Compact course notes

Pure Math 800 (Section 2) Winter 2014

Riemann surfaces

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Contents

1 Foundations

What is a Riemann surface? We first need to define what a topological surface is.

1.1 Definitions and notation

Definition 1.1.1. A topological space X is *Hausdorff* if its topology separates points. That is, for all $p, q \in X$ with $p \neq q$, there exist open sets $U_p \ni p, U_q \ni q$ such that $U_p \cap U_q = \emptyset$.

Example 1.1.2. Consider the following examples of topological spaces:

· $\mathbf{R}^n$ with the metric topology is Hausdorff. To see this, take d to be the distance between $p, q \in \mathbf{R}^n$, and let $U_p = B_p(d/3)$ and $U_q = B_q(d/3)$.

· $\mathbf{R}^n$ with the cofinite topology is not Hausdorff. In the cofinite topology all open sets are of the form $\mathbf{R}^n \setminus \{p_1, \ldots, p_m\}$ for $m \in \mathbf{N}$ and $p_i \in \mathbf{R}^n$.

Definition 1.1.3. A (*topological*) *surface* X is a Hausdorff topological space that is locally homeomorphic to $\mathbf{C}$ (or $\mathbf{R}^2$). That is, for all $p \in X$, there exist open sets $U \ni p$, $V \subset \mathbf{C}$, and a homeomorphism $\varphi : U \to V$.

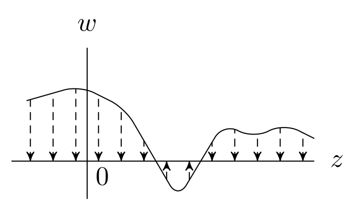
Remark 1.1.4. The topology on $\mathbf{C}$ (or $\mathbf{R}^2$) is always assumed to be the metric topology. Also note that topological surfaces may be considered as topological 2-manifolds.

Example 1.1.5. Consider the following examples of topological surfaces:

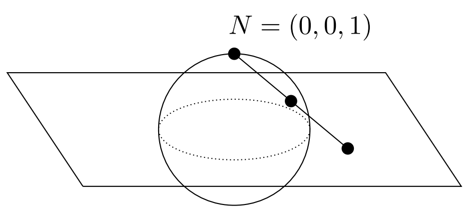
· $\mathbf{C} \cong \mathbf{R}^2$ is a topological surface. For all $p \in \mathbf{C}$, pick $U = V = \mathbf{C}$ and $\varphi = \mathrm{id}$.

· $U \subset \mathbf{C}$ open is a topological surface, as above.

· the graph of a continuous function $f : W \to \mathbf{C}$ for $W \subset \mathbf{C}$ open is a topological surface. The graph is $G = \{(z, w) \in \mathbf{C}^2 \ : \ w = f(z)\}$. For all $p \in G$, pick $U = G$ and $V = W$ with $\varphi(z, f(z)) = z$. This projection is a homeomorphism.



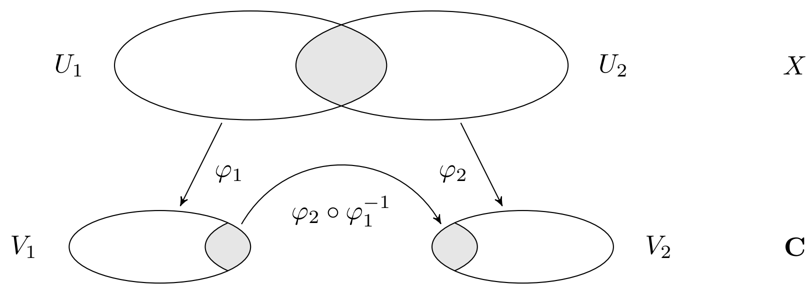
· The Riemann sphere $\mathbf{S}^2 \cong \mathbf{C} \cup \{\infty\} := \mathbf{P}^1$ is a topological surface by using stereographic projection. Place it as below and let $N = (0, 0, 1)$ be the north pole.



$$\begin{array}{rrcl} \varphi : & \mathbf{S}^2 \setminus N & \to & \mathbf{C} \\ & (x, y, z) & \mapsto & \left(\frac{x}{1-z}, \frac{y}{1-z}\right) \\ & N & \mapsto & \infty \end{array}$$

The map φ is a homeomorphism.

Definition 1.1.6. A homeomorphism $\varphi : U \to V$ for $U \subset X$ open and $V \subset \mathbf{C}$ open is a (*complex*) *chart*. Two charts $\varphi_i : U_i \to V_i$ are *holomorphically compatible* if the map $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ is biholomorphic (i.e. is holomorphic with a holomorphic inverse).



Note that $\varphi_1(U_1 \cap U_2)$ and $\varphi_2(U_1 \cap U_2)$ are components in $\mathbf{C}$, since they are the images of $U_1 \cap U_2$ under homeomorphisms. So, $\varphi_2 \circ \varphi_1^{-1}$ is biholomorphic if it is holomorphic from $\mathbf{C}$ to $\mathbf{C}$ in the usual sense.

Example 1.1.7. Consider $\mathbf{P}^1 = \mathbf{C} \cup \{\infty\} = \mathbf{S}^2$. The topology on $\mathbf{P}^1$ is given as follows: Let $U \subset \mathbf{P}^1$. If $U \subset \mathbf{C}$, then U is open iff it is open in the metric topology on $\mathbf{C}$. Otherwise, U is open if it can be described as $U = (\mathbf{C} \setminus K) \cup \{\infty\}$ for some compact subset $K \subset \mathbf{C}$ (with respect to the metric topology).

Now let us describe some complex charts on $\mathbf{P}^1$. For $p \in \mathbf{P}^1$, if $p \in \mathbf{C}$, pick $U = \mathbf{C}$ and $\varphi = \mathrm{id} : \mathbf{C} \to \mathbf{C}$. If $p = \infty$, then $U = \mathbf{C}^* \cup \{\infty\} = (\mathbf{C} \setminus \{0\}) \cup \{\infty\}$ and

$$\begin{array}{rrcl} \psi: & \mathbf{C}^* \cup \{\infty\} & \to & \mathbf{C} \\ & z & \mapsto & \begin{cases} \frac{1}{z} & \text{if } z \neq \infty \\ 0 & \text{if } z = \infty \end{cases} \end{array} \quad \text{with inverse} \quad \begin{array}{rrcl} \psi^{-1}: & \mathbf{C} & \to & \mathbf{C}^* \cup \{\infty\} = U \\ & w & \mapsto & \begin{cases} 1/w & \text{if } w \neq 0 \\ \infty & \text{if } w = 0 \end{cases} \end{array}.$$

Now φ and ψ are complex charts on X that contain every point. Also, $U_1 = \mathbf{C} \cap U_2 = \mathbf{C}^* \cup \{\infty\} = \mathbf{C}^*$ and $\varphi(U_1 \cap U_2) \cap \psi(U_1 \cap U_2) = \mathbf{C}^*$. Moreover $\varphi \circ \varphi^{-1} : \mathbf{C}^* \to \mathbf{C}^*$ by $z \mapsto 1/z$ is biholomorphic. Hence the charts φ and ψ are holomorphically equivalent (note that we should have checked that X is Hausdorff).

1.2 Atlases and lattices

Definition 1.2.1. An *atlas* on X is a collection $\mathcal{U} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ of charts $\varphi_\alpha : U_\alpha \subset X \to \varphi_\alpha(U_\alpha) \subset \mathbf{C}$ that cover X, i.e. $X = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$. An atlas $\mathcal{U}$ on X is called *complex* (or *holomorphic*) if any 2 charts in $\mathcal{U}$ are holomorphically compatible.

Example 1.2.2. Let $X = U \subset \mathbf{C}$ be open. Then for all $p \in U$, the map $\varphi = \mathrm{id} : U \to U$ is a chart containing p. Since we only need one chart in this case, and $\varphi \circ \varphi^{-1} = \mathrm{id}_U$ is biholomorphic, we have that $\mathcal{U} = \{(U, \mathrm{id}_U)\}$ is a holomorphic atlas on X that contains only one chart.

Remark 1.2.3.

· If a topological surface X can be covered by an atlas that contains only one chart (for example, $U \subset \mathbf{C}$ open on the graph G of a continuous function $f : U \subset \mathbf{C} \to \mathbf{C}$), then U is a holomorphic atlas.

· A topological surface X admits many distinct atlases, which may contain an infinite number of charts.

· The space $\mathbf{P}^1 = \mathbf{C} \cup \{\infty\}$ admits the following holomrphic atlas: $\mathcal{U} = \{(U_1, \varphi), (U_2, \psi)\}$ as described above. Note that any atlas of $\mathbf{P}^1$ must contain at least 2 charts, otherwise there exists a homeomorphism $\varphi : \mathbf{P}^1 \to V \subset \mathbf{C}$ for V open, which is impossible, as $\mathbf{P}^1 \cong \mathbf{S}^2$ is compact, whereas V is not.

Definition 1.2.4. Two holomorphic atlases $\mathcal{U}$ and $\mathcal{U}'$ on X are called *analytically equivalent* if every chart in $\mathcal{U}$ is holomorphically compatible with every chart in $\mathcal{U}'$. Since the composition of any 2 biholomorphic maps is biholomorphic, analytic equivalence is an equivalence relation.

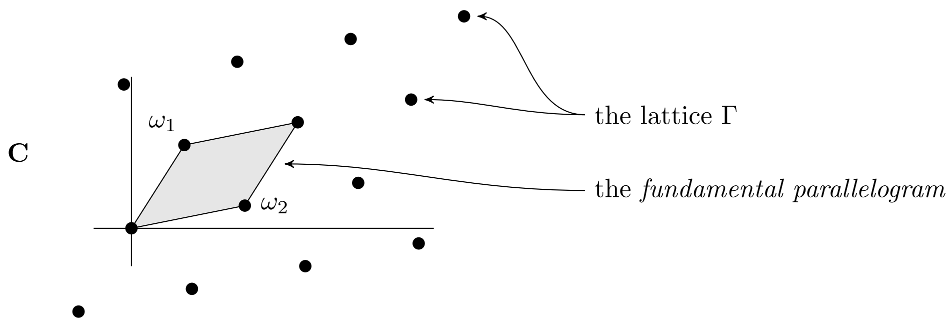
A *complex structure* Σ on X is an equivalence class of analytically equivalent atlases.

Note that any complex structure can be given by a choice of holomorphic atlas (by taking the equivalence class of that atlas). Moreover, every complex structure Σ contains a maximal atlas $\mathcal{U}^*$, where maximality is with respect to inclusion.

Definition 1.2.5. A *Riemann surface* is a pair (X, Σ), where X is a connected surface and Σ is a complex structure on X. Note that not all authors require X to be connected.

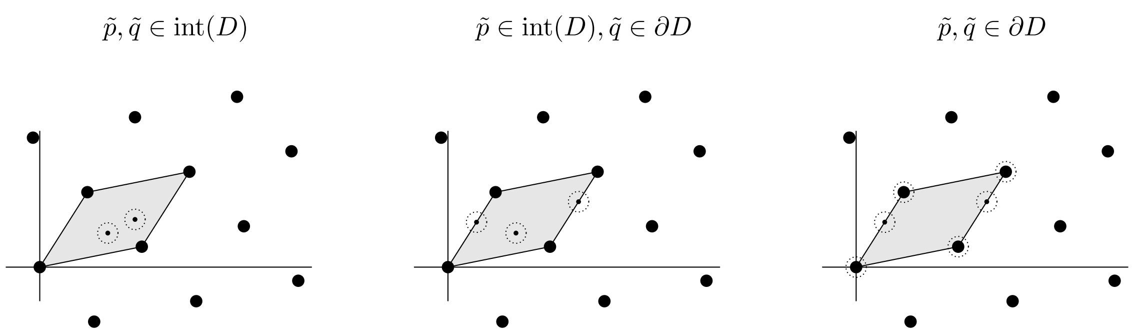
Example 1.2.6. With $\mathcal{U}$ as above, for each example respectively, each of $(\mathbf{C}, [\mathcal{U}])$, $(U, [\mathcal{U}])$ for any $U \subset \mathbf{C}$ open, and $(\mathbf{P}^1, [\mathcal{U}])$ is a Riemann surface.

Example 1.2.7. Consider the torus, described as $X = \mathbf{C}/\Gamma$, where $\Gamma = \{m\omega_1 + n\omega_2 \ : \ m, n \in \mathbf{Z}\}$ is a *lattice* for some fixed, linearly independent (over $\mathbf{R}$) $\omega_1, \omega_2 \in \mathbf{C}$. So in $\mathbf{C}/\Gamma$, $z_1 \sim z_2$ iff $z_1 = z_2 + \omega$ for $\omega \in \Gamma$. We would like to show that X is a Riemann surface and give it a topology.



We may express X as $X = \{[z] \ : \ z \in \mathbf{C}\}$ where $[z] = \{z' \in \mathbf{C} \ : \ z' = z + \omega, \omega \in \Gamma\}$. Next we need a topology on X. We are going to use the quotient topology. Consider the projection map $\pi : \mathbf{C} \to \mathbf{C}/\Gamma = X$, so $U \subset X$ is open iff $\pi^{-1}(U)$ is open in $\mathbf{C}$. By definition, π is continuous. So X is connected since it is the continuous image of $\mathbf{C}$, which is connected. Further, X is compact since it is the continuous image of a compact set.

To prove that the quotient topology is Hausdorff, we use the fact that π is an open map. Indeed, for all open sets $U \subset \mathbf{C}$, $\pi(U)$ is open because $\pi^{-1}(\pi(U)) = \bigcup_{\omega \in \Gamma}(U + \omega)$, each of which is open in $\mathbf{C}$, so $\pi^{-1}(\pi(U))$ is open in $\mathbf{C}$. Now choose $p, q \in X$ distinct points, We need to find open neighborhoods $V_p \ni p$, $V_q \ni q$ such that $V_p \cap V_q = \emptyset$. Let $\tilde{p}$, $\tilde{q}$ be the the associated points to p, q, respectively, on the fundamental parallelogram, so $\pi(\tilde{p}) = p$ and $\pi(\tilde{q}) = q$. We do a proof by picture for each case, for D the fundamental parallelogram.



The indicated area around each point is its neighborhood. From the pictures, it is clear that we can always pick a small enough neighborhood, so X is Hausdorff. It remains to find a complex atlas on X.

For all $p \in X$, find an open set $U_p \subset X$ containing p and a homeomorphism $\varphi : U_p \to \varphi(U_p) \subset \mathbf{C}$. As before, pick the point $\tilde{p}$ in the fundamental parallelogram mapping to p under π, and an open disk D centered at $\tilde{p}$ that's small enough so that $\pi : D \subset \mathbf{C} \to \pi(D) \subset X$ is a homeomorphism. Set $U_p = \pi(D)$ which is open, and $\varphi_p = \pi^{-1}|_{U_p} : U_p \to D$. Now (U_p, φ_p) is a chart containing p. Since $\bigcup_{p \in X} U_p = X$, we get an atlas

$\mathcal{U} = \{(U_p, \varphi_p) \ : \ p \in X\}$.

