(D) = 1$. If $D \equiv$ pt, then $h^0(D) < 1$, since otherwise $X \cong \mathbb{P}^1$, which is impossible, as $g = 2$. In general, either $D \equiv$ pt, in which case $h^0(D) = 1$, or $D \not\equiv$ pt, in which case $h^0(D) = 0$.

Suppose that $\deg(D) = 2$. Then by Riemann–Roch $h^0(D) - h^0(K - D) = 2 + 1 - 2 = 1$, implying that

$$h^0(D) = 1 + h^0(K - D) = \begin{cases} 1 & \text{if } K - D \not\equiv 0 \\ 2 & \text{if } K - D \equiv 0 \end{cases}.$$

So D corresponds to a morphism iff $D \equiv K$.

Suppose that $\deg(D) = 3$. Then by Riemann–Roch, $h^0(D) - h^0(K - D) = 3 + 1 - 2 = 2$, implying that $h^0(D) = 2$, as $h^0(K - D) = 0$. For this case, let $\psi : X \to \mathbb{P}^1$ be the morphism we build out of D. It is surjective and has degree at least 2, so there are $P, Q \in X$ such that $\psi(P) = \psi(Q)$ (we do not exclude the case $P = Q$). So for all $f \in \mathcal{L}(D)$, $f(P) = f(Q)$. In other words, $h^0(D - P) = h^0(D - P - Q)$. So $h^0(D - P) = h^0(D - P - Q) = 1$, and $D - P - Q = R$, and $D - R \not\equiv K$. For all P, there exists a Q such that this is the case. So if ψ corresponds to D, then $D - K \not\equiv$ pt. So D corresponds to a 3-1 map to $\mathbb{P}^1$. If $D \equiv K+$ pt, then ψ doesn't correspond to D, so ψ is φ.

Suppose that $\deg(D) = 4$. Then $h^0(D) = 3$, and

$$h^0(D - P - Q) = \begin{cases} 1 & \text{if } D - P - Q \not\equiv K \\ 2 & \text{if } D - P - Q \equiv K \end{cases}.$$

Let ψ be a morphism built from D. If ψ corresponds to D (i.e. $\psi^*\mathcal{O}(1) \cong \mathcal{L}(D)$), then ψ is an embedding iff $h^0(D - P) \not\equiv h^0(D - P - Q)$ for all P, Q (including $P = Q$). But $h^0(D) \neq h^0(D - P)$, so ψ does correspond to D. Then is it true that for all $P, Q \in X$, $h^0(D-P) \neq h^0(D-P-Q)$? The answer is no. To see this, choose P, Q so that $D - K \equiv P - Q$, which is possible because $D - K$ is effective, as $h^0(D-K) \geqslant 1$. Note that if $D \not\equiv 2K$, then ψ almost embeds X in $\mathbb{P}^2$ as a singular curve of degree 4, with $\psi(P) = \psi(Q)$, for P, Q satisfying $D - K \equiv P + Q$. If $D \equiv 2K$, then ψ is a 2-1 map onto a conic in $\mathbb{P}^2$.

Suppose that $\deg(D) = 5$. Then we always get an embedding.

3 Schemes

3.1 Definitions

✠ *Lecture 32 - 25.11.2013*

Definition 3.1.1. Let A be a commutative ring with identity. Let $\operatorname{Spec}(A)$ be the set of prime ideals of A. We put the Zariski topology on A, by defining the closed sets to be, for any ideal $I \subset A$,

$$V(A) = \{\text{prime ideals containing } I\}$$

and finite unions and arbitrary intersections thereof. Note that as usual, the Zariski topology has open sets exactly the sets whose complements are finite, a type of cofinite topology.

> An ideal generated by an element x will be denoted by (x). Recall more definitions from ring theory:
>
> **Definition 3.1.2.** Let I be a non-trivial ideal of R (i.e. $I \neq R$) with no ideal J of R with $I \subsetneq J \subsetneq R$. Then I is termed a *maximal ideal* of R.
>
> Let R be a commutative ring with an ideal I. If for all $a, b \in R$ with $ab \in I$ either $a \in I$ or $b \in I$, then I is termed a *prime ideal* of R.

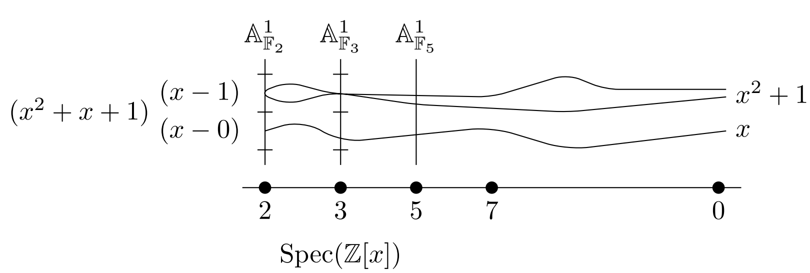
Example 3.1.3. Consider the following examples:

· Let $A = \mathbb{C}$. Then $\operatorname{Spec}(A) = \{0\}$ has one point, and is endowed with the unique topology.
· Let $A = \mathbb{C}[t]$. Then $\operatorname{Spec}(A) = \{(t-a), (0)\}$, where the first is closed and the second is not closed.