The last thing is to check that for any 2 charts (U_p, φ_p), (U_q, φ_q) we have $\psi = \varphi_p \circ \varphi_q^{-1} : \varphi_q(U_p \cap U_q) \to \varphi_p(U_p \cap U_q)$ is a biholomorphic morphism. So let $z \in \varphi_q(U_p \cap U_q)$, so

$$\pi(\psi(z)) = \pi((\varphi_p \varphi_q^{-1})(z)) = \varphi_q^{-1}(z) = \pi(z),$$

and $\pi(\psi(z)) = \pi(z)$ iff $\psi(z) - z \in \Gamma$. Now we have that $h(z) = \psi(z) - z$ is constant on the connected components of $\varphi_q(U_p \cap U_q)$, implying that $\psi(z) = z + c$ for c some constant on any connected component of $\varphi_q(U_p \cap U_q)$. Then ψ is biholomorphic, and $\mathcal{U}$ is a complex structure on X.

Definition 1.2.8. For $\Gamma = \{m\omega_1 + n\omega_2 \ : \ m, n \in \mathbf{Z}\}$ a lattice in $\mathbf{C}$, an *elliptic function* relative to the lattice Γ is a doubly-periodic meromorphic function $f : \mathbf{C} \to \mathbf{C}$. That is, for all $z \in \mathbf{C}$,

$$f(z + \omega_1) = f(z + \omega_2) = f(z).$$

1.3 Algebraic curves

Definition 1.3.1. Let $p(z, w)$ be a non-constant polynomial in 2 complex variables. Then $C = \{(z, w) \in \mathbf{C}^2 \ : \ p(z, w) = 0\}$ is called the *algebraic curve* defined by p. Further, C is *smooth* at (z_0, w_0) if

$$\nabla p(z_0, w_0) = \left(\frac{\partial p}{\partial z}(z_0, w_0), \frac{\partial p}{\partial w}(z_0, w_0) \right) \neq (0, 0),$$

otherwise it is *singular*. The space $C \setminus \{\text{singular points of } C\}$ is a Riemann surface.

Example 1.3.2. Consider the following examples of algebraic curves.
· $p(z, w) = w^2 - z$, then $\nabla p = (-1, 2w) \neq (0, 0)$, so $C = \{(z, w) \ : \ z = w^2\}$ is smooth at every point.
· $p(z, w) = w^2 - z^3$, then $\nabla p = (-3z^2, 2w) = (0, 0)$ iff $(z, w) = (0, 0)$, so $C = \{(z, w) \ : \ z^3 = w^2\}$ is smooth everywhere except at $(0, 0)$

Proposition 1.3.3. Let C be an algebraic curve. Each connected component of $S = C \setminus \{\text{singular points of } C\}$ admits a natural complex structure, making it into a Riemann surface.

Proof: Follows directly from the implicit function theorem. ■

Remark 1.3.4. Recall that a complex function $f(z)$ on 1 variable is holomorphic at z_0 if one of the following equivalent properties holds:
· f is complex differentiable at z_0 and in a neighborhood of z_0
· f admits a convergent power series expansion at z_0
· the Cauchy–Riemann equations hold at (x_0, y_0), where $z_0 = x_0 + iy_0$
· f is continuous at z_0 and $\partial f / \partial \bar{z}(z_0) = 0$, where $f(z) = f(z, \bar{z})$

Example 1.3.5. Differentiating by $\bar{z}$ is exactly the same as differentiating by a different variable. For example,

$$\frac{\partial z^2}{\partial \bar{z}} = 0 \qquad \text{and} \qquad \frac{\partial \bar{z}}{\partial \bar{z}} = 1,$$

So the first function is everywhere holomorphic, and the second is nowhere holomorphic.

Theorem 1.3.6. [Implicit function theorem]
Let $p(z, w)$ be a non-constant holomorphic function of 2 variables, and consider $C = \{(z, w) \ : \ p(z, w) = 0\} \subset \mathbf{C}^2$. Suppose that $(z_0, w_0) \in C$ is such that $\frac{\partial p}{\partial w}(z_0, w_0) \neq (0, 0)$. Then there exists a disk $D_1 \subset \mathbf{C}$ centered at z_0 and a disc $D_2 \subset \mathbf{C}$ centered at w_0, and a holomorphic function $\varphi : D_1 \subset \mathbf{C} \to D_2 \subset \mathbf{C}$ with $\varphi(z_0) = w_0$ and $C \cap (D_1 \times D_2) = \{(z, p(z)) \ : \ z \in D_1\}$.

That is, $C \cap (D_1 \times D_2)$ is the graph of φ. Note that if $\frac{\partial p}{\partial z}(z_0, w_0) \neq (0, 0)$, then there exists $\psi = \psi(w)$ such that $C \cap (D_1 \times D_2) = \{(\psi(w), w) \ : \ w \in D_2\}$.

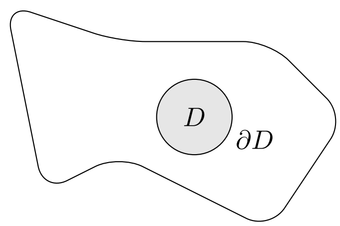
Proof: We will need the following claim: Let g be a holomorphic function of 1 variable on an open set containing a disk D such that g does not vanish on ∂D. Then

$$\frac{1}{2\pi i}\int_{\partial D}\frac{g'(w)}{g(w)}dw = \begin{pmatrix} \#\text{ of zeros of } g \text{ inside} \\ D \text{ counting multiplicities}\end{pmatrix}.$$

Moreover, if g has only 1 zero inside D (say w_1), and that zero has multiplicity 1, then

$$w_1 = \frac{1}{2\pi i}\int_{\partial D}\frac{wg'(w)}{g(w)}dw.$$

Both follow from the residue theorem, so we will not prove them. Now consider the following family of functions of the variable w: $f_z(w) = p(z,w)$ for all z (here z is considered a parameter). At $z = z_0$, we have $f'_{z_0}(w_0) = \frac{\partial p}{\partial w_0}(z_0,w_0) \neq (0,0)$. This implies that f_{z_0} is 1-1 in an open neighborhood of w_0 in $\mathbf{C}$. Let D_2 be a disk centered at w_0 such that the closure $\overline{D_2}$ is contained in this neighborhood. This means in particular that f_{z_0} is 1-1 on $\overline{D_2}$. But $f_{z_0}(w_0) = p(z_0,w_0) = 0$ (since $(z_0,w_0) \in C$), so w_0 is the only zero of f_{z_0} on $\overline{D_2}$, and f_{z_0} does not vanish on ∂D_2.



Note that $|f_{z_0}|$ is a continuous function of w on ∂D_2, since f_{z_0} is holomorphic. Therefore, by the EVT, since ∂D_2 is compact, $|f_{z_0}|$ addains a minimum value on ∂D_2, which must be > 0. So there exists $\delta > 0$ with $|f_{z_0}| > \delta$ on ∂D_2. By the continuity of $p(z,w)$ on z, we must also have that $|f_z| > \delta'$ on ∂D_2 for some $\delta' > 0$. Then by the claim,

$$N(z) = \frac{1}{2\pi i}\int_{\partial D_2}\frac{f1_z(w)}{f_z(w)}dw = \begin{pmatrix} \#\text{ of zeros of } f_z \text{ in } D_2 \\ \text{counting multiplicities}\end{pmatrix} = \frac{1}{2\pi i}\int_{\partial D_2}\frac{\frac{\partial p}{\partial w}(z,w)}{p(z,w)}dw,$$

so $N(z)$ is continuous in z. But $N(z)$ takes values in $\mathbf{Z}_{\geqslant 0}$ and $N(z_0) = 1$, so by continuity of $N(z)$ in D_1, we must have that $N(z) = 1$ for all $z \in D_1$. For every $z \in D_1$, set $\varphi(z) =$ (unique zero of $f_z(w)$ in D_2). Then φ is a complex function defined on D_1 such that $p(z,\varphi(z)) = 0$, Moreover, by the claim

$$\varphi(z) = \frac{1}{2\pi i}\int_{\partial D_2}\frac{wf'_z(w)}{f_z(w)}dw = \frac{1}{2\pi i}\int_{\partial D_2}\frac{w\frac{\partial p}{\partial w}(z,w)}{p(z,w)}dw,$$

which is holomorphic in z since $p(z,w)$ and $\frac{\partial p}{\partial w}(z,w)$ are. Hence $\varphi(z)$ is holomorphic on D_1. ■

Proposition 1.3.7. Let $p(z,w)$ be a polynomial in 2 complex variables, and $C = \{(z,w) : p(z,w) = 0\}$. Then, every connected component of $S = C \setminus \{\text{singular points of } C\}$ is a Riemann surface.

Proof: For simplicity, we assume that S is connected. Let $(z_0,w_0) \in S$. Since C is smooth at (z_0,w_0),

$$\nabla p(z_0,w_0) = \left(\frac{\partial p}{\partial w}(z,w_0), \frac{\partial p}{\partial z}(z_0,w_0)\right) \neq (0,0).$$

If $\frac{\partial p}{\partial z}(z_0,w_0) \neq 0$, then S is locally the graph of a holomorphic function $\psi(w) = \{(\psi(w),w) : w \in U_2\}$ for U_2 open. Similarly, if $\frac{\partial p}{\partial w}(z_0,w_0) \neq 0$, then S is locally the graph of a holomorphic function $\varphi(z) = \{(z,\varphi(z)) : z \in U_1\}$ by the IFT. This gives corresponding charts for the two maps, namely

$$\begin{array}{rrcl} \varphi_2 : & \{(\psi(w),w) : w \in U_2\} & \to & U_2 \\ & (\psi(w),w) & \mapsto & w \end{array} \quad \text{and} \quad \begin{array}{rrcl} \varphi_1 : & \{(z,\varphi(z)) : z \in U_1\} & \to & U_1 \\ & (z,\varphi(z)) & \mapsto & z \end{array}.$$

The union of all such charts, for all $(z_0, w_0) \in S$ over S gives us an atlas. The only thing left to check is that every two charts are holomorphically compatible. If we pick $\theta, \tilde{\theta}$ charts both of the form φ_1 or φ_2, then $\theta \circ \tilde{\theta}^{-1} = \text{id}$, which is biholomprihc, so $\theta, \tilde{\theta}$ are holomorphically compatible. If θ is of the type φ_1 and $\tilde{\theta}$ is of the type φ_2, then

$$\begin{aligned} \varphi_1 \circ \varphi_2^{-1}(w) &= \varphi_1(\psi(w), w) = \psi(w) \\ \text{and} \quad \varphi_2 \circ \varphi_1^{-1}(z) &= \varphi_2(z, \varphi(z)) = \varphi(z), \end{aligned}$$

which are both holomorphic. Hence $\theta, \tilde{\theta}$ are holomorphically compatible. As for Hausdorffness, we note that S is Hausdorff because we are endowing it with the induced metric topology on $\mathbf{C}^2$, which is Hausdorff. ■

2 Holomorphic mappings on Riemann surfaces

From now on, the topology on $\mathbf{C}^n$ is always assumed to be the metric topology, and any subset of $\mathbf{C}^n$ to be endowed with the induced metric topology.

2.1 Subsection 1

Definition 2.1.1. Let X be a Riemann surface and $Y \subset X$ any open subset of X. A complex form $f : Y \to \mathbf{C}$ is called *holomorphic* if for all charts $\psi : U \subset X \to V \subset \mathbf{C}$ on X, the map $f \circ \psi^{-1} : \psi(Y \cap U) \to \mathbf{C}$ is holomorphic in the usual sense on the open set $\psi(Y \cap U) \subset \mathbf{C}$.

The set of all holomorphic functions on Y is denoted by $\mathcal{O}(Y)$.

Remark 2.1.2. Consider the following:

a. Constant functions are holomorphic.

b. For all $f, g \in \mathcal{O}(Y)$ and $\alpha \in \mathbf{C}$, $\alpha f + g \in \mathcal{O}(Y)$ and $fg \in \mathcal{O}(Y)$, hence $\mathcal{O}(Y)$ is a $\mathbf{C}$-algebra, i.e. a $\mathbf{C}$-vector space.

c. It is enough to verify the condition of the definition of any family of charts covering Y. Indeed, we let $\varphi : \tilde{U} \subset X \to \tilde{V} \subset \mathbf{C}$ be any chart in the maximal atlas of Y and $\psi : U \subset X \to V \subset \mathbf{C}$ be a chart in such a family. Since ψ is in the maximal atlas, it is holomorphically comaptible with φ, so that $\psi \otimes \varphi^{-1}$ is biholomorphic. Then $f \circ \varphi^{-1} = (f \circ \psi^{-1}) \circ (\psi \circ \varphi^{-1})$, hence $f \circ \varphi^{-1}$ is holomorphic.

d. Every chart $\psi : U \subset X \to U \subset \mathbf{C}$ on X is trivially holomorphic with respect to the complex structure on X, since $\psi \circ \varphi^{-1}$ is biholomorphic for any chart $\varphi : \tilde{U} \subset X \to \tilde{V} \subset \mathbf{C}$ in the maximal atlas. One also calls ψ a *local coordinate* or *uniformization parameter*, and (U, ψ) a *coordinate neighborhood* of any point in U. In this context, we also write $z = \psi$.

Example 2.1.3. Let $U \subset X$ be open. Then any holomorphic function $f : U \to \mathbf{C}$ is also holomorphic as a function thought of on a Riemann surface.

Proposition 2.1.4. Let $f : \mathbf{P}^1 \to \mathbf{C}$ be holomorphic. Then f is constant.

<u>*Proof:*</u> We use two charts, namely

$$\begin{array}{rccc} \varphi : & U = \mathbf{C} & \to & \mathbf{C} \\ & z & \mapsto & z \end{array} \qquad \text{and} \qquad \begin{array}{rccc} \psi : & U' = \mathbf{C}^* \cup \{\infty\} & \to & \mathbf{C} \\ & z & \mapsto & \begin{cases} 1/z & \text{if } z \neq \infty \\ 0 & \text{if } z = \infty \end{cases} \end{array}.$$

Since $f : \mathbf{P}^1 \to \mathbf{C}$ is holomorphic, we have $f \circ \varphi^{-1} : \mathbf{C} \to \mathbf{C}$ and $f \circ \psi^{-1} : \mathbf{C} \to \mathbf{C}$ are holomorphic. Then we can write $f \circ \varphi^{-1} = \sum_{n=0}^{\infty} a_n z^n$ on $\varphi(U)$, and $f \circ \psi^{-1}(w) = \sum_{n=0}^{\infty} b_n w_n$ on $\psi(U)$. But on $\varphi(U \cap U') = \mathbf{C}^*$, we have that

$$f \circ \psi^{-1}(w) = (f \circ \varphi^{-1}) \circ (\varphi \circ \psi^{-1})(w) = (f \circ \varphi^{-1})(1/w) = \sum_{n=0}^{\infty} a_n (1/w)^n = \sum_{n=0}^{\infty} a_n w^{-n}.$$

But Laurent series expansions on $\mathbf{C}^*$ are unique, forcing $a_n = b_n = 0$ if $n > 0$, and $a_0 = b_0$. Hence $f = a_o = b_0$ is constant. ∎

Note the above result is not surprising, as $\mathbf{P}^1$ is compact, and functions on compact subsets of $\mathbf{C}$ are bounded.