In general, if $Z(f_1, \ldots, f_r) \subset \mathbb{A}^n_{\mathbb{C}}$ is an affine variety, then the closed points of $\operatorname{Spec}(\mathbb{C}[t_1, \ldots, t_n])/(f_1, \ldots, f_r)$ are in a 1-1 correspondence with the points of $\mathbb{Z}(f_1, \ldots, f_r)$. That is, a closed point of $\operatorname{Spec}(A)$ is a maximal ideal of A.

· Let $A = \mathbb{C}[t]/(t^2)$. Then $\operatorname{Spec}(A) = \{(t)\}$.
· Let $A = \mathbb{C} \oplus \mathbb{C}$. Then $\operatorname{Spec}(A) = \{0 \oplus \mathbb{C}, \mathbb{C} \oplus 0\}$ with the discrete topology, as all points in it are closed.
· Let $A = \mathbb{Z}$. Then $\operatorname{Spec}(A) = \{(0), (P)\}$ for every positive prime P.
· Let $A = \mathbb{Z}[x]$. Then $\operatorname{Spec}(A) = \{(0), (f(x)), (P), (P, f(x))\}$ for P as above and every irreducible f.
· Let $A = \mathbb{Z}[x]/(f(x)) \cong \mathbb{Z}[\alpha] \subset \mathbb{C}$ for α a root of f. This is more complicated.

Consider the last example above. We have that $(x - a, f(x)) \subset \mathbb{Z}[x]$ for some $a \in \mathbb{Z}$ and $f(x) = a_0 + a_1 x + \cdots + a_n x^n$. Suppose that $P \mid a_0$. Then $(f(x)) \subset (x, P)$. This allows us to draw $\operatorname{Spec}(A)$ in a sort of diagram, which will not be explained:



✠ *Lecture 33 - 27.11.2013*

Remark 3.1.4. Consider an affine variety $Z(I) \subset \mathbb{A}^n$ that is the zero set of some ideal. The coordinate ring is $k[x_1, \dots, x_n]/I = A$. A prime ideal P of A corresponds to an irreducible subvariety Y of Z. And a point $x \in Z$ lies on Y iff $I(\{x\})$ contains P. So $x \in Y$ iff $I(\{x\}) \in V(P)$.

Note that $V(P)$ consists of all the prime ideals that correspond to irreducible subvarieties of Y. Further, closed points of $\text{Spec}(A)$ correspond to actual points. Open points correspond to subvarieties.

Definition 3.1.5. Let A be a commutative ring and $X = \text{Spec}(A)$. Define a sheaf $\mathcal{O}_X$ on X by $\mathcal{O}_X(X - V(f)) = A_{(f)}$ (for f non-zero), which equals $A[1/f]$ if A is a domain or f is not a zero divisor. Note that $V(f)$ represents V of the principal ideal generated by f. For the inclusion maps, note that

$$\begin{aligned} X - V(g) \subset X - V(f) &\text{ iff } V(f) \subset V(g) \\ &\text{ iff } g \in \sqrt{(f)} \\ &\text{ iff } g = fh \text{ for some } h, \end{aligned}$$

so res : $A_{(f)} \to A_{(g)}$ is the natural inclusion map. In particular, it is functorial, so it plays nice with the structure of the sheaf. The stalk of $\mathcal{O}_X$ at $P \in X$ is $\varinjlim_{f \not\in P}[A[1/f]] = A_P$, which is a local ring. When $P = (0)$, then A_P is the fraction field of A, or X.

Example 3.1.6. Let $X = \text{Spec}(\mathbb{Z})$. Then $A_{(P)} = \mathbb{Z}?(P) = \{\frac{a}{b} \ : \ P \nmid b\}$, and $A_{(0)} \cong \mathbb{Q}$. So 3/4 would be a rational function on $\text{Spec}(\mathbb{Z})$ with a zero at 3 and a pole of order 2 at 2. Note that 2 is a uniformizer for the local ring $\mathbb{Z}_2$. That is, any $\frac{a}{b} \in \{\frac{a}{b} \ : \ 2 \nmid b\}$ is $2^n \frac{a'}{b'}$, where $2 \nmid a'b'$. Then $\text{ord}_{(2)}(3/4) = -2$ because $\frac{3}{4} = 2^{-2}\frac{3}{1}$.

Remark 3.1.7. The "value" of a "rational function" on $\text{Spec}(A)$ at a point P is an element of the residue field A_P/P. This is an affine structure of the form $(\text{Spec}(A), \mathcal{O}_X)$. Using this we can make an identification between schemes and manifolds.

✠ *Lecture 34 - 29.11.2013*

Let A be a commutative ring with identity and $\text{Spec}(A) = \{P \subset A \text{ is a prime ideal}\}$. Note that (1) is not an ideal. Let $\mathcal{O}_X$ be the sheaf of regular functions on $X = \text{Spec}(A)$, and $(\mathcal{O}_X)_P = A_P$, so the stalk of $\mathcal{O}_X$ at P is the ring localized at the point P. Recall that

$$\mathcal{O}_X(U) = \{f \ : \ f \text{ is a function defined at all } P \in U\} = \bigcap_{P \in U} A_P.$$

If $U = \text{Spec}(A) - V(I)$ (and A is a domain), then $\mathcal{O}_X(U) = \bigcap_{P \not\supset I} A_P \subset \mathcal{K}(A)$. If $I = (f)$, then $\mathcal{O}_X(U) = \bigcap_{P \not\ni f} A_P = A[1/f]$.

Example 3.1.8. The spaces $X = \text{Spec}(\mathbb{R})$ and $Y = \text{Spec}(Y)$ are single points as topological spaces, as they are both fields, andfields have only 1 prime ideal. But $\mathcal{O}_X \not\cong \mathcal{O}_Y$ as sheaves on that point, as $\mathcal{O}_X(\text{pt}) = \mathbb{R}$ and $\mathcal{O}_Y(\text{pt}) = \mathbb{C}$.

Also, $Z = \text{Spec}(\mathbb{C}[t])/(t^2)$ (essentially describes Taylor series up to order 1) is topologically a single point.

Note that an affine scheme to a scheme is like an n-dimensional ball to an n-dimensional manifold.

Definition 3.1.9. A *ringed space* is a topological space X with a sheaf of rings $\mathcal{O}_X$. A *locally ringed space* is a ringed space for which the stalks $(\mathcal{O}_X)_P$ are local rings, for every $P \in X$.

A *morphism of ringed spaces* from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$, where $f : X \to Y$ is continuous and $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves, where $(f_*\mathcal{O}_X)(U) = \mathcal{O}_X(f^{-1}(U))$.

A *morphism of locally ringed spaces* from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a morphism of ringed spaces $(f, f^\#)$ such that the induced map on stalks $f_P^\# : (\mathcal{O}_Y)_P \to (f_*\mathcal{O}_X)_P$ satisfies $(f_\#^P)^{-1}(f_*\mathcal{M}_P) = \mathcal{M}_{f(P)}$, where $\mathcal{M}$ denotes the maximal ideal.

Example 3.1.10. The pair $(\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)})$ is a locally ringed space.

Definition 3.1.11. A locally ringed space $(X, \mathcal{O}_X)$ is a *scheme* iff every point $P \in X$ has a neighborhood $(U, \mathcal{O}_X|_U)$ that is locally isomorphic to $(\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)})$ for some ring A. Note that A does not have to be the same for all $P \in X$.

3.2 Fundamental construction of schemes

✠ *Lecture 35 - 2.12.2013*

Remark 3.2.1. Recall some definitions:

$$\begin{aligned}\text{Spec}(A) = \{P \subset A \text{ prime ideals}\} &\longleftrightarrow \text{ affine sheaves } \longleftrightarrow \text{ affine varieties}\\ \text{Proj}(A) = \{P \subset A \text{ homogeneous prime ideals, } A \not\subset P\} &\longleftrightarrow \text{ projective schemes } \longleftrightarrow \text{ projective varieties}\end{aligned}$$

The ring A is a graded ring, so $A = A_0 \oplus A_1 \oplus \cdots$ for abelian groups A_i with $A_i A_j \subset A_{i+j}$. An element $\alpha \in A$ is homogeneous iff $\alpha \in A_i$ for some i. An ideal $I \subset A$ is homogeneous iff it is generated by homogeneous elements.

The ideal $A_+ = A_1 \oplus \cdots$ is the *irrelevant ideal*. It does not correspond to a subset of projective space. Let $V(I) = \{P \in \text{Proj}(A) \ : \ I \subset P\}$. The Zariski topology on $\text{Proj}(A)$ is the one for which $\{V(I)\}$ is the set of closed sets.

Definition 3.2.2. To make $\text{Proj}(A)$ a scheme, we need $\mathcal{O}_X$, a sheaf of rings. Let $P \in \text{Proj}(A)$. Define $A_{(P)}$ to be the degree-0 piece of the homogeneous localization of A at P. If A is a domain, then $A_{(P)} = \{\frac{f}{g} \ : \ f, g \in A_i$ for some i, $g \notin P\}$. Say $U \subset X$ is open, and $U \neq \emptyset$. Then define

$$\mathcal{O}_X(U) = \left\{ f : U \to \bigsqcup_{P \in U} A_{(P)} \ : \ f(P) \in A_{(P)}, \ f \text{ is locally a quotient} \right\}.$$

The map f is *locally a quotient* iff for every $P \in U$ there is some neighborhood V of P with $V \subset U$ and $a, b \in A$ homogeneous of the same degree such that for all $Q \in V$, $b \notin Q$ and $f(Q) = \frac{a}{b} \in A_{(Q)}$.

The above is a very general construction of a sheaf. The stalks are $A_{(P)}$. Note that X with such a sheaf construction is a locally ringed space.

Definition 3.2.3. In order to show $\text{Proj}(A)$ is coherent, every point must have a neighborhood that is locally isomorphic to $\text{Spec}(A)$. So let $f \in A$ be non-zero and homogeneous. Define

$$D_+(f) = \{P \in \text{Proj}(A) \ : \ f \notin P\}.$$

Note that $D_+(f) \cong \text{Spec}(A_{(f)})$, where $A_{(f)}$ is the degree-0 piece of the homogeneous localization of A at (f), which is $A[1/f]$.

Example 3.2.4. Note that $\mathbb{P}^1 = \text{Proj}(k[t, u])$ so

$$\begin{aligned} k[t,u] &= k \oplus (kt + ku) \oplus (kt^2 + ktu + ku^2) \oplus \cdots \\ \text{and} \quad k[t,u]_{(t)} &= k \oplus k(u/t) \oplus k(u/t)^2 \oplus \cdots \\ &= k[u/t]. \end{aligned}$$

These are all related to $\mathcal{O}(1)$.