Theorem 2.1.5. [Removable singularities theorem - Riemann]
Let U be an open subset of a Riemann surface and $a \in U$. Suppose that $f \in \mathcal{O}(U \setminus \{0\})$ is bounded in some neighborhood of a. Then f can be extended uniquely to a function $\tilde{f} \in \mathcal{O}(U)$.

Proof: This follows from Riemann's removable singularities theorem on $\mathbf{C}$.

Definition 2.1.6. Let X, Y be Riemann surfaces. A continuous mapping $f : X \to Y$ is *holomorphic* if for all pairs of charts

$$\psi_1 : U_1 \subset \mathbf{C} \to V_1 \subset \mathbf{C} \qquad \text{and} \qquad \psi_2 : U_2 \subset \mathbf{C} \to V_2 \subset \mathbf{C}$$

the mapping $\psi_2 \circ f \circ \psi_1^{-1} : V_1 \to V_2$ is holomorphic in the usual sense. The mapping f is called *biholomorphic* if it is bijective and both $f : X \to Y$ and $f^{-1} : Y \to X$ are holomorphic.

Further, two Riemann surfaces are *isomorphic* if there exists a biholomorphic map $f : X \to Y$. If $X = Y$ and $f : X \to X$ is biholomorphic, then f is called an *automorphism*.

Example 2.1.7.

a. Any holomorphic function $f : X \to \mathbf{C}$ is a holomorphic mapping from X to $\mathbf{C}$ considered as a Riemann surface

b. If X, Y, Z are Riemann surfaces and $f : X \to Y$, $g : Y \to Z$ are holomorphic mappings, then so is $g \circ f : X \to Z$.

c. Given two tori $X = \mathbf{C}/\Gamma$ and $X' = \mathbf{C}/\Gamma'$, for Γ generated by ω_1, ω_2 and Γ' generated by ω_1' and ω_2', X is isomorphic to X' iff $\alpha\Gamma = \Gamma'$ for some $\alpha \in \mathbf{C}^*$.

d. The following map is an examples of a non-trivial holomorphic map between two Riemann surfaces:

$$\begin{array}{rccl} f : & \mathbf{P}^1 & \to & \mathbf{P}^1 \\ & z & \mapsto & \begin{cases} \frac{az+b}{cz+d} & \text{if } z \neq \infty \\ \frac{a}{c} & \text{if } z = \infty \end{cases} , \end{array}$$

with $ad - bc \neq 0$. This is the extension of the Möbius transformation $z \mapsto \frac{az+b}{cz+d}$ to $\mathbf{P}^1$. Thus f is an automorphism.

Theorem 2.1.8. [Identity theorem in $\mathbf{C}$]
Let D be a domain on $\mathbf{C}$ and suppose that f, g are holomorphic functions on D such that $f = g$ on a subset $A \subset D$ that has a limit point $a \in D$. Then $f = g$ on D.

Recall that $a \in D$ is a limit point if for any open neighborhood W of a in D, $(A \setminus \{a\}) \cap W \neq \emptyset$.

Corollary 2.1.9. If $f = g$ on an open subset $W \subset D$, then $f = g$ on D.

Proof: Note that since f, g are holomorphic on D, they are constant on D, and so is $f - g$. Moreover, $f - g = 0$ on W, so $f - g = 0$ on $\overline{W}$, where $\overline{W}$ is the closure of W in D, by continuity of $f - g$. Hence $f = g$ on $\overline{W}$, where $\overline{W}$ has a lmiti point in D since it is closed. This implies that $f = g$ on D by the identity theorem. ∎

Theorem 2.1.10. [Identity theorem]
Suppose X, Y are Riemann surfaces, and $f, g : X \to Y$ are holomorphic maps. If f and g coincide on a set $A \subset X$ that has a limit point at $a \in X$, then $f = g$ on X.

Proof: We begin with assumption that $Y = \mathbf{C}$. Let $G = \{x \in X \ : \ f|_{W_x} = g|_{W_x}$ for some open neighborhood W_x of x in $X\}$. The idea is to prove that G is open, closed, and non-empty, which will force $G = X$ since X is connected.

· G is open: Observe that $G = \bigcup_{x\in G} W_x$ since $W_x \subset G$ for all $x \in G$. So G is open since W_x is open.

· G is closed: We will do this by showing that $\partial G \subset G$. Let $b \in \partial G$. We want to show that there exists an open neighborhood W_b of b in X such that $f|_{W_b} = g|_{W_b}$. Let us first remark that $f(b) = f(g)$. Since f, g are holomorphic, they are continuous on X atherefore so is $f - g$. Thus $(f - g)^{-1}(\{0\})$ is closed in X. But, by definition, $G \subset (f - g)^{-1}(\{0\})$, implying that $\partial G \subset \overline{G} \subset (f - g)(\{0\})$ since $(f - g)^{-1}(\{0\})$ is closed. Hence $f(b) = g(b)$. Now let U be a connected open neighborhood of b in X and $\varphi : U \to V \subset \mathbf{C}$ be a homeomorphism (so that (U, φ) is a chart containing b). Since we are assuming f, g to be holomorphic, we have that $f \circ \varphi^{-1}$, $g \circ \varphi^{-1} : \varphi(U) \subset \mathbf{C} \to \mathbf{C}$ are holomorphic functions. But $b \in \partial G$ and U is an open neighborhood of b, so $G \cap U \neq \emptyset$, implying $f|_W = g|_W$ for some open set $(W \cap U) \subset U$ (by the definition of g). Hence

$$f \circ \varphi^{-1}\big|_{\varphi(W\cap U)} = g \circ \varphi^{-1}\big|_{\varphi(W\cap U)} \qquad \text{on the open set} \qquad \varphi(W \cap U) \subset \varphi(U),$$

implying that $f \circ \varphi^{-1} = g \circ \varphi^{-1}$ on $\varphi(U)$ by the identity theorem in $\mathbf{C}$. Hence $f|_U = g|_U$ with $b \in U$. Hence $b \in G$, so $\partial G \subset G$, and G is closed.

· G is non-empty: As before, one can show that $\overline{A} \subset (f - g)^{-1}(\{0\})$, forcing $f(a) = g(a)$, as $a \in \partial A$. We gan then find a neighborhood of a on which f, g agree, so $a \in G$. ■

2.2 Meromorphic functions

Definition 2.2.1. Let X be a Remann surface and $Y \subset X$ be open. A *meromorphic function* f on Y is a holomorphic function $f|_{Y'} = f' : Y' \to \mathbf{C}$ such that $Y' \subset Y$ is open, $Y \setminus Y'$ contains only isolated points, and for all $p \in Y \setminus Y'$ we have $\lim_{x\to p}[|f(x)|] = \infty$. The points of $Y \setminus Y'$ are called the *poles* of f. The set of all meromorphic functions on Y is denoted by $\mathcal{M}(Y)$.

Example 2.2.2. Consider the map

$$\begin{array}{rccl} f: & \mathbf{P}^1 & \to & \mathbf{C} \\ & z & \mapsto & \begin{cases} \frac{az+b}{cz+d} & \text{if } z \neq \infty \\ \frac{a}{c} & \text{if } z = \infty \end{cases}, \end{array}$$

where $ad - bc \neq 0$. Then f is holomorphic for $z \neq -d/c$, and has a pole at $z = -d/c$. Hence $f \in \mathcal{M}(\mathbf{P}^1)$.

Consider a polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ with $a_i \in \mathbf{C}$ for all i. Then p is a holomorphic function from $\mathbf{C}$ to $\mathbf{C}$. Extend f to $\mathbf{P}^1$ by setting $p(\infty) = \infty$. Then

$$\begin{array}{rccl} p: & \mathbf{P}^1 & \to & \mathbf{C} \\ & z & \mapsto & \begin{cases} p(z) & \text{if } z \neq \infty \\ \infty & \text{if } z = \infty \end{cases} \end{array}$$

is meromorphic on $\mathbf{P}^1$ since $\lim_{z\to\infty}[|p(z)|] = \infty$.

Definition 2.2.3. Let $S \subset X$ be a subset and $x \in S$. Then x is an *isolated point* of S if there exists an open neighborhood W of x in X such that $(S \setminus \{x\}) \cap W = \emptyset$. Otherwise, x is a *limit point* of S.

Example 2.2.4. Consider the map

$$\begin{array}{rccl} f: & \mathbf{C} & \to & \mathbf{C} \\ & z & \mapsto & e^{1/z} \end{array},$$

for which $z = 0$ is an isolated singularity of f. However, 0 is not a pole, because it does not satisfy the limit condition, as

$$\left| e^{1/z} \right| = \left| e^{(x-iy)/(x^2+y^2)} \right| = \left| e^{x/(x^2+y^2)} \right| \left| e^{i(-y/(x^2+y^2))} \right| = e^{x/(x^2+y^2)}.$$

Hence

$$\lim_{\substack{x \to 0^+ \\ y=0}} [|f(z)|] = \infty \quad \text{and} \quad \lim_{\substack{x \to 0^- \\ y=0}} [|f(z)|] = 0 \quad \text{implying} \quad \lim_{z \to 0} [|f(z)|] \neq \infty.$$

Therefore $z = 0$ is not a pole and is an essential singularity. This example shows that the limit condition is necessary to ensure that the point in $Y \setminus Y'$ is indeed a pole.

Theorem 2.2.5. Suppose that X is a Riemann surface and $f \in \mathcal{M}(X)$. Then for each pole of f, define $f(p) = \infty$. Then $f : X \to \mathbf{P}^1$ is a holomorphic mapping. Conversely, if $f : X \to \mathbf{P}^1$ is a holomorphic mapping, then f is either identically ∞ on X or else $f^{-1}(\infty)$ consists of isolated points and $f : X \setminus f^{-1}(\infty) \to \mathbf{C}$ is a meromorphic function on X.

Proof: Let $f \in \mathcal{M}(X)$. Then there exists $X' \subset X$ open with $f : X' \to \mathbf{C}$ holomorphic, and $X \setminus X'$ satisfying the conditions above. Set

$$\begin{array}{rccl} f: & X & \to & \mathbf{P}^1 \\ & z & \mapsto & \begin{cases} f(z) & \text{if } z \in X' \\ \infty & \text{if } z \in X \setminus X' \end{cases}. \end{array}$$

Then by the conditions above, for all $p \in X \setminus X'$,

$$\lim_{z \to p} [|f(z)|] = \infty \quad \iff \quad \lim_{z \to p} [f(z)] = \infty.$$

We now need to check that $f : X \to \mathbf{P}^1$ is in fact holomorphic. To do this, pick two charts $\varphi : U \subset X \to V \subset \mathbf{C}$ and $\psi : U' \subset \mathbf{P}^1 \to V' \subset \mathbf{C}$ of X and $\mathbf{P}^1$ with $f(U) \subset U'$. Now we check that $g = \psi \circ f \circ \varphi^{-1} : V \subset \mathbf{C} \to V' \subset \mathbf{C}$ is holomorphic. Let $P = X \setminus X'$ be the set of poles of f. Since f is holomprihc on $X' = X \setminus P$, then g is holomorphic on $V \setminus \varphi(P)$. Let $p \in P$. If $\varphi(p) \notin P$, then we're fine. If $\varphi(p) \in V$, then

$$g(\varphi(p)) = \psi \circ f(p) = \psi(\infty) \subset V' \subset \mathbf{C}.$$

This tells us that there exists an open neighborhood W of $\varphi(p)$ on $\mathbf{C}$ such that $g \in \mathcal{O}(W \setminus \varphi(p))$ and g is bounded on W. Hence by Riemann's removable singularity theorem, $g \in \mathcal{O}(W)$. Therefore f is holomorphic at p, so f is holomorphic on V, finally implying that $f : X \to \mathbf{P}^1$ is holomrphic.

Conversely, suppose that $f : X \to \mathbf{P}^1$ is a holomorphic mapping. Then by the identity theorem, if $f^{-1}(\infty)$ does not consist of isolated points, then $f = \infty$ on all of X, because $f^{-1}(\infty)$ must contain a limit point. Hence either $f = \infty$ on X or $f : X \setminus f^{-1}(\infty) \to \mathbf{C}$ is a meromorphic function on X. ■

From now on, we identify meromorphic functions on X with their corresponding holomorphic mappings $f : X \to \mathbf{P}^1$.

Theorem 2.2.6. [Local behavior of holomorphic mappings]
Suppose that X, Y are Riemann surfaces and $f : X \to Y$ is a non-constant holomorphic mapping. Suppose that $a \in X$ and $b = f(a)$. Then there exists an integer $k \geqslant 1$ and charts $\varphi : U \subset X \to V \subset \mathbf{C}$, $\psi : U' \subset Y \to V' \subset \mathbf{C}$ such that

1. $a \in U$ with $\varphi(a) = 0$ and $b \in U'$ with $\psi(b) = 0$,
2. $f(U) \subset U'$, and
3. the map $f : \psi \circ f \circ \varphi^{-1} : V \subset \mathbf{C} \to V' \subset \mathbf{C}$ is given by $f(z) = z^k$ for all $z \in V$.

Proof: Let $\varphi_1 : U_1 \subset X \to V_1 \subset \mathbf{C}$ be a chart of X with $a \in U_1$, and $\psi_1 : U' \subset Y \to V' \subset \mathbf{C}$ be a chart of Y with $b \in U'$. If $\varphi(a) \neq 0$, replace φ by $\varphi_1 : U_1 \subset X \to (V_1 \setminus \varphi_1(a)) \subset \mathbf{C}$ so we may assume that $\varphi_1(a) = 0$.

Similarly, we may assume that $\psi(b) = 0$. We also need $f(U_1) \subset U'$. If $f(U_1) \not\subset U'$, replace (U_1, φ_1) by $(U_1 \cap f^{-1}(U'), \varphi_1|_{U_1 \cap f^{-1}(U')})$. The set $U_1 \cap f^{-1}(U')$ is open since f is holomorphic and so continuous. Hence we may assume that $f(U_1) \subset U'$. Next consider $f_1 = \psi \circ f \circ \varphi_1^{-1} : V_1 \subset \mathbf{C} \to V' \subset \mathbf{C}$. Since f is holomorphic and non-constant, we have that f_1 is a non-constant holomrphic function on $\mathbf{C}$. Then

$$f_1(0) = \psi \circ f(\varphi_1^{-1}(0)) = \psi(f(a)) = \psi(b) = 0,$$

so we may write $f_1(w) = w^k g(w)$ for some integer $k \geqslant 1$ and $g \in \mathcal{O}(V_1)$ with $g(0) \neq 0$. Since $g(0) \neq 0$, we can find a neighborhood W of 0 and $h \in \mathcal{O}(M)$ such that $h^k = g$ on W. Hence

$$f_1(w) = (wh(w))^k = (\alpha(w))^k \ \forall\ w \in W \qquad \text{for} \qquad \begin{array}{rccc} \alpha: & W \subset \mathbf{C} & \to & \mathbf{C} \\ & w & \mapsto & wh(w) \end{array}.$$

Note that α is holomorphic at $\alpha'(w) = h(w) + wh'(w)$, meaning that $\alpha(0) = h(0) \neq 0$, as $(h(0))^k = g(0) \neq 0$. So by the inverse function theorem, α is invertible with a holomrphic inverse in some neighborhood $V_2 \subset V_1$ of 0. Finally, define the objects

$$\begin{aligned} &U = \varphi_1^{-1}(V_2) \quad \text{which is open,} \\ &V = \alpha(V_2) \quad \text{which is open since } \alpha \text{ is biholomorphic, and} \\ &\varphi = \alpha \circ (\varphi_1|_U) : U \subset X \to V \subset \mathbf{C}. \end{aligned}$$

Then (U_1, φ) and (U', ψ) satisfy conditions **1.** and **2.**. For **3.**, note that

$$\begin{aligned} f(z) &= \psi \circ f \circ \varphi^{-1}(z) \\ &= (\psi \circ f \circ \varphi_1^{-1})(\alpha^{-1}(z)) \\ &= f_1(\alpha^{-1}(z)) \\ &= \alpha(\alpha^{-1}(z)) \\ &= z^k. \end{aligned}$$

This completes the proof. ■

Remark 2.2.7. The above theorem tells us that for all $y \in U'$, $|f^{-1}(y) \cap U| = k$. We call k the *multiplicity* with which f takes the value at a.