Example 3.2.5. There are also examples related to $\mathcal{O}(2)$, for example

$$k[x,y,z]/(xy - z^2) = k \oplus (kx + ky + kz) \oplus (kx^2 + ky^2 + kz^2 + kxz + kyz) \oplus \cdots$$

4 Exercises

4.1 Exercise 1

Question

Let C be the curve $y^2z = x^3 - xz^2$ in $\mathbb{P}^2_{\mathbb{C}}$, and let $p = [0 : 0 : 1] \in C$. Find a morphism from C to projective space corresponding to the rational divisor $2p$. Remember to prove that it's a morphism, and not just a rational map.

Answer

First, since a divisor has codimension 1, the morphism will be to $\mathbb{P}^1$, where the subvarieties of codimension 1 are points. Therefore we are looking for homogeneous polynomials f, g such that $\varphi : [x : y : z] \mapsto [f : g]$ is our map.

Next, we note that p is on the curve C, and the tangent line to C at P will have multiplicity (at least) 2 at p, so we use that. We dehomogenize at p to get $y^2 - x^2 + x = 0$, and the tangent line is the degree-one part, so the tangent line to C at p is $x = 0$, denoted as the curve C'.

By Bezout's theorem, C' and C intersect at $1 \cdot 3 = 3$ places, and at least 2 of them are at p (because of the tangency). We note that for C, if $x = 0$, then $y = 0$ or $z = 0$, so C' intersects C also at $[0 : 1 : 0] = q$. Therefore the divisor D' of C' is given by $D' = 2p + q$.

Let us try the map for which f is x and g is z, so $\varphi : [x : y : z] \mapsto [x : z]$. We choose x to be the first coordinate, because that will give us the divisor $2p$. The second coordinate is z because that vanishes at q once. It remains to check that it is a morphism. Observe that this presentation is fine on $\mathbb{P}^2 - [0 : 1 : 0]$, or all points on C except $[0 : 1 : 0]$, where $[x : z]$ is not defined. We rearrange the equation for C to get

$$\frac{x}{z} = \frac{y^2}{x^2 - z^2},$$

so $\varphi : [x : y : z] \mapsto [y^2 : x^2 - z^2]$ on $\mathbb{P}^2 - \{y = 0\}$ is an expression that is defined on $[0 : 1 : 0]$ and is compatible with $[x : z]$ everywhere else, exactly because of the rearrangement.

What is the divisor of φ? In the first representation, the first coordinate is zero twice at p. In the second representation, the first coordinate does not vanish.

Note that the divisor changes if we pick a different representation. However, the divisor class is independent of the representation, so the "different" representations will all be linearly equivalent. So the map φ is a morphism with divisor $2p$. ■

4.2 Exercise 2

Question

Prove that $\text{Pic}(\mathbb{P}^n)$ is isomorphic to $\mathbb{Z}$ for $n \geqslant 1$, and for each element of $\text{Pic}(\mathbb{P}^n)$ that corresponds to a morphism, find a corresponding morphism.

Answer

To prove that the two spaces are isomorphic, we construct an appropriate isomorphism. First we define formally the Picard group of $\mathbb{P}^n$ as

$$\text{Pic}(\mathbb{P}^n) = (\text{divisors of } \mathbb{P}^n) \Big/ (\text{principal divisors of } \mathbb{P}^n).$$

Note that a divisor in $\mathbb{P}^n$ may be expressed as a finite sum $\sum(\text{integer}) \cdot (\text{irreducible divisor}) = \sum a_i D_i$, where $\deg(D_i) = d_i$. Further, there is some homogeneous polynomial p_i of degree d_i such that $\text{div}(p_i) = D_i$. So for the subvariety $x_0 = 0$ with divisor D', we have that

$$D_i - d_i H = \text{div}\left(\frac{p_i}{x_0^{d_i}}\right)$$

This means that D_i is linearly equivalent to $d_i H$. So consider the map φ from $\text{Div}(\mathbb{P}^n)$ to $\mathbb{Z}$ given by

$$\begin{array}{rccc} \varphi : & (\text{divisors of } \mathbb{P}^n) & \to & \mathbb{Z} \\ & \sum a_i D_i & \mapsto & \sum a_i d_i \end{array}.$$

This map is a homomorphism - surjectivity is given by a map with a divisor that's just a point, since $\mathbb{Z}$ is 1-dimensional. Hence $\text{Im}(\varphi) \cong \mathbb{Z}$.

Now recall that a principal divisor of $\mathbb{P}^n$ is one that is a divisor of a rational function. Rational divisors have homogeneous polynomials of the same degree in the numerator and the denominator, so the sum of the coefficients of the terms in the divisor add up to zero (i.e. such functions have the same number of zeros as poles). Therefore if f is a principal divisor of $\mathbb{P}^n$, then it is in the kernel of φ.

For the other direction, some more work needs to be done to show $\ker(\varphi) \subset \text{Prin}(P^n)$. Now the kernel of φ is exactly the principal divisors of $\mathbb{P}^n$. Then the first isomorphism theorem gives us the second-last isomorphism below:

$$\text{Pic}(\mathbb{P}^n) = \text{Div}(\mathbb{P}^n)/\text{Prin}(\mathbb{P}^n) \cong \text{Div}(\mathbb{P}^n)/\ker(\varphi) \cong \text{Im}(\varphi) \cong \mathbb{Z}$$

Therefore φ is an isomorphism between $\text{Pic}(\mathbb{P}^n)$ and $\mathbb{Z}$. ■

4.3 Exercise 3

Question

Let S be the graded ring $k[x,y,z]/(y^2z - x^3 + xz^2)$, and let M be the graded S-module $S[1]$. Prove that the sheaf $\tilde{M}$ on the curve X given by $y^2z = x^3 - xz^2$ associated to M is an invertible sheaf, and find a Weil divisor D that is associated to $\tilde{M}$.