Remark 2.2.8. Let X be a Riemann surface and $Y \subset X$ an open subset. Let $f \in M(Y)$. Then f can be identified with a holomorphic mapping $f : Y \to \mathbf{P}^1$. By the identity theorem, $f^{-1}(U)$ is an isolated set of points unless $f = 0$, so $1/f \in \mathcal{M}(Y)$. Further, p is a pole of f iff p is an isolated vero of multiplicity k where k is the integer appearing in the theorem. Hence the definition of poles coincides with the usual definition of poles of cemplex functions on $\mathbf{C}$. This means that in particular meromorphic functions admit laurent series expansions after composing them with a chart. Let (U, φ) be a chart of Y containing a pole p of $f \in \mathcal{M}(Y)$. Then if $\varphi(p) = 0$, on $\varphi(U)$ we can write

$$f \circ \varphi^{-1}(z) = \sum_{j=-k}^{\infty} c_j z^j.$$

Thus, for all $f, g \in \mathcal{M}(Y)$, fg, $f + g$, $1/f \in \mathcal{M}(Y)$, so $\mathcal{M}(Y)$ is a field.

Theorem 2.2.9. [Open mapping theorem]
Let X, Y be Riemann surfaces and $f : X \to Y$ a non-constant holomrphic mapping. Then f is open (i.e. $f(U) \subset Y$ is open for all $U \subset X$ open).

Proof: Left as an exercise. ■

Corollary 2.2.10. Let X, Y be Riemann surfaces with X compact. If $f : X \to Y$ is a non-constant holomrphic mapping, then Y is compact and f is surjective.

Proof: Since f is open, $f(X)$ is open in Y. Also, $f(X)$ is compact in Y since f is continuous. Hence $f(X)$ is closed in Y since Y is Hausdorff. So $f(X) \neq 0$, is open and closed in Y, which is connected. Therefore $F(X) = Y$ and Y is compact. ∎

Corollary 2.2.11. Every holomrphic function on a compact Riemann surface is constant.

Proof: Let $f : X \to \mathbf{C}$ be holomorphic. If f is not constant, then $\mathbf{C}$ is compact, a contradiction. ∎

Corollary 2.2.12. [Fundamental theorem of algebra]
If $f(z) = a_0 + a_1 z + \cdots + a_n z^n$, then f has at least 1 zero.

Proof: Extend f to a holomorphic mapping $\mathbf{P}^1 \to \mathbf{P}^1$ by setting $f(\infty) = \infty$. Then $f : \mathbf{P}^1 \to \mathbf{P}^1$ is surjective since $\mathbf{P}^1$ is compact. ∎

2.3 Meromorphic functions on Riemann surfaces

Proposition 2.3.1. Let $f \in \mathcal{M}(\mathbf{P}^1)$. Then f is rational (i.e. can be expressed as a quotient of polynomials).

Proof: Let $f \in \mathcal{M}(\mathbf{P}^1)$ and let's identify it with the corresponding holomrphic function $f : \mathbf{P}^1 \to \mathbf{P}^1$. Then f can only have a finite number of poles (otherwise $f^{-1}(\infty)$ would be an infinite subset of $\mathbf{P}^1$ and so would have a limit point since $\mathbf{P}^1$ being compact implies $f = \infty$ by the identity theorem). Note that we can assume that none of the poles of f is ∞ (i.e. $\infty \notin f^{-1}(\infty)$), otherwise we just work with $1/f$ so that ∞ is now a zero. Let $a_1, \ldots, a_m \in \mathbf{C}$ be the poles of f. Then f admits a Laurent series expansion about each a_i which has principal part

$$h_i(z) = \sum_{j=-k_i}^{-1} (z - a_i)^j.$$

Then $g = f - (h_1 + \cdots + h_m)$ is holomrphic on $\mathbf{P}^1$, so g is constant since $\mathbf{P}^1$ is compact. Hence $f = c + (h - 1 + \cdots + h_m)$ for all h_i rational. So f is rational. ∎

Let $\Gamma = \{m\omega_1 + n\omega_2 \ : \ n, m \in \mathbf{Z}\}$ be a lattice in $\mathbf{C}$. Let us describe meromorphic functions on the torus $X = \mathbf{C}/\Gamma$.

Definition 2.3.2. A meromorphic function $f : \mathbf{C} \to \mathbf{P}^1$ is called *doubly-periodic* with respect to Γ if $f(z + \omega) = f(z)$ for all $\omega \in \Gamma$.

Let $\pi : \mathbf{C} \to \mathbf{C}/\Gamma = X$ be the canonical map. Then if $f : \mathbf{C} \to \mathbf{P}^1$ is doubly-periodic with respect to Γ, it descends to the function $F : X \to \mathbf{P}^1$, where $f = F \circ \pi$. Then $F \in \mathcal{M}(X)$. Conversely, given any $F \in \mathcal{M}(X)$, $f := F \circ \pi$ is doubly-periodic with respect to Γ. The main point is making the identification

$$\mathcal{M}(\mathbf{C}/\Gamma) \quad \longleftrightarrow \quad \begin{pmatrix} \text{doubly periodic} \\ \text{functions wrt } \Gamma \end{pmatrix}.$$

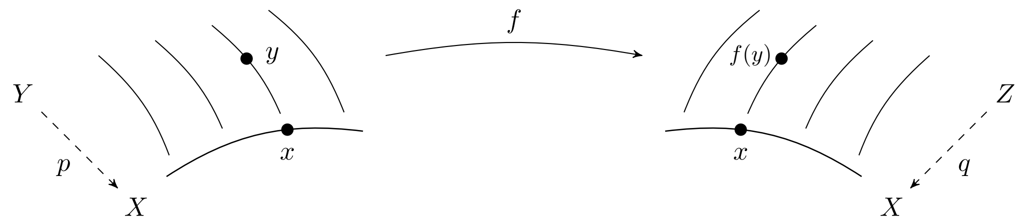
Theorem 2.3.3.
1. Every holomrphic doubly-periodic $f : \mathbf{C} \to \mathbf{C}$ is constant.
2. Every non-constant doubly-periodic $f : \mathbf{C} \to \mathbf{P}^1$ is surjective.

Proof: **1.** The map f corresponds to a holomorphic mapping $f : X = \mathbf{C}/\Gamma \to \mathbf{C}$ ond X is compact. **2.** The map f corresponds to a non-constant holomorphic mapping $f : X \to \mathbf{P}^1$ with X compact. ∎

3 Branched and unbranched coverings

3.1 Definitions

Definition 3.1.1. Let X, Y, Z be topological spaces and $p : Y \to X$ a continuous map. For all $x \in X$, $p^{-1}(x)$ is called the *fiber* of p over x. If $y \in p^{-1}(x)$, then y *lies over* x. If $p : Y \to X$ and $q : Z \to X$ are continuous, a continuous map $f : Y \to Z$ is *fiber-preserving* if $p = q \circ f$. That is, if for all $x \in X$, $f(p^{-1}(x)) \subset q^{-1}(x)$.



Recall that a subset $A \subset Y$ is called *discrete* if for all $a \in A$, there exists an open neighborhood V of a with $V \cap A = \{a\}$. Then $p : Y \to X$ is called *discrete* if every fiber $p^{-1}(x)$ is discrete in Y.

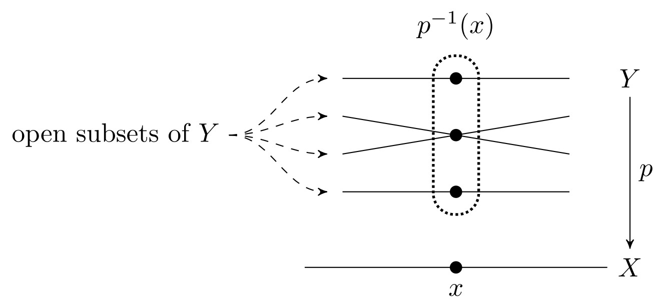
Theorem 3.1.2. Let X, Y be riemann surfaces and $p : Y \to X$ a non-constant holomorphic mapping. Then p is open and discrete.

Proof: The map p is open by the open mapping theorem. Also, suppose that p has a fiber $p^{-1}(x)$ that is not discrete. Then there exists $a \in A = f^{-1}(z)$ such that $V \cap (A \setminus \{a\}) \neq \emptyset$ for all open neighborhoods V of a. Hence a is a limit point of A, so $f(y) = f(a) = x$ for all $y \in Y$ by the identity theorem. ■

Corollary 3.1.3. If Y is compact, then every fiber of $p : Y \to X$ is finite.

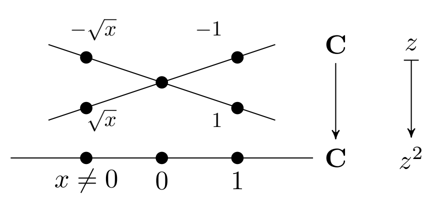
Proof: This follows as $p^{-1}(x)$ is discrete in Y and Y is compact. ■

Definition 3.1.4. Let X, Y be Riemann surfaces, and $p : X \to Y$ a non-constant holomorphic map. A point $y \in Y$ is called a *branch point* or *ramification point* of p if there does not exist an open neighborhood V of p with $p|_V$ injective. Also, p is *unbranched* if p has no branch points, and *branched at y* if y is a branch point.



The middle point above is a branch point.

Remark 3.1.5. Consider the map $\mathbf{C} \to \mathbf{C}$ given by $z \mapsto z^2$. This is branched at $z = 0$.



However, the map $p : \mathbf{C}^* \to \mathbf{C}^*$ with $z \mapsto z^2$ is unbranched, since $p'(z) = 2z \neq 0$ on $\mathbf{C}^*$. Hence p is locally invertible at every point on $\mathbf{C}^*$, so p is locally injective on $\mathbf{C}^*$. In general, $p : \mathbf{C} \to \mathbf{C}$ with $z \mapsto z^k$, for $k \geqslant 2$ is branched at $z = 0$ but unbranched away from $z = 0$. When $k = 1$, the map is unbranched everywhere on $\mathbf{C}$.

Example 3.1.6. The map $\exp : \mathbf{C} \to \mathbf{C}^*$ is unbranched on $\mathbf{C}$, since $\exp'(z) = \exp(z) \neq 0$ on $\mathbf{C}$.

The map $\pi : \mathbf{C} \to \mathbf{C}/\Gamma$, where Γ is a lattice in $\mathbf{C}$, is also unbranched.

In general, let $p : Y \to X$ be any non-constant holomorphic map between Riemann surfaces X, Y. Then for all $y \in Y$, there exist charts (U, φ) of Y containing y and (U', ψ) of X containing $p(y)$ such that $\psi \circ p \circ \varphi^{-1}(z) = z^k$. So y is a branch point iff $k \geqslant 2$.

Theorem 3.1.7. Let X, Y be Riemann surfaces and $p : X \to Y$ a non-constant holomorphic map. Then p is unbranched iff p is a local homeomorphism (i.e. for all $y \in Y$ there exists an open neighborhood V of y such that $p|_V : V \to p(V)$ is a homeomorphism).

Proof: Suppose that p is unbranched. Then for all $y \in Y$, there exists an open neighborhood V of y such that $p|_V : V \to p(V)$ is injective, so we have a well-defined inverse $(p|_V)^{-1} : p(V) \to V$. Now, $p|_V$ is continuous since p is holomorphic. Also, p is open by the open mapping theorem. So $p(V)$ is open and for all $U \subset V$ open, $((p|_V)^{-1})^{-1}(U) = (p|_V)(U)$ is open since p is open. So $p|_V$ and $(p|_V)^{-1}$ are continuous, meaning that $p|_V$ is a homeomorphism. This implies that p is a local homeomorphism.

If p is a local homeomorphism, then p is locally injective, so p is unbranched. ■

3.2 Covering maps

Definition 3.2.1. Let X, Y be topological spaces. A mapping $p : X \to Y$ is called a *covering map* if for all $x \in X$ there exists an open neighborhood $U \subset X$ of x such that $p^{-1}(U) = \bigcup_{j \in J} V_j$, where

- $V_j \subset Y$ is open for all $j \in J$,
- $V_j \cap V_i = \emptyset$ if $j \neq i$, and
- $p|_{V_j} : V_j \to U$ is a homeomorphism for all $j \in J$.

In particular, p is a local homeomorphism. This is because for all $y \in Y$, if $x = p(y)$, then $y \in V_j$ for some $j \in J$.

Example 3.2.2. The map $p : \mathbf{C}^* \to \mathbf{C}^*$ given by $z \mapsto z^k$ is a covering map. To see this, let $a \in \mathbf{C}^*$ and choose $b \in \mathbf{C}^*$ such that $b^k = a$. Let ω be a primitive root of unity. Then $p^{-1}(a) = \{b, b\omega, \ldots, b\omega^{k-1}\}$, which are the k roots of $z^k - a$. But p is locally invertible, so we can find an open neighborhood V_0 of b in $\mathbf{C}^*$ such that $p|_{V_0} : V_0 \to p(V_0)$ is biholomorphic. Set $V_j = \omega^j V_0 = \{\omega^j w \ : \ w \in V_0\} \subset \mathbf{C}$. Then $b\omega^j \in V_j$ for all j, and $p|_{V_j} : V_j \to p(V_0)$ is a homeomorphism for all j. Moreover, $a \in p(V_0)$ and $V_j \cap V_i = \emptyset$ if $i \neq j$, since $\omega^i \neq \omega^j$.

Note that $p(V_0) = U$ is open since p is an open map. Hence if we set $U = p(V_0)$, then U is an open neighborhood of a satisfying all the conditions of a covering map.

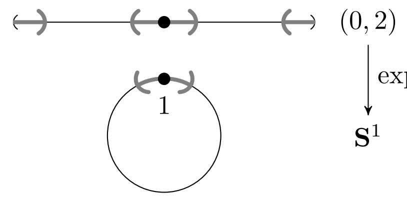
Example 3.2.3. The map $\exp : \mathbf{C} \to \mathbf{C}^*$ is a covering map. To see this, choose $a \in \mathbf{C}^*$ and $b \in \mathbf{C}$ with $\exp(b) = a$. Since exp is locally invertible, there exists an open neighborhood V_0 of b such that $p|_{V_0} : V_0 \to p(V_)0$ is o homeomorphism. Then, set $V_m = V_0 + 2\pi i m$ for all $m \in \mathbf{Z}$, and $U = p(V_0)$. So $a \in p(V_0)$ is open and $p^{-1}(U) = \bigcup_{m \in Z} V_m$ with $V_m \cap V_n = \emptyset$ if $m \neq n$, and $p|_{V_m} : V_m \to U$ is a homeomorphism.

The map $\pi : \mathbf{C} \to \mathbf{C}/\Gamma$ is also a covering map.

Remark 3.2.4. Although covering maps are local homeomorphisms. not every local homeomorphism is a covering map. For example, let $D = \{z \in \mathbf{C} \ : \ |z| = 1\} \subset \mathbf{C}$ and $i : D \hookrightarrow \mathbf{C}$ the inclusion map. This is not a covering map. To see this, let $a \in \mathbf{C}$. Then $U = D$ and $p^{-1}(U) = D$ on D. If $n \notin \overline{D}$, then $p^{-1}(A) = \emptyset$.

But if $a \in \partial D$, then any open neighborhood U of a is such that $p^{-1}(U) \not\subset D$, so we cannot describe $p^{-1}(U)$ as a union of open sets U_j in D. Similarly, the map

$$\begin{array}{rccc} \exp: & (0,2) & \to & \mathbf{S}^1 \subset \mathbf{C} \\ & t & \mapsto & e^{2\pi i t} \end{array}$$



For a small open neighborhood U of 1 in $\mathbf{S}^1$, $\exp^{-1}(U)$ is the union of open sets in $(0,2)$, but some of these open sets map homeomorphically onto open sots in $\mathbf{S}^1$ that do not contain 1. Nonetheless, since $\exp'(z) \neq 0$ on $(0,2)$, exp is a local homeomorphism and is surjective. But it is not a covering map.

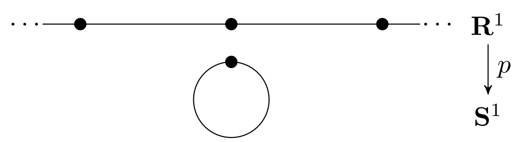
Theorem 3.2.5. Suppose X, Y are topological spaces with X connected. Let $p : X \to Y$ be a covering map. Then for all $x_0, x_1 \in X$, the fibers $p^{-1}(x_0)$ and $p^{-1}(x_1)$ have the same cardinality. In particular, if $y \neq \emptyset$, then p is surjective.

Before we begin the proof, let us define cardinality and look at some examples.

Definition 3.2.6. The cardinality of $p^{-1}(x)$ is the *number of sheets* of the covering map, and may be either finite or infinite.

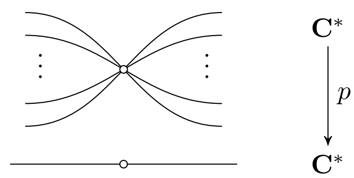
Example 3.2.7. Consider the following map, which has infinitely many sheets.

$$\begin{array}{rccc} p: & \mathbf{R}^1 & \to & \mathbf{S}^1 \subset \mathbf{C} \\ & t & \mapsto & e^{2\pi i t} \end{array}$$



Consider this map, which has k sheets.

$$\begin{array}{rccc} p: & \mathbf{C}^* & \to & \mathbf{C}^* \\ & z & \mapsto & z^k \end{array}$$



<u>*Proof:*</u> (of Theorem 3.2.5) Let $x_0 \in X$. Then there exists an open neighborhood U of x_0 in X such that $p^{-1}(U) = \bigcup_{j \in J} V_j$ with $V_j \subset Y$ open, $V_i \cap V_j = \emptyset$ if $i \neq j$, and $p|_{V_j} : V_j \to U$ a homeomorphism for all $j \in J$.Then for all $x \in U$, there exists a unique $y_j \in V_j$ such that $p(y_j) = x$ with $y_j \neq y_i$ if $j \neq i$, as $V_i \cap V_j = \emptyset$. Therefore $|p^{-1}(x)| = |J|$ for all $x \in U$.

Let us now show that $|p^{-1}(x)| = |J|$ for all $x \in X$. Set $A = \{x \in X \ : \ |p^{-1}(x)| = |J|\}$. Then $A \neq \emptyset$ since $x_0 \in A$. Also, A is open. Indeed, for all $\tilde{x} \in A$, there exists an open neighborhood $\tilde{U}$ of $\tilde{x}$ in X such that $p^{-1}(\tilde{U}) = \bigcup_{j \in \tilde{J}} \tilde{V}_j$, with the $\tilde{V}_j$s satisfying the conditions of a covering map. Then $|p^{-1}(x)| = |\tilde{J}|$, implying that $|\tilde{J}| = |J|$. And as before, for all $x \in \tilde{U}$, $|p^{-1}(x)| = |\tilde{J}| = |J|$, implying that $\tilde{U} \subset A$, so A is open. Finally, $X = \bigcup U_{\tilde{J}} A_{\tilde{J}}$, where $A_{\tilde{J}} = \{x \in X \ : \ |p^{-1}(x)| = |\tilde{J}|\}$ with each $A_{\tilde{J}}$ open. Hence $X = A$ since A is connected. ■

3.3 Proper holomorphic mappings

Definition 3.3.1. Let X, Y be Riemann surfaces. A holomorphic mapping $p : X \to Y$ is *proper* if $p^{-1}(K)$ is compact for all $K \subset X$ compact.

Example 3.3.2. Consider a map $p : Y \to X$ with Y compact. Then p is proper because for all $K \subset X$ compact, K is closed, so $p^{-1}(K)$ is closed in Y, hence compact, so $p^{-1}(K)$ is compact.

Consider the map $f : \mathbf{C} \to \mathbf{C}$ with $z \mapsto a$ for $a \in \mathbf{C}$. This map is not proper.

Proposition 3.3.3. Let X, Y be Riemann surfaces and $p : Y \to X$ a non-constant holomorphic mapping. Then

1. the set A of branch points of p is closed and discrete in Y.

Moreover, if p is proper, then

2. $p^{-1}(x)$ is finite for all $x \in X$,

3. $p(D)$ is closed and discrete in X for any discrete closed set $D \subset Y$. In particular, $B = p(A)$ is closed and discrete in X. And

4. if p is unbranched (so that $A = \emptyset$), then p is a covering map.

Proof: **1.** Let $W = \{y \in Y \;:\; y \text{ is not a branch point of } p\}$. Then W is open. Indeed, if $y \in Y$, then there exists an open neighborhood V of y in Y such that $p|_V$ is injective. Hence p is not branched at any $y \in V$, so $V \subset W$. Therefore the set A of all branch points of p is $Y \setminus W$, which is closed. To see that A is discrete, recall that p is locally looks like $z \mapsto z^k$ with $k \geqslant 1$ (since p is non-constant). And the map $z \mapsto z^k$ has an isolated branch point, namely $z = 0$.

2. We have already seen that $p^{-1}(x)$ is a discrete subset oy Y for all $x \in X$. But, $\{x\}$ is a compact subset of X, so $p^{-1}(x)$ is a compact, discrete subset by the preperness of p. Hence $p^{-1}(x)$ is finite.

3. Since p is open, $p(D)$ is closed in X. Now, to prove that $p(D)$ is discrete in X, it is enough to show that for all compact sets $K \subset X$, $p(D) \cap K$ is finite (otherwise $p(D) \cap K$ would have a limit point, since it would be an infinite subset of K, which is compact). But $p(D) \cap K = p(D \cap p^{-1}(K))$ and $p^{-1}(K)$ is compact by the properness of p. So $D \cap p^{-1}(K)$ is finite because D is discrete, and hence $p(D) \cap K = p(D \cap p^{-1}(K))$ is finite.

4. Suppose p is a proper and unbranched non-constant holomorphic map. We need to show that there exists an open neighborhood U of x in X such that $p^{-1}(U) = \bigcup_{j \in J} V_j$ with the V_js satisfying the properties of a covering map. Before we proceed, we need a lemma.

Lemma 3.3.4. If $V \subset Y$ is an open neighborhood of $p^{-1}(x)$, then there exists an open neighborhood U of x in X such that $p^{-1}(U) \subset V$.

Proof: Since p is open, $p(Y \setminus V)$ is closed in X because $Y \setminus V$ is closed in Y. Set $U = X \setminus p(Y \setminus V)$. Then $x \in U$ since $x \notin p(Y \setminus V)$ (because $p^{-1}(x) \subset V$). Also, $p^{-1}(U) \subset V$ by definition of U. ■

Note that although p is open, so that $p(V)$ is an open neighborhood of x in X, we may not have $p^{-1}(p(V)) \subset V$. We now return to the unfinished proof.

Proof: (of Proposition 3.3.3, **4.** continued) First note that since p is proper, the fiber $p^{-1}(x)$ is finite. So $p^{-1}(x) = \{y_1, \ldots, y_n\}$ with $y_i \neq y_j$ if $i \neq j$. Since p is unbranched, it is a local homeomorphism, so there exists an open neighborhood W_j of y_j in Y such that

$$p|_{W_j} : W_j \xrightarrow{homeom.} p(W_j) = U_j.$$

Note that $x = p(y_j) \in U_j$ for all j. Moreover, since Y is Hausdorff and the W_js are open neighborhoods of the y_js, which are pairwise distinct, we may assume that $W_i \cap W_j = \emptyset$ if $i \neq j$. Let $W = \bigcup_i W_i$, which is open, and $p^{-1}(x) = \{y_1, \ldots, y_n\} \subset W$. Note that although $x \in \bigcap_i U_i$, which is open in X, we may not have $p^{-1}(\bigcap_i U_i) \subset W$. But, by the lemma, there exists an open neighborhood U of x in X such that $p^{-1}(U) \subset W$. Set $V_j = p^{-1}(U) \cap W_j$. Then

- V_j is open for all j,
- $V_j \cap V_i = \emptyset$ if $j \neq i$ because $W_i \cap W_j = \emptyset$, and
- $p|_{V_j} : V_j \to U$ is a homeomorphism because we assume that $p|_{W_j}$ is a homeomorphism.

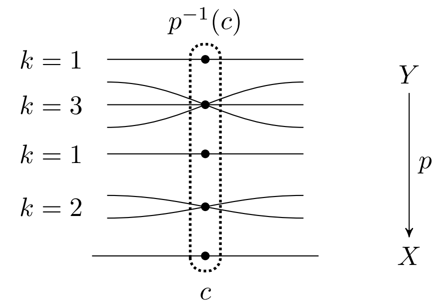
Hence p is a covering map. ■

Example 3.3.5. Consider the map $\exp : \mathbf{C} \to \mathbf{C}^*$ given by $z \mapsto e^{2\pi i z}$. This is not a proper non-constant holomorphic map, because the fibers of exp are not finite. For example, $p^{-1}(1) = \mathbf{Z}$.

Definition 3.3.6. Let $p : Y \to X$ be a proper non-constant holomorphic map. Let A be the set of branch points of p in Y, and set $B = p(A) \subset X$, which is called the *set of critical values*.

Remark 3.3.7. Note that A and B are closed and discrete subsets of X and Y, respectively, by the proposition. Moreover, $p : Y' = Y \setminus A \to X \setminus B = X'$ is an unbranched non-constant holomorphic map. Then $p : Y' \to X'$ is a covering map by the proposition, and therefore has a well-defined finite number of sheets, say n. For every $y \in Y$, set $v(p, y) = k$, the multiplicity that p takes at y, so that k is a positive integer and p looks like $z \mapsto z^k$ in a neighborhood of y centered at y. Then, for all $c \in X$, set

$$m_c = \sum_{y \in p^{-1}(c)} v(p, y) = \left(\begin{matrix} \text{multiplicity} \\ \text{of the value } c \end{matrix} \right).$$



The following theorem shows that $m_c = n$ for all $c \in X$.

Theorem 3.3.8. Let $p : Y \to X$ be a proper non-constant holomorphic map. Then there exists $n \in \mathbf{N}$ such that p takes every value $c \in X$, counting multiplicity, n times.

Proof: Let n be the number of sheets of the unbranched non-constant holomorphic map $; Y' \to X'$. Suppose that $b \in B \subset X$ is a critical value of p with $p^{-1}(b) = \{y_1, \ldots, y_r\}$, and $k_j = v(p, y_j)$. Then there exist open neighborhoods V_j of y_j in Y that are pairwise disjoint and are such that p looks like z^{k_j} an V_j. Then $p^{-1}(b) \subset V = \bigcap_i V_i$. Then by the lemma above, there exists an open neighborhood U of b in X such that $p^{-1}(U) \subset V$. Let $c \in U \cap Y'$. Then $p^{-1}(c) = \bigcup_i (p^{-1}(c) \cap V_j)$ with $p^{-1}(c) \cap V_i$ and $p^{-1}(c) \cap V_j$ disjoint if $i \neq j$. Then

$$\left| p^{-1}(c) \right| = \sum_{j=1}^{r} \left| p^{-1}(c) \cap V_j \right| \qquad \text{and} \qquad \left| p^{-1}(c) \cap V_j \right| = k_j,$$

since $y_j \notin (p^{-1}(c) \cap V_j)$ and $p|_{V_j}$ looks like $z \mapsto z^{k_j}$. Hence $k_1 + \cdots + k_r = |p^{-1}(c)| = n$, since $c \in Y'$. ■

Corollary 3.3.9. Let X be a compact Riemann surface. If $f \in M(X)$, then f has the same number of zeros as poles, counting multiplicities.

The above corolarry follows from properness.

Corollary 3.3.10. Let X be a compact Riemann surface. If there exists $f \in M(X)$ such that f has only one pole, and that pole has multiplicity 1, then $X \cong \mathbf{P}^1$.

4 Sheaves and analytic continuation

4.1 Sheaves

Definition 4.1.1. Let (X, τ) be a topological space. A *presheaf* of abelian groups on X is a pair $(\mathcal{F}, \rho)$ consisting of

1. a family $\mathcal{F} = \{\mathcal{F}(U)\}_{U \in \tau}$ of abelian groups, and

2. a family $\rho = \{\rho_V^U\}_{U,V\in\tau, V\subset U}$ of group homomorphisms $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ such that
$\rho_U^U = \mathrm{id}_{\mathcal{F}(U)}$ for all $U \in \tau$, and
$\rho_W^V \circ \rho_V^U = \rho_W^U$ for all $U, V, W \in \tau$ and $W \subset V \subset U$.

Remark 4.1.2. One usually writes $\mathcal{F}$ instead of $(\mathcal{F}, \rho)$. Further, the group homomorphisms ρ_V^U are called *restriction homomorphisms.* Instead of $\rho_V^U(f)$ for $f \in \mathcal{F}(U)$, we write $f|_V$. The elements of $\mathcal{F}(U)$ are called the *sections* of $\mathcal{F}$ over U. If $U = X$, then $\mathcal{F}$ is a *global section* of $\mathcal{F}$. Usually, we also set $\mathcal{F}(\emptyset) = \{0\}$.