Answer

Let $\tilde{M}$ be the sheaf associated to M, and M' the presheaf associated to M. Then

$$M'(U) = \left\{ \frac{m}{s} \ : \ m \in M, s \in S, \deg(m) = \deg(s), s \text{ is non-vanishing on } U \right\}.$$

For any point p on the curve X, if the stalk $\tilde{M}_p$ is a free rank-one module over the local ring $\mathcal{O}_p$, then $\tilde{M}$ will be invertible, by definition of invertibility. As the stalks of $\tilde{M}$ and M' are identical, we have that

$$\begin{aligned}\tilde{M}_p = M'_p &= \left\{ \frac{m}{s} \ : \ m \in M, s \in S, \deg(m) = \deg(s), s(p) \neq 0 \right\} \\ &= \left\{ \frac{m}{s} \ : \ m, s \in S, \deg(m) = \deg(s) + 1, s(p) \neq 0 \right\}.\end{aligned}$$

For $p \neq [0:1:0]$, note that $z\mathcal{O}_p \subseteq \tilde{M}_p$, as $a \in z\mathcal{O}_p$ implies $a = zf/g$, for $f, g \in S$ of the same degree. Then $\deg(zf) = \deg(g) + 1$, so $a \in \tilde{M}_p$. Further, $\tilde{M}_p \subseteq z\mathcal{O}_p$, as $b \in \tilde{M}_p$ implies $b = h/k$, for $\deg(h) = \deg(k) + 1$. Then $b = zh/zk = z(h/zk) \in z\mathcal{O}_p$, with $\deg(zh) = \deg(zk) + 1$. If $p = [0:1:0]$, we have that $y\mathcal{O}_p = \tilde{M}_p$, proved in the same manner. Hence $\tilde{M}_p = z\mathcal{O}_p$, so $\tilde{M}$ is invertible.

It remains to find a Weil divisor associated to $\tilde{M}$. We let the map φ that embeds $\tilde{M}$ in the constant sheaf be "division" by z, as z is a local generator OR IS IT? WHY Z?. As mentioned above, z will not work at the point $[0:1:0]$ (note that z vanishes only at $[0:1:0]$), so there we take the generator y, also as mentioned. Hence we have the Cartier divisor

$$\left\{ (1, X - [0:1:0]), \left(\frac{z}{y}, X - \{y = 0\} \right) \right\}.$$

Since z/y has a zero at $[0:1:0]$, the corresponding Weil divisor is $d[0:1:0]$, where d is the order of vanishing of z/y at $[0:1:0]$. Note that the tangent line to X at $[0:1:0]$ is $z = 0$ (by dehomogenizing the curve at $[0:1:0]$ and taking the linear part), so neither $x = 0$, nor $x + z = 0$ nor $x - z = 0$ are tangent to X at p. As

$$\frac{z}{y} = \frac{x}{y} \cdot \frac{x-z}{y} \cdot \frac{x+z}{y},$$

it follows that z/y vanishes to order 3 at $[0:1:0]$, so $d = 3$. We had to break up the expression z/y into factors, since z/y is not defined at $[0:1:0]$. Hence the Weil divisor of $\tilde{M}$ is $3[0:1:0]$. ■

4.4 Exercise 4

Question

Compute the canonical sheaf of $\mathbb{P}^1$.

Answer

Let $\mathbb{P}^1$ be given in x, y, and its canonical sheaf by $\bigwedge^1 \Omega_{\mathbb{P}^1} = \Omega_{\mathbb{P}^1}$. Consider the differential $\omega = d(x/y)$. We use ω to embed $\Omega_{\mathbb{P}^1}$ into the constant sheaf, by the map

$$\begin{array}{rccc} \varphi: & \Omega_{\mathbb{P}^1} & \to & K \\ & \alpha & \mapsto & f \end{array}$$

where f is the unique function such that $\alpha = f\omega$. As $\Omega_{\mathbb{P}^1}$ is an invertible sheaf, any two non-zero elements are related by a rational function, so f exists and is unique. Now find local generators for $\Omega_{\mathbb{P}^1}$. Note the whole space may be covered by two open sets on which $y \neq 0$ and $x \neq 0$, respectively. For φ acting on α,

- · on $\{y \neq 0\}$: $d(x/y) = d(x/1) = dx$, so ω generates the stalk $\mathcal{O}_{\mathbb{P}^1}$ at α
- · on $\{x \neq 0\}$: dy generates the stalk $\mathcal{O}_{\mathbb{P}^1}$ at α

Note that the only point where $y = 0$ is $[1 : 0]$.

Then if $xy \neq 0$, ω will generate the stalk of $\Omega_{\mathbb{P}^1}$ as an $\mathcal{O}_{\mathbb{P}^1}$-module because $\frac{x}{y} - \frac{x_0}{y_0} = \frac{xy_0 - yx_0}{yy_0}$ is a uniformizer at $[x_0 : y_0]$ if $x_0 y_0 \neq 0$. That is, it vanishes to order one at $[x_0 : y_0]$. We may immediately generate the associated Cartier divisor.

$$\cdot \text{ on } \{y \neq 0\}\text{: } \left(\frac{dx}{\omega}\right)^{-1} = \left(\frac{dx}{d(x/1)}\right)^{-1} = \left(\frac{dx}{dx}\right)^{-1} = 1$$

$$\cdot \text{ on } \{x \neq 0\}\text{: } \left(\frac{dy}{\omega}\right)^{-1} = \left(\frac{dy}{d(1/y)}\right)^{-1} = \left(\frac{dy}{-dy/y^2}\right)^{-1} = \left(-y^2\right)^{-1} = -\frac{1}{y^2}$$

So the Cartier divisor is given by

$$\left\{ \left(1, \mathbb{P}^1 - [1:0]\right), \left(-\frac{1}{y^2}, \mathbb{P}^1 - [0:1]\right) \right\}$$

The associated Weil divisor is then clear: 1 has no zeros or poles, and $-1/y^2$ vanishes to order -2 at $[1 : 0]$. Hence the associated Weil divisor is $D = -2[1 : 0]$. Since $\deg(D) = -2$, it follows that $\Omega_{\mathbb{P}^1} \cong \mathcal{O}(-2)$. ■

4.5 Exercise 5

Question

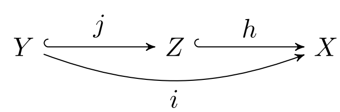
Find a Weil divisor corresponding to the canonical sheaf for the twisted cubic curve Y in $\mathbb{P}^3$, given as the solution set of the equations $xz = y^2$, $xw = yz$, $yw = z^2$. [You may assume that the canonical sheaf of a smooth subvariety of $\mathbb{P}^n$ defined by a single equation of degree d is isomorphic to $i^*\mathcal{O}(d-n-1)$, where i is the inclusion map.]

Answer

The Weil divisor is given by

$$\begin{aligned} \omega_Y &\cong i^*(\mathcal{I}_Y/\mathcal{I}_Y^2)^\vee \\ &\cong j^*\omega_Z \otimes (\mathcal{J}_Y/\mathcal{J}_Y^2)^\vee \\ &\cong j^*h^*\omega_X \otimes (\mathcal{J}_Y/\mathcal{J}_Y^2)^\vee \\ &\cong j^*h^*\mathcal{O}(-2) \otimes (\mathcal{J}_Y/\mathcal{J}_Y^2)^\vee \end{aligned} \qquad \text{where} \qquad \begin{aligned} \omega_Z &\cong h^*\omega_X \otimes (\mathcal{I}_Z/\mathcal{I}_Z^2)^\vee \\ &\cong h^*\mathcal{O}(-2). \end{aligned}$$

The variety Z is the solution set of the equation $xw = yz$. Note that $\omega_Z \cong h^*\mathcal{O}(-2)$ from the previous question. The relations among the maps i, j, h is as below.



We look for a non-zero element in $\mathcal{J}_Y/\mathcal{J}_Y^2$. Such an element is $\frac{y^2-xz}{x^2} + \mathcal{J}_Y^2$ on $\{x \neq 0\}$. Therefore, our map $\varphi : \mathcal{J}_Y/\mathcal{J}_Y^2 \to K$ will be "division" by this element.

When are $xw = yz$ and $y^2 = xz$ enough to define Y near p? That is, where is this a local generator for $\mathcal{J}_Y/\mathcal{J}_Y^2$? Well, this will happen anwhere that $\frac{z^2-yw}{x^2}$ is in the ring $\mathcal{O}_{Z,p}\left[\frac{y^2-xz}{x^2}\right]$. This gives

$$xw = yz \implies w = yz/x \implies \frac{z^2-yw}{x^2} = \frac{z^2 - y(yz/x)}{x^2} = \frac{z^2x - y^2z}{x} = \frac{-z(y^2-xz)}{x^3} = \frac{-z}{x}\left(\frac{y^2-xz}{x^2}\right)$$

So now we have the function that multiplies this non-zero element to get $xw = yz$. So the local generator is $\frac{y^2-xz}{x^2} + \mathcal{J}_Y^2$ on $\{x \neq 0\}$. Note that the only point on Y where $x = 0$ is $[0:0:0:1]$. As $w \neq 0$ here, we consider the element $\frac{z^2-yw}{w^2} + \mathcal{J}_Y^2$. Then

$$x = yz/w \implies \frac{y^2-xz}{w^2} = \frac{y^2 - (yz/w)z}{w^2} = \frac{y^2w - yz^2}{w^3} = \frac{-y}{w}\left(\frac{z^2-yw}{w^2}\right)$$

Here also we have the function that multiplies this non-zero element to get $xw = yz$, so the local generator is $\frac{z^2-yw}{w^2} + \mathcal{J}_Y^2$ on $\{w \neq 0\}$. To find where φ maps this element, we divide by the first element:

$$\frac{\frac{z^2-yw}{w^2}}{\frac{y^2-xz}{x^2}} = \frac{x^2z^2 - ywx^2}{w^2y^2 - w^2xz} = \frac{x^2z^2 - y\left(\frac{yz}{x}\right)x^2}{\left(\frac{yz}{x}\right)^2 y^2 - \left(\frac{yz}{x}\right)^2 xz} = \frac{x^4z^2 - y^2zx^3}{y^4z^2 - y^2z^3x} = \frac{-x^3}{y^2z}\left(\frac{y^2z - z^2x}{y^2z - z^2x}\right) = \frac{-x^3}{y^2z} = \frac{-x^2}{yw}$$