Finally, we do not need to use abelian groups; we may also define presheaves of sets, rings, modules, etc.

Example 4.1.3. Consider the following examples of sheaves.

1. Let X be a topological space. Set $\mathcal{F}(U) = \{\text{continuous functions on } U \text{ for all open } U \subset X\}$. Set ρ_V^U to be the usual setriction functions, i.e. $\rho_V^U(f) = f|_V$.

2. Suppose X is a Riemann surface. Consider $\mathcal{O}(U)$, the holomorphic functions on U and ρ_V^U the usual restriction maps. Then $\mathcal{O}$ is the presheaf of holomorphic functions on X. Similarly, we have $\mathcal{O}^*$ the presheaf of nowhere-vanishing holomorphic functions on X, and $\mathcal{M}(U)$ and $\mathcal{M}^*(U)$.

Definition 4.1.4. Let $\mathcal{F}$ be a presheaf on X. Then $\mathcal{F}$ is called a *sheaf* of X if it satisfies the following properties for all open $U \subset X$ and open open cover $\{U_i\}_{i\in I}$:
1. Locality: If $f, g \in \mathcal{F}(U)$ are such that $f|_{U_i} = g|_{U_i}$ for all $i \in I$, then $f = g$.
2. Gluing: If $f_i \in \mathcal{F}(U_i)$ for all $i \in I$ such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$, then there exists $f \in \mathcal{F}(U)$ such that $f|_{U_i} = f_i$ for all $i \in I$.
Note that the function f whose existence is assured by **2.** is uniquely determined by **1.**.

Example 4.1.5. Consider the following examples of sheaves.
a. The presheaf of continuous functions on a topological space is a sheaf.
b. If X is a Riemann surface, then $\mathcal{O}$, $\mathcal{O}^*$, $\mathcal{M}$, $\mathcal{M}^*$ are sheaves.
c. Let X be a Riemann surface and $p \in X$. We define $\mathbf{C}_p(U) = \left\{\begin{smallmatrix} \mathbf{C} & p\in U \\ 0 & p\notin U \end{smallmatrix}\right.$ to be the *skyscraper sheaf.* The restriction maps are the usual ones.

Remark 4.1.6. Note that not every presheaf is a sheaf. Take $X = \{x, y\}$ with the discrete topology. The open sets in X are $\emptyset, \{x\}, \{y\}, X$. We define the presheaf $\mathcal{F}$ as follows:

$$\begin{array}{rcl} \mathcal{F}(\emptyset) & = & \{0\} \\ \mathcal{F}(\{x\}) & = & \mathbf{R} \\ \mathcal{F}(\{y\}) & = & \mathbf{R} \\ \mathcal{F}(\{x,y\}) & = & \mathbf{R}\times\mathbf{R}\times\mathbf{R} \end{array} \qquad \text{with} \qquad \begin{array}{rrcl} \rho_\emptyset^U : & \mathcal{F}(U) & \to & \mathcal{F}(\emptyset) \\ & x & \mapsto & 0 \\ \rho_{\{x\}}^X : & \mathbf{R}\times\mathbf{R}\times\mathbf{R} & \to & \mathbf{R} \\ & (a,b,c) & \mapsto & a \\ \rho_{\{y\}}^X : & \mathbf{R}\times\mathbf{R}\times\mathbf{R} & \to & \mathbf{R} \\ & (a,b,c) & \mapsto & b \end{array}.$$

Then $\mathcal{F}$ is not a sheaf because axiom **1.** fails. Indeed, consider $U = X = \{x\} \cup \{y\} = U_1 \cup U_2$ and pick $f = (a, b, c)$ and $f' = (a, b, c')$ with $c \neq c'$, so $f \neq f'$. Then

$$f|_{\{x\}} = a = f'|_{\{x\}} \qquad \text{and} \qquad f|_{\{y\}} = b = f'|_{\{y\}},$$

but $f \neq f'$.

Example 4.1.7. For another example, consider $X = \mathbf{R}^1$ with the usual topology. Set $\mathcal{F}(U)$ to be the bounded continuous functions on U, and ρ_V^U the usual restriction functions. Then, although $\mathcal{F}$ is a presheaf, it fails to satisfy axiom **2.**. Indeed, taking $X = \bigcup_{i\in N}(-i, i) = U_i$ and $f_i(x) = x$, f_i is clearly bounded and contuous on U_i for all i. However, there does not exist $f \in \mathcal{F}(X)$ with $f|_{U_i} = f_i$, since the only function f such that $f|_{U_i} = f_i$ is $f(x) = x$, which is unbounded on $X = \mathbf{R}^1$.

4.2 Stalks

Definition 4.2.1. Let $\mathcal{F}$ be a presheaf on a topological space $X \ni a$. Consider the formal disjoint union $\bigcup_{a\in U}\mathcal{F}(U)$, where the union is taken over every open neighborhood U of a. Introduce an equivalence relation on this disjoint union by letting, for U, V open neighborhoods of a with $f \in \mathcal{F}(U)$ and $g \in \mathcal{F}(V)$, $f \sim_a g$ iff there exists an open neighborhood W of a such that $W \subset (U \cap V)$ and $f|_W = g|_W$. Set the *stalk* of $\mathcal{F}$ at a to be

$$\mathcal{F}_a = \left(\bigcup_{a\in U}\mathcal{F}(U)\right) \Big/ \sim_a .$$

Further, let $\rho_a : \mathcal{F}(U) \to \mathcal{F}_a$, given by $f \mapsto [f]$ for any open neighborhood U of a, be the *germ* of f at a.

Example 4.2.2. Consider the following examples.

a. If $\mathcal{F} = \mathcal{O}$ on a Riemann surface X, then $\mathcal{O}_a = \{$germs of holomorphic functions on X at $a\}$, for all $a \in X$.The same thing happens if we consider the sheaf of continuous, or C^∞, or $\mathcal{M}(X)$.

b. For $\mathcal{F} = \mathbf{C}_p$ the skyscraper sheaf, $(\mathbf{C}_p)_a = \left\{\begin{smallmatrix}\mathbf{C} & a=p\\ 0 & a\neq p\end{smallmatrix}\right.$.

Note that by definition, if $\varphi \in \mathcal{F}_a$, then $\varphi = \rho_a(f)$ for some $f \in \mathcal{F}(U)$ with U an open neighborhood of a.

Definition 4.2.3. Let $\mathcal{F}$ be a presheaf on X. Set $|\mathcal{F}| = \bigsqcup_{x\in X}\mathcal{F}_x$, the disjoint union of all the stalks, with map $p : |\mathcal{F}| \to X$ given by $\varphi \in \mathcal{F}_x \mapsto x$. We endow $|\mathcal{F}|$ with a topology as follows: for all $U \subset X$ open and $f \in \mathcal{F}(U)$, let $[U, f] = \{\rho_x(f) \ : \ x \in U\} \subset |\mathcal{F}|$ be open. Then note that

- $p([U, f]) = U$, and in fact $p|_{[U,f]} : [U, f] \to U$ is a bijection.
- if $[V, g] \subset [U, f]$ for $V \subset X$ open and $g \in \mathcal{F}(V)$, then $V \subset U$, as $V = p([V, g]) \subset p([V, f]) = U$.
- if $\varphi \in [U, f]$, then $\varphi = \rho_x(f)$ with $p(\varphi) = x \in U$.

Theorem 4.2.4. Let $B = \{[U, f] \ : \ U \subset X$ open and $f \in \mathcal{F}(U)\}$. Then

1. B is the basis of a topology on $|\mathcal{F}|$, and

2. $p : |\mathcal{F}| \to X$ is a local homeomorphism.

Proof: **1.** We need to show that for all $\varphi \in |\mathcal{F}|$, there exists $[U, f] \in B$ with $\varphi \in [U, f]$. To see this, note that if $\varphi \in \mathcal{F}_x$, then there exists an open neighborhood U of x and $f \in \mathcal{F}(U)$ such that $\varphi = \rho_x(f) \in [U, f]$.

We also need to check that if $\varphi \in [U - f] \cap [V, g]$, then there exists $[W, h] \in B$ with $\varphi \in [W, h] \subset ([U, f] \cap [V, g])$. To see this, note that if $\varphi \in ([U, f] \cap [V, g])$, then $\varphi = \rho_x(f) = \rho_x(g)$ ot $x = p(\varphi)$. This means in particular that $x \in U \cap V$. We then have $\rho_x(f) = \rho_x(g)$, implying that $f \sim_a g$, so there exists an open neighborhood W of x such that $f|_W = g|_W = h$. Hence $\varphi = \rho_x(h) \in [W, h]$ and $\rho_y(h) = \rho_y(f) = \rho_y(g)$ for all $y \in W$, meaning that $[W, h] \subset ([U, f] \cap [V, g])$ Therefore B is the basis of a topology on $|\mathcal{F}|$.

2. Let us first check that $p : |\mathcal{F}| \to X$ is continuous. Let $U \subset X$ be open and pick $\varphi \in p^{-1}(U)$. Then $\varphi = \rho_x(f)$ for some $f \in \mathcal{F}(V)$, with V an open neighborhood of $x = p(\varphi) \in U$. Therefore $x \in (U \cap V)$ and $\varphi \in [U \cap V, f|_{U\cap V}]$. This is contained in $p^{-1}(U)$ since $p([U \cap V, f|_{U\cap V}]) = (U \cap V) \subset U$. Hence $p^{-1}(U)$ is open, so p is continuous.

To show that p is a local homeomorphism, let $\varphi \in |\mathcal{F}|$, and find an open neighborhood $\tilde{U}$ of φ such that $p|_{\tilde{U}}$ is a homeomorphism. As before, $\varphi = \rho_x(f)$ for some $f \in \mathcal{F}(U)$ with U an open neighborhood of $x = p(\varphi)$. Then $\varphi \in [U, f]$ and $p|_{[U,f]} : [U, f] \to U$ is a continuous bijection. Also, $(p|_{[U,f]})^{-1} : U \to [U, f]$ is continuous, since for any $[V, g] \subset [U, f]$, we have that $((p|_{[U,f]})^{-1})^{-1}([V, g]) = p([V, g]) = V \subset U$, which is open in U. ■

Remark 4.2.5. If $f \in \mathcal{F}(U)$, then the map

$$\begin{array}{rccc} \hat{f}: & U & \to & \mathcal{F} \\ & x & \mapsto & \rho_x(f) \end{array}$$

is continuous, with $p \circ \hat{f} = \mathrm{id}_U$. It is called a *section* of f over U.

Theorem 4.2.6. Let X be a Riemann surface. Then $|\mathcal{O}(X)| = |\mathcal{O}|$ is Hausdorff.

Proof: Let $\varphi_1, \varphi_2 \in |\mathcal{O}|$ be such that $\varphi_1 \neq \varphi_2$. Suppose that $p(\varphi_1) = x_1 \neq x_2 = p(\varphi_2)$, so $x_1, x_2 \in X$. So there exist disjoint open neighborhoods $U_1, U_2 \subset X$ of x_1, x_2, respectively. Then $\varphi_i \in p^{-1}(U_i)$ for $i = 1, 2$ by continuity of $p : |\mathcal{O}| \to X$. But $p^{-1}(U_1) \cap p^{-1}(U_2) = \emptyset$, since $U_1 \cap U_2 = \emptyset$. Hence we can separate φ_1 and φ_2.

If that case does not hold, we may suppose that $p(\varphi_1) = p(\varphi_2) = x$. Then there exist open neighborhoods U_i of x and $f_i \in \mathcal{O}(U_i)$ with $\varphi_i = \rho_x(f_1)$ for $i = 1, 2$. Then $\varphi_i \in [U_i, f_i]$ and $x \in U_1 \cap U_2$. Let U be the connected component of $U_1 \cap U_2$ containing x. Then $\varphi_i \in [U, f_i|_U] \in B$ for $i = 1, 2$. So the $[U, f_i|_U]$ are open neighborhoods of the φ_is. Suppose that we do not have $[U, f_1|_U] \cap [U, f_2|_U] \neq \emptyset$. Then $\rho_y(f_1) = \rho_y(f_2)$ for some $y \in U$. Thus $f_1 \sim_y f_2$, implying that there exists an open neighborhood W of y such that $W \subset U$ and $f_1|_W = f_2|_W$. Then by the identity theorem, $f_1 = f_2$ on U, since U is connected (and therefore a Riemann surface). Hence $\varphi_1 = \rho_x(f_1) = \rho_x(f_2) = \varphi_2$, a contradiction. So the intersection is indeed empty. Therefore we can separate φ_1 and φ_2, so the space is Hausdorff. ■

Theorem 4.2.7. Let X be a Riemann surface, Y a Hausdorff topological space, and $p : Y \to X$ a local homeomorphism. Then there exists a unique complex structure on Y such that p is holomorphic.

Corollary 4.2.8. Let X be a Riemann surface. Then there exists a unique complex structure on $|\mathcal{O}|$ such that $p : |\mathcal{O}| \to X$ is holomorphic. In fact, if Y is any conneted component of $|\mathcal{O}|$, then Y is a Riemann surface and $p|_Y : Y \to X$ is an unbranched holomorphic mapping.

Proof: Let $y_0 \in Y$. Then there exists an open neighborhood $\tilde{U} \subset Y$ of y_0 such that $p|_{\tilde{U}} : \tilde{U} \to p(\tilde{U})$ is a homeomorphism. Now, $p(y_0) \in p(\tilde{U}) \subset X$, for $p(\tilde{U})$ open in X. Let $(\tilde{U}_1, \tilde{\varphi}_1)$ be a chart of X with $p(y_0) \in \tilde{U}_1$. Now set

$$\begin{aligned} U_1 &= \tilde{U}_1 \cap p(\tilde{U}) \subset X, \\ V &= \tilde{\varphi}_1(U_1) \subset \mathbf{C}, \\ \varphi_1 &= \tilde{\varphi}_1|_{U_1} : U_1 \subset X \to V \subset \mathbf{C}. \end{aligned}$$

Then (U_1, φ_1) is a chart of X with $p(y_0) \in U_1$. Set $U = p^{-1}(U_1)$. Then $p|_U : U \to U_1$ is a homeomorphism. Now, $\varphi = \varphi_1 \circ p : U \subset Y \to V \subset \mathbf{C}$ is a homeomorphism with $y_0 \in U$. Hence (U, φ) is a chart of Y with $y_0 \in U$. Given two such charts $(U, \varphi = \varphi_1 \circ p)$ and $(U', \psi = \psi_1 \circ p)$, $\psi \circ \varphi^{-1} = (\psi_1 \circ p) \circ (p^{-1} \circ \varphi_1^{-1}) = \psi_1 \circ \varphi_1^{-1}$ is holomorphic. Hence $\{(U, \varphi)\}$ is a complex structure on Y.

Uniqueness is left as an exercise. ■

4.3 The Riemann surface of a holomorphic function

Suppose that f is a holomorphic function on an open set $U \subset \mathbf{C}$. What is the biggest subset of $\mathbf{C}$ in which f exists? In other words, what is the biggest subset $W \subset \mathbf{C}$ in which f can be extended holomorphically?