Hence the desired Cartier divisor is

$$\left\{(1, \{x \neq 0\}), \left(\frac{-yw}{x^2}, \{w \neq 0\}\right)\right\}$$

and the associated Weil divisor is

$$\mathrm{div}(y) - \mathrm{div}(x^2) = \mathrm{div}(y) - 2\mathrm{div}(x).$$

As $w \neq 0$, the Weil divisor will be some multiple of the point $[0:0:0:1]$. We calculate the tangent line to Y at $[0:0:0:1]$ by finding the Jacobian:

$$J = \begin{pmatrix} -z & 2y & -x & 0 \\ w & -z & -y & x \\ 0 & -w & 2z & -y \end{pmatrix} \qquad \text{and} \qquad J_{[0:0:0:1]} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Hence both $x = 0$ and $y = 0$ contain the tangent line at $p = [0:0:0:1]$, but $z = 0$ does not, so $\mathrm{ord}_p(z) = 1$. This allows us to calculate the orders at p:

$$y = \frac{z^2}{w} \implies \mathrm{ord}_p(y) = 2 \qquad \text{and} \qquad x = \frac{y^2}{z} \implies \mathrm{ord}_p(x) = 4 - 1 = 3.$$

So our Weil divisor is $-4[0:0:0:1]$ for $\mathcal{J}_Y/\mathcal{J}_Y^2$. So $\omega_Y \cong i^*\mathcal{O}(-2) \otimes \mathcal{L}(4[0:0:0:1])$ and

$$-2(x=0) + 4[0:0:0:1] = -6[0:0:0:1] + 4[0:0:0:1] = -2[0:0:0:1]$$

Therefore the Weil divisor for ω_Y is $-2[0:0:0:1]$. ■

4.6 Exercise 6

Question

Assuming the Serre duality theorem and general theory of cohomology of sheaves, prove the Riemann-Roch Theorem for curves. That is, prove that for an arbitrary Weil divisor D on a smooth projective curve X, we have $h^0(D) - h^0(K-D) = \deg(D) + 1 - g$, where K is any Weil divisor representing the canonical sheaf, and $g = h^0(K)$ is the genus of the curve.

Answer

This question will be done by induction on D. Start with $D = 0$, for which

$$h^0(D) = \dim(H^0(\mathcal{L}(D))) = \dim(\{f \in k(X)^* \ : \ \operatorname{div}(f) + D \geqslant 0\} \cup \{0\}) = \dim(\{f \in k(X)^* \ : \ \operatorname{div}(f) \geqslant 0\} \cup \{0\}) = \dim(k) = 1.$$

As $\deg(D) = 0$, the desired equation then reduces to $h^0(K) = g$, which is true by definition of g, so the base case is complete. Now let D be a non-zero Weil divisor on X. Let p be a point in X, and consider the sequence

$$0 \longrightarrow \mathcal{I}_p \xrightarrow{\ \iota\ } \mathcal{O}_X \xrightarrow{\ \pi\ } \mathcal{O}_p \longrightarrow 0$$

where $\pi : \mathcal{O}_X(U) \to \mathcal{O}_p(U)$ is defined by $(f : \mathcal{O}_X(U) \to k) \mapsto f(p)$ if $p \in U$ and 0 if $p \notin U$. This map is surjective because $\mathcal{O}_p$ is the skyscraper sheaf, or simply k, at p, and there is at least one non-zero function f at p. The sequence would then be exact if in place of $\mathcal{I}_p$ we had $\ker(\pi)$, by letting ι be the standard inclusion. However, note the following two cases:

$$p \notin U: \qquad \begin{aligned} (\ker(\pi))(U) &= \{g \in \mathcal{O}_X(U) \ : \ \pi_U(g) = 0\} = \mathcal{O}_X(U) \\ \mathcal{I}_p(U) &= \mathcal{O}_X(U) \end{aligned}$$

$$p \in U: \qquad \begin{aligned} (\ker(\pi))(U) &= \{g \in \mathcal{O}_X(U) \ : \ \pi_U(g) = 0\} \\ &= \{g \in \mathcal{O}_X(U) \ : \ g(p) = 0\} \\ \mathcal{I}_p(U) &= \{g \in \mathcal{O}_X(U) \ : \ g(p) = 0\} \end{aligned}$$

Hence $\ker(\psi) = \mathcal{I}_p$, so ι is injective, and the sequence is indeed short exact. By the above and by noting that $\mathcal{I}_p \cong \mathcal{L}(-p)$ by definition, this sequence simplifies to

$$0 \longrightarrow \mathcal{L}(-p) \longrightarrow \mathcal{O}_X \longrightarrow k(p) \longrightarrow 0$$

Tensor the sequence with $\mathcal{L}(D)$ to get a new sequence:

$$0 \longrightarrow \mathcal{L}(-p) \otimes \mathcal{L}(D) \longrightarrow \mathcal{O}_X \otimes \mathcal{L}(D) \longrightarrow k(p) \otimes \mathcal{L}(D) \longrightarrow 0$$

This sequence is still exact because it is exact at the stalks. The stalks of $\mathcal{L}(D)$ are isomorphic to the stalks of $\mathcal{O}_X$, as $\mathcal{L}(D)$ is an invertible sheaf. Note that $\mathcal{L}(-p) \otimes \mathcal{L}(D) \cong \mathcal{L}(D-p)$ and $\mathcal{O}_X \otimes \mathcal{L}(D) \cong \mathcal{L}(D)$ by the nature of $\mathcal{O}_X$. Further, $k(p) \otimes \mathcal{L}(D) \cong k(p)$ as we can make the sets U small enough so that $\mathcal{L}(D)$ looks like $\mathcal{O}_X$ on them, and use the aforementioned nature of $\mathcal{O}_X$ as the structure sheaf. Hence we get equivalently:

$$0 \longrightarrow \mathcal{L}(D-p) \longrightarrow \mathcal{L}(D) \longrightarrow k(p) \longrightarrow 0$$

Now use the zig-zag lemma to get an exact sequence:

$$0 \longrightarrow H^0(D-p) \longrightarrow H^0(D) \longrightarrow H^0(k(p)) \longrightarrow H^1(D-p) \longrightarrow H^1(D) \longrightarrow H^1(k(p)) \longrightarrow 0$$

Since the alternating sum of the dimensions of the spaces in an exact sequences is zero, We have that

$$h^0(D-p) - h^0(D) + h^0(k(p)) - h^1(D-p) + h^1(D) - h^1(k(p)) = 0$$

Note that $h^0(k(p)) = \dim(k) = 1$ and $h^1(k(p)) = 0$ since the support of $k(p)$ is a p, a 0-dimensional variety. Hence the above simplifies to the following:

$$h^0(D-p) - h^1(D-p) = h^0(D) - h^1(D) - 1$$

Now suppose that Riemann–Roch holds for D. Add the above to RR:

$$\begin{aligned} h^0(D) - h^1(D) + (h^0(D-p) - h^1(D-p)) &= \deg(D) + 1 - g + (h^0(D) - h^1(D) - 1) \\ \iff h^0(D-p) - h^1(D-p) &= (\deg(D) - 1) + 1 - g \\ \iff h^0(D-p) - h^1(D-p) &= \deg(D-p) + 1 - g \end{aligned}$$

So RR holds for $D-p$. Now suppose RR holds for $D-p$, and subtract the same equation:

$$\begin{aligned} h^0(D-p) - h^1(D-p) - (h^0(D-p) - h^1(D-p)) &= \deg(D-p) + 1 - g - (h^0(D) - h^1(D) - 1) \\ \iff h^0(D) - h^1(D) &= (\deg(D-p) + 1) + 1 - g \\ \iff h^0(D) - h^1(D) &= \deg(D) + 1 - g \end{aligned}$$

Hence RR holds for D. Since every Weil divisor is a finite sum of points, by induction RR holds for all Weil divisors D of curves. ■

4.7 Exercise 7

Question

For any divisor D on a smooth projective curve X of genus one, compute $h^0(D)$ and $h^1(D)$.

Answer

Since X is a genus 1 curve, $g = 1$. To make the calculations easier, consider first Riemann–Roch for $D = K$:

$$\underbrace{h^0(K)}_{=1} - \underbrace{h^0(K-K)}_{=h^0(0)=1} = \deg(K) + 1 - 1 \quad \Longrightarrow \quad \deg(K) = 0$$

This follows as $h^0(K) = g$ by definition. Since the degree of the constant sheaf is 0, we must have that $K = 0$. Then by Serre duality, specialized to curves, we have that $h^0(D) = h^1(-D)$. So we only need to compute $h^0(D)$. There are three cases to do, one each for $\deg(D) < 0, = 0, > 0$.

If $\deg(D) > 0$, then by Riemann-Roch we have that

$$h^0(D) - \underbrace{h^0(K-D)}_{=0} = \deg(K) + \underbrace{1-g}_{=0} \quad \Longrightarrow \quad H^0(D) = \deg(D)$$

by Kodaira vanishing.

If $\deg(D) < 0$, then by Kodaira vanishing ($h^1(-D) = 0$) and Serre duality ($h^0(D) = h^1(-D)$), $h^0(D) = 0$.

If $\deg(D) = 0$ and $D \equiv 0$, then by Riemann–Roch, $h^0(D) - h^0(K) = \deg(D) + 1 - g$, or $h^0(D) = 1$. If $D \not\equiv 0$, then as $H^0(D) = \{f \in k(X)^* \ : \ \operatorname{div}(f) + D \geqslant 0\}$, a function $f \in H^0(D)$ is rational, so

$$\deg(\operatorname{div}(f) + D)) = \deg(\operatorname{div}(f)) + \deg(D) = 0 + 0 = 0.$$

So since $\operatorname{div}(f) + D = \sum a_p p$ is degree zero with $a_p \geqslant 0$, we must have that $a_p = 0$ for all a_p. Therefore

$$\operatorname{div}(f) + D = 0 \quad \Longleftrightarrow \quad D = -\operatorname{div}(f) \quad \Longleftrightarrow \quad D = \operatorname{div}(1/f)$$

which implies that D is linearly equivalent to 0, as $1/f \in k(X)^*$. Since this contradicts the assumption, no such f exists, hence $H^0(D)$ is empty, or $H^0(D) = 0$. This completes the answer. ■

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