Remark 4.3.1. If W is a domain containing U, then such an extension must be unique by the identity theorem. Indeed, if f_1 and f_2 are extensions of f to W, then $f_1|_U = f = f_2|_U$, so $f_1 = f_2$, because U is connected. Then problem with such an extension is that it may lead to multivalued functions.

Example 4.3.2. Consider $f(z) = \sqrt{z}$ and pick the analytic branches. For $z = re^{i\theta}$,

$$f_1(z) = \sqrt{r}e^{i\theta/2},\ \theta \in (-\pi, \pi) \qquad \text{and} \qquad f_2(z) = \sqrt{r}e^{i\theta/2},\ \theta \in (0, 2\pi).$$

Note that $f_1(z)$ is not continuous along the negative x-axis, and $f_2(z)$ is not continuous along the positive x-axis. So we cannot piece them together to get an analytic function on $\mathbf{C}^*$ because we will get a multivalued function. To reconcile this, we replace the complex plane by a potential domain of f by its graph. Let $w = \sqrt{z}$. Then set

$$S = \{(z, w) \ : \ z = w^2\} = \{(z, w) \ : \ p(z, w) = z - w^2 = 0\} \subset \mathbf{C},$$

with $\nabla p \neq 0$ on S. Then S is a Riemann surface, and f may be thought of as a projection of S onto the w-axis, $(z, w) \mapsto w$, which is single-valued.

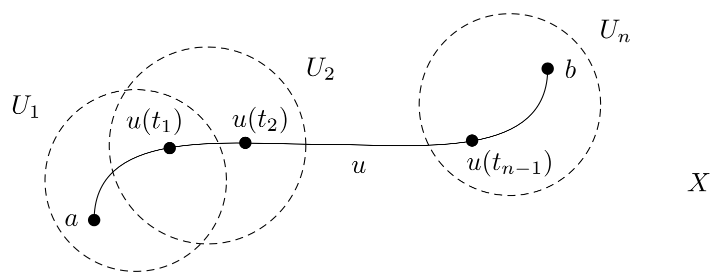
Definition 4.3.3. Let X be a Riemann surface, and $u : [0, 1] \to X$ a curve in X, with $a = u(0)$ and $b = u(1)$ (we assume u to be continuous). The holomorphic germ $\psi \in \mathcal{O}_b$ is said to be the result of an *analytic continuation* along the curve u of the holomorphic germ $\varphi \in \mathcal{O}_a$, if there exist:

· a partition $0 = t_0 < \cdots < t_n = 1$ of $[0, 1]$,

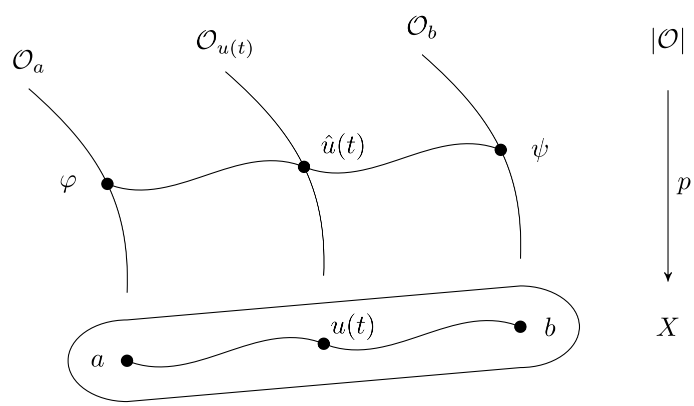
· connected open sets $U_i \subset X$ with $u([t_{i-1}, t_i]) \subset U_i$,

· $f_i \in \mathcal{O}(U_i)$ such that $\varphi = \rho_a(f_1)$, $\psi = \rho_b(f_n)$, and $f_i|_{V_i} = f_{i+1}|_{V_i}$, where V_i is the connected component of $U_i \cap U_{i+1}$ containing $u(t_i)$

for all i, as in the diagram below.



Lemma 4.3.4. Let X be a Riemann surface and $u : [0, 1] \to X$ a curve with $a = u(0)$ and $b = u(1)$. Then $\psi \in \mathcal{O}_b$ is the analytic continuation of $\varphi \in \mathcal{O}_a$ along u iff there exists a curve $\hat{u} : [0, 1] \to |\mathcal{O}|$ such that $\hat{u}(0) = \varphi$, $\hat{u}(1) = \psi$, and $p \circ \hat{u} = u$ (that is, $\hat{u}$ is a lifting of U to $p : |\mathcal{O}| \to X$).



Proof: ($\Rightarrow$) Suppose that $\psi \in \mathcal{O}_b$ is an analytic continuation of $\varphi \in \mathcal{O}_a$ along u. Set $\hat{u}(t) = \rho_{u(t)}(f_i)$ if $u(t) \in u([t_{i-1}, t_i]) \subset U_i$ (that is, $t \in [t_{i-1} t_i]$). Then $\hat{u}$ is well-defined, since $f_i|_{V_i} = f_{i+1}|_{V_i}$ for all i. Also, $\hat{u}$ is continuous. It is enough to show that $\hat{u}^{-1}([U, f]) \subset [0, 1]$ is open for any $[U, f] \in B$, where B is the basis of the topology on $|\mathcal{O}|$. Let $t \in \hat{u}^{-1}([U, f])$. Then $\hat{u}(1) \in [U, f]$, so $\rho_{u(t)}(f_i) = \hat{u}(t) \in [U, f]$. This implies that $\rho_{u(t)}(f_i) = \rho_{u(t)}(f)$ with $u(t) \in U$ (and $u(t) \in U_i$), so $f_i \sim_{u(t)} f$. Hence there exists an open neighborhood W of $u(t)$ such that $W \subset (U_i \cap U)$ and $f_i|_W = f|_W$. Therefore

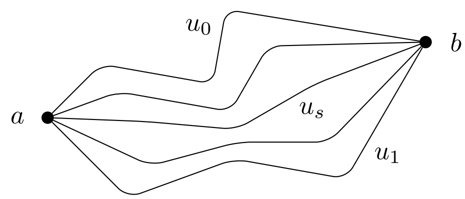
$$\rho_{u(s)}(f_i) = \rho_{u(s)}(f) \ \ \forall \ s \in u^{-1}(W). \tag{1}$$

But, by continuity of u, $u^{-1}(W) \subset [0, 1]$ is open. Then by (1), we have that $u^{-1}(W) \subset \hat{u}^{-1}([U, f])$ with $u^{-1}(W)$ an open neighborhood of t. Therefore $\hat{u}^{-1}([U, f])$ is open, so $\hat{u}$ is continuous.

($\Leftarrow$) Suppose that $\hat{u} : [0, 1] \to |\mathcal{O}|$ is a lifting of u, so that $\hat{u}(0) = \varphi$, $\hat{u}(1) = \psi$, and $p \circ \hat{u} = u$. Then for all $t \in [0,1]$, we have that $\hat{u}(t) = \rho_{u(t)}(f_t)$ for some $f_t \in \mathcal{O}(U_t)$, with U_t an open neighborhood of $u(t)$, so $\hat{u}(t) \in [U_t, f_t]$ for all $t \in [0,1]$. Hence $\{[U_t, f_t] \ : \ t \in [0,1]\}$ is an open cover of $\hat{u}([0,1]) \subset |\mathcal{O}|$. However, $\hat{u}([0,1])$ is compact since it is the continuous image of a compact set. Thus there exists a finite subcover $\{[U_i, f_i] \ : \ i = 1, \dots, n\}$ of $\hat{u}([0,1])$, and a portition $0 = t_0 < \cdots < t_n = 1$ of $[0,1]$ that satisfies the condition of analytic continuation along u. The rest of the details are left as an exercise. ■

Remark 4.3.5. The lemma tells us that there is a 1-1 correspondence between analytic continuation on S along curves in X, and curves in $|\mathcal{O}|$.

Theorem 4.3.6. [Monodromy theorem]
Let X be a Riemann surface and $u_0, u_1 : [0,1] \to X$ holomorphic curves from a to b, such that there exists a continuous map : $[0,1] \times [0,1] \to X$ with $A(t,0) = u_0(t)$ and $A(t,1) = u_1(t)$ for all t, with fixed endpoints $A(0,s) = a$ and $A(1,s) = b$. Set $u_s(t) = A(t,s)$, so it is a deformation retract of u_0 onto u_1.

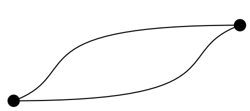


Suppose that $\varphi \in \mathcal{O}_a$ admits an analytic continuation along every curve u_s. Then the analytic continuations of φ along u_0 and u_1 yield the same germ $\psi \in \mathcal{O}_b$.

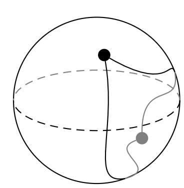
Proof: By the lemma, the analytic continuation of φ along u_s corresponds to the lift $\hat{u}_s : [0,1] \to |\mathcal{O}|$ with $\hat{u}_s(0) = \varphi$ and $p \circ \hat{u}_s = u_s$. Also, $\hat{u}_s(1) \in \mathcal{O}_b$, implying that $\hat{u}_s(1) \in p^{-1}(\{b\})$. Note that each $\hat{u}_s$ lives in the connected component of $|\mathcal{O}|$ containing φ (since $\hat{u}_s(0) = \varphi$ for all s). Let Y be the connected component of $|\mathcal{O}|$ containing φ. Then Y is a Riemann surface and $p = p|_Y : Y \to X$ is an unbranched holomorphic map. Hence $\hat{u}_s(a) \in p^{-1}(\{b\})$ for all $s \in [0,1]$, and $p^{-1}(\{b\})$ is discrete. Therefore $A(\{1\} \times [0,1]) \subset p^{-1}(\{b\})$. But $A(\{1\} \times [0,1])$ is the continuous image of the connected set $\{1\} \times [0,1]$, so $A(\{1\} \times [0,1])$ is connected. Therefore $A(\{1\} \times [0,1]) = \{\psi\}$ for some $\psi \in \mathcal{O}_b$, since $p^{-1}(\{b\})$ is discrete. Therefore $\hat{u}_s(1) = \psi$ for all $s \in [0,1]$. ■

Definition 4.3.7. A topological space X is called *simply connected* if any two curves in X with the same inital point and the same endpoint are homotopic.

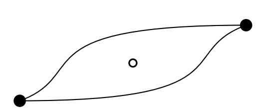
Example 4.3.8. Consider the following spaces:



$\mathbf{C}$ is simply connected,



$\mathbf{P}^1 \cong S^2$ is simply connected,



$\mathbf{C}^*$ is not simply connected.

Corollary 4.3.9. Let $X \ni a$ be a simply connected Riemann surface and $\varphi \in \mathcal{O}_a$ a germ that admits an analytic continuation along any curve starting at a. Then there exists a globally-defined holomorphic function $f \in \mathcal{O}_(X)$ such that $\rho_a(f) = \varphi$.

Proof: Since X is simply connected, any two curves in X with the same endpoints are homotopic. Therefore the analytic contuation of φ alonf the curves will yield the same germ $\psi_x \in \mathcal{O}_x$ (that is, analytic continuation is path independent) by the monodromy theorem. Set $f(x) = \psi_x(x)$, so $\psi_x = [g]$ for some holomorphic function $g \in \mathcal{O}(U)$, with U an open neighborhood of x. Then $\psi_x(x) = g(x)$ is well-defined, becaues if $g \sim_x g'$ for some $g' \in \mathcal{O}(U')$, with U' and open neighborhood of x, then there exists an open neighborhood W or x with $W \subset (U \cap U')$ and $g|_W = g'|_W$, meaning that $g(x) = g'(x)$.

So we have that f is holomorphic at x. Let us check that f is holomorphic at any $x \in X$. So $f(x) = \psi_x(x)$ with ψ_x the analytic continuation of φ along a curve $u : [0,1] \to X$ from a to x. This analytic continuation is given by a partition $0 = t_0 < \cdots < t_n = 1$ of $[0,1]$, connected open sets $U_i \subset X$ and $f_i \in \mathcal{O}(U_i)$. Note that we have $\psi_x = \rho_x(f_n)$.

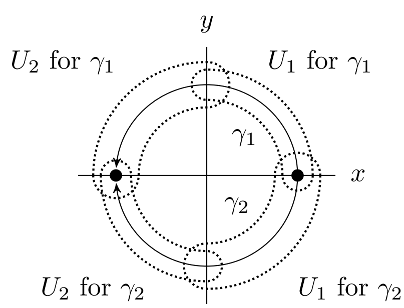
Next, we claim that $\psi_{x'} = \rho_{x'}(f_n)$ for all $x' \in U_n$, f is given by f_n in U_n, and so f is holomorphic on U_n and at x. To see this, for all $x' \in U_n$ consider the curve

$$\begin{array}{rccl} \hat{u}: & [0,1] & \to & X \\ & t & \mapsto & \begin{cases} u(t) & \text{if } t \in [0, t_{n-1}] \\ v(t) & \text{if } t \in [t_{n-1}, t_n] \end{cases} \end{array} .$$

So the partition $0 = t_0 < \cdots < t_n = 1$, the connected open sets $U_i \subset X$, and the $f_i \in \mathcal{O}(U_i)$ will determine the analytic continuation of φ along $\hat{u}$, which must be ψ_x, since analytic continuation is path independent in X. This proves the claim and the theorem. ■

Note that in general, if X is not simply connected, then analytic continuation along two curves with the same initial point and the same endpoint may yield non-identical germs.

Example 4.3.10. Consider the function $f(z) = \sqrt{z}$. Near 1, we can analytically continue



Along γ_1, using U_1 and U_2 as indicated, we gut

$$f_1(z) = r^{1/2}e^{i\theta/2},\ \theta \in (-\pi, \pi) \qquad \text{and} \qquad f_2(z) = r^{1/2}e^{i\theta/2},\ \theta \in (0, 2\pi),$$

meaning that $\psi = (-1)^{1/2} = i$. However, along γ_2, using the indicated U_1 and U_2, while we get the same f_1, for f_2 we have $f_2 = r^{1/2}e^{i\theta/2}$ for $\theta \in (-2\pi, 0)$, meaning that $\psi = (-1)^{1/2} = -i$. Thus, if we consider all the germs obtained by analytic continuation, we get a multivalued function.

Remark 4.3.11. Recall that if $p : Y \to X$ is an unbranched holomorphic map between Riemann surfaces, then for all $s \in Y$, there exists an open neighborhood V of y such that $p' = p|_V : V \to p(V) \subset X$ is biholomorphic. We define

$$\begin{array}{rccc} p^*: & \mathcal{O}_{X,p(y)} & \to & \mathcal{O}_{Y,y} \\ & [v] & \mapsto & [v \circ p] \end{array} \qquad \text{and} \qquad \begin{array}{rccc} p_*: & \mathcal{O}_{Y,y} & \to & \mathcal{O}_{X,p(y)} \\ & [u] & \mapsto & [u \circ p^{-1}] \end{array},$$

which are well-defined isomorphisms. Note that by definition, $p^*([y]) = [u \circ p]$, which can be written as $p^*(\rho_{p(y)}(g)) = \rho_y(g \circ p) = \rho_y(p^*(g))$. Similarly, $p_*([h]) = [h \circ (p|_V)^{-1}]$ can be written as $p_*(\rho_y(h)) = \rho_{p(y)}((p^{-1})^*(h))$.

4.4 Analytic continuation

Definition 4.4.1. Let $X \ni a$ be a Riemann surface and $\varphi \in \mathcal{O}_a$. A 4-tuple (Y, p, F, b) is called an *analytic continuation* of φ if

i. Y is a Riemann surface and $p : Y \to X$ is an unbranched holomorphic map,

ii. F is a holomorphic function on Y, and

iii. $b \in Y$ is such that $p(b) = a$ and $p_*(\rho_b(F)) = \varphi$.

Theorem 4.4.2. Analytic continuations always exist.

Proof: Take Y to be the connected component of $|\mathcal{O}|$ containing φ and $p = p|_Y : Y \to X$ given by $\varphi \in \mathcal{O}_x \mapsto x = p(\varphi)$. For η a germ, set

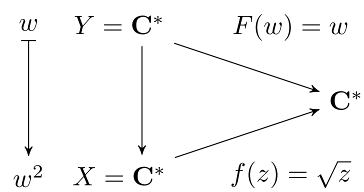
$$\begin{array}{rccc} F: & Y \subset |\mathcal{O}| & \to & \mathbf{C} \\ & \eta & \mapsto & \eta(p(\eta)) \end{array}.$$

We claim that F is holomorphic. Let $(U, \varphi = \varphi_1 \circ p|_U)$ be a chart on Y, where $(p(U), \varphi_1)$ is a chart on X. We want to show that $F \circ \varphi^{-1} : \varphi(U) \subset \mathbf{C} \to \mathbf{C}$ is holomorphic. So let $\eta \in U$. Then there exists $[V, f] \in B$ (the basis of topology on a set of germs $|\mathcal{O}|$) with $\eta \in [V, f] = \{\rho_x(f) \ : \ x \in V\}$. Then $[V, f] \subset U$ WLOG, since U is generated by X. Then for al $\alpha \in [V, f]$, $\alpha = \rho_{p(\alpha)}(f)$ and $\alpha(p(\alpha)) = f(p(\alpha))$. Hence $F(\alpha) = f \circ p(\alpha)$, so $F = p^*(f)$ on $[V, f]$, telling us that $f = (p^{-1})^*(F)$ on V. Then

$$F \circ \varphi^{-1} = F \circ (\varphi_1 \circ p)^{-1} = (F \circ p^{-1}) \circ \varphi_1^{-1} = (p^{-1})^*(F) \circ \varphi_1^{-1} = f \circ \varphi_1^{-1},$$

which is holomorphic, since $f \in \mathcal{O}(V)$. This proves the claim. Finally, set $b = \varphi$. Then (Y, p, F, b) is an analytic continuation of φ because $p_*(\rho_b(F)) = \rho_{p(b)}((p^{-1})^*(F)) = \rho_a(f) = \varphi$. ■

Example 4.4.3. The elements in the diagram



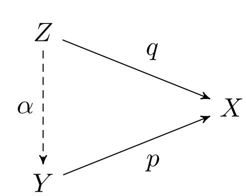
describe an analytic continuation of the germ of $f(z) = \sqrt{z}$ at any point.

Lemma 4.4.4. Let $X \ni a$ be a Riemann surface with $\varphi \in \mathcal{O}_a$ and (Y, p, F, b) an analytic continuation of φ. If $v : [0, 1] \to Y$ is any curve with $v(0) = b$, $v(1) = y$, then $\psi = p_*(\rho_y(F)) \in \mathcal{O}_{p(y)}$ is an analytic continuation of φ along $u = p \circ v$.

Proof: We have $u(t) = p(v(t))$ for all $t \in [0, 1]$. For every $t \in [0, 1]$, let $\hat{u}(t) = p_*(\rho_{v(t)}(F)) \in \mathcal{O}_{p(v(t))} = \mathcal{O}_{v(t)}$. Then $\hat{u}(0) = p_*(\rho_{v(0)}(F)) = p_*(\rho_b(F)) = \varphi$, by the definition of (Y, p, F, b), and $\hat{u}(1) = p_*(\rho_{v(1)}(F)) = \psi$. One can check that $\hat{u}$ is continuous (as in the previous lemma). And $p \circ \hat{u} = u$, so $\hat{u}$ is a lift of u, meaning that ψ corresponds to analytic continuation of φ along u. ■

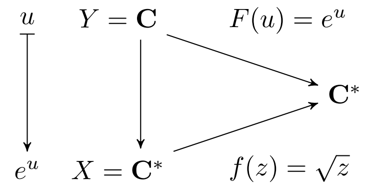
Remark 4.4.5. The lemma tells us that we recover the notion of analytic continuation along a curve from the definition of analytic continuation.

Definition 4.4.6. An analytic continuation (Y, p, F, b) is called *maximal* if it has the following universal property: if (Z, q, G, c) is any other analytic continuation of φ, then there exists a fiber-preserving holomorphic map $\alpha : Z \to Y$ such that $\alpha(c) = b$, $\alpha^*(F) = G$, and the diagram below commutes.

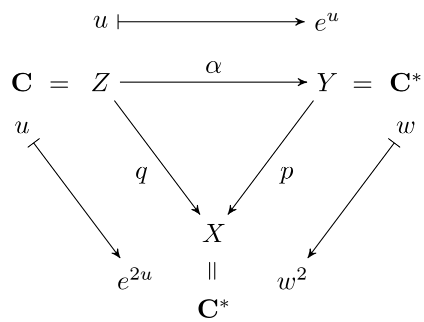


Theorem 4.4.7. Maximal analytic continuations always exist. They are in fact given by the 4-tuple (Y, p, F, b) where Y is the connected component of $|\mathcal{O}|$ containing φ, and the rest of the conditions as in the previous theorem.

Example 4.4.8. The diagram



describes an analytic continuation of the germ of f at any pont $a \in \mathbf{C}^*$. Note that we have the diagram



commuting.

5 Section 5

5.1 Calculus on Riemann surfaces

Definition 5.1.1. On $\mathbf{C}$, let $U \subset \mathbf{C}$ be open. We identify $\mathbf{C}$ with $\mathbf{R}^2$ by writing $z = x + iy$, where (x, y) are the coordinates of $\mathbf{R}^2$. Define

$$\mathcal{E}(U) = \{f : U \to \mathbf{C} \ : \ f \text{ is } \infty\text{-differentiable with respect to } x,\, y\}.$$

Then $\mathcal{E}(U)$ is a $\mathbf{C}$-algebra, and also, in particular, an abelian group. Set $\mathcal{E} = (\mathcal{E}(U), \rho)$, where ρ is the natural restriction of functions. Then $\mathcal{E}$ is a sheaf, called the *sheaf of differentiable functions* on $\mathbf{C}$. Here

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Further, $\mathcal{O}(U) = \ker(\frac{\partial}{\partial z} : \mathcal{E}(U) \to \mathcal{E}(U))$. In fact, $\mathcal{O}$ is the kernel of the map of sheaves $\frac{\partial}{\partial z} : \mathcal{E} \to \mathcal{E}$. Now we can extend these differential operators to any Riemann surface via charts.

Definition 5.1.2. Let X be a Riemann surface and $Y \subset X$ an open subset. A function $f : Y \to \mathbf{C}$ is said to be *differentiable* on Y if for all charts $\varphi : U \subset Y \to V \subset \mathbf{C}$ on X, $f \circ \varphi^{-1} : V \to \mathbf{C}$ is in $\mathcal{E}(U)$. Note that U, V are open.

Remark 5.1.3. Note that:
· Holomorphic functions are differentiable, since $\mathcal{O}(V) \subset \mathcal{E}(V)$.
· The sum, product, and composition of differentiable functions is differentiable.
· We only need to check is f is differentiable on a set of charts $\{(U_\alpha, \varphi_\alpha)\}$ such that $X = \bigcup_\alpha U_\alpha$. Indeed, if

(U,φ) is another chart, then $U \cap U_\alpha \neq \emptyset$ for some α and $\varphi_\alpha \circ \varphi^{-1}$ is holomorphic on U, so $\varphi_\alpha \circ \varphi^{-1} \in \mathcal{E}(\varphi(U))$. Hence

$$f \circ \varphi^{-1} = \underbrace{\left(f \circ \varphi_\alpha^{-1}\right)}_{\in\, \mathcal{E}(\varphi_\alpha(U_\alpha))} \circ \underbrace{(\varphi_\alpha \circ \varphi_\alpha)}_{\in\, \mathcal{E}(\varphi(U))},$$

so $f \circ \varphi^{-1} \in \mathcal{E}(\varphi(U))$. Let (U,φ) be a chart on X and $f \in \mathcal{E}(U)$. Suppose that $\varphi = z = x + iy$, where $x = \mathrm{Re}(\varphi)$ and $y = \mathrm{Im}(\varphi)$, since $\varphi : U \to \mathbf{C}$.

Definition 5.1.4. Define $\frac{\partial f}{\partial x}$ as below, and similarly define $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$, $\frac{\partial f}{\partial \overline{z}}$:

$$\frac{\partial f}{\partial x} = \varphi^* \left(\frac{\partial}{\partial x}(f \circ \varphi^{-1}) \right).$$

Note that as on $\mathbf{C}$, the differential operators $\frac{\partial}{\partial x}$, etc, are $\mathbf{C}$-linear and admit the usual product and chain rules. Further, let $a \in X$. Then

$$\begin{aligned}
\mathcal{E}_a &= \{\text{germs of differentiable functions on } X \text{ at } a\}, \\
m_a &= \{\eta \in \mathcal{E}_a \ : \ \eta \text{ is the germ of a differentiable function that vanishes at } a\}, \\
m_a^2 &= \left\{\eta \in m_a \ : \ \eta = [f] \text{ with } \frac{\partial f}{\partial x} a = \frac{\partial f}{\partial y}(a) = 0 \text{ in any chart } (U \ni a, \varphi = x + iy)\right\} \\
&= \{\text{germs of differentiable functions that vanish to 2nd order at } a\}.
\end{aligned}$$

Note that $\mathcal{E}_a$, m_a, and m_a^2 are $\mathbf{C}$-vector spaces with m_a a subspace of $\mathcal{E}_a$ and m_a^2 a subspace of m_a.

Proposition 5.1.5. The definition of m_a^2 is independent of the representative f of η and of the chart $(U \ni a, \varphi)$.

Proof: Suppose that $\eta = [f] = [g]$ with $f \in \mathcal{O}(U)$ and $g \in \mathcal{O}(V)$, where U, V are open sets containing a. Then $f \sim_a g$, implying that there exists an open set $W \subset U \cap V$ with $a \in W$ such that $f|_W = g|_W$. Then $(\partial f/\partial x)(a) = (\partial g/\partial x)(a)$ (and similarly for y). If $(U', \varphi' = z' = x' + iy')$ is another chart with $a \in U'$, then

$$\frac{\partial f}{\partial x'}(a) = \frac{\partial f}{\partial x}(a)\frac{\partial f}{\partial x'}(a) + \frac{\partial f}{\partial y}(a)\frac{\partial f}{\partial y'}(a) = 0.$$

■

Definition 5.1.6. The quotient vector space $T_a^{(1)} = m_a/m_a^2$ is called the *cotangent space* of X at a, and the elements of $T_a^{(1)}$ are called *cotangent vectors* to X at a. Also, if U is an open neighborhood of a and $f \in \mathcal{E}(U)$, then the differential of f at a is the element

$$d_a f = [f - f(a)] \mod m_a^2.$$

Note that $[f - f(a)] \in m_a$ since $(f - f(a))(a) = 0$.

Theorem 5.1.7. Let $(U, \varphi = z = x + iy)$ be a chart with $a \in U$. Then $\{d_a x, d_a y\}$ is a basis of $T_a^{(1)}$, and so is $\{d_a z, d_a \overline{z}\}$. Moreover, if $f \in \mathcal{E}(U)$, then

$$d_a f = \frac{\partial f}{\partial x}(a) d_a x + \frac{\partial f}{\partial y}(a) d_a y \qquad \text{and} \qquad d_a f = \frac{\partial f}{\partial z}(a) d_a z + \frac{\partial f}{\partial \overline{z}}(a) d_a \overline{z}.$$

Proof: So we first show that $T_a^{(1)} = \mathrm{span}_{\mathbf{C}}\{d_a x, d_a y\}$. Let $t \in T_a^{(1)}$, and suppose that $t = \varphi \mod m_a^2$ for some $\varphi \in m_a$. Also, $\varphi = [f]$ with $f \in \mathcal{E}(U)$ and $f(a) = 0$. Then by Taylor around a,

$$\begin{aligned}
\varphi &= [f] \\
&= \underbrace{[c_1(x - x(a)) + c_2(y - y(a))]}_{\in\, m_a} + \underbrace{(\text{higher order terms})}_{\in\, m_a^2} \\
&= [c_1(x - x(a)) + c_2(y - y(a))] \mod m_a^2,
\end{aligned}$$

with $c_1, c_2 \in \mathbf{C}$. Hence $t = c_1 d_a x + c_2 d_a y$. Now we need to show that $d_a x$ and $d_a y$ are linearly independent. Suppose that $c_1, c_2 \in \mathbf{C}$ are such that $c_1 d_a x + c_2 d_a y = 0$. Set $f = c_1(x - x(a)) + c_2(y - y(a))$. Then $f(a) = 0$, and $(\partial f/\partial x)(a) = c_1$ and $(\partial f/\partial y)(a) = c_2$. So $f \in m_a^2$ iff $c_1 = c_2 = 0$, but $f \in m_a^2$ iff $c_1 d_a x + c_2 d_a y = 0$. Hence $d_a x$, $d_a y$ are linearly independent. Next, note that if $f \in \mathcal{E}(U)$ then

$$[f - f(a)] = \left[\frac{\partial f}{\partial x}(a)(x - x(a)) + \frac{\partial f}{\partial y}(a)(y - y(a))\right] + (\text{higher order terms}),$$

so

$$d_a f = \frac{\partial f}{\partial x}(a)[x - x(a)] + \frac{\partial f}{\partial y}(a)[y - y(a)] \mod m_a^2 = \frac{\partial f}{\partial x}(a) d_a x + \frac{\partial f}{\partial y}(a) d_a y.$$

The proof is similar for $d_a z$ and $d_a \overline{z}$. ■

Remark 5.1.8. If we think of X as a 2-dimensional real manifold, then $T_a^{(1)} = (T_a^* X) \otimes \mathbf{C}$. That is, $T_a^{(1)}$ is the complexification of $T_a^* X = \operatorname{span}_{\mathbf{R}}\{d_a x, d_a y\}$.

Definition 5.1.9. Suppose that $(U, \varphi = z)$ is local chart with $a \in U$. By the above theorem, we have that $T_a^{(1)} = \operatorname{span}_{\mathbf{C}}\{d_a z, d_a \overline{z}\}$. Set

$$\begin{aligned} T_a^{1,0} &= \operatorname{span}_{\mathbf{C}}\{d_a z\} = \text{ cotangent vectors of type } (1,0)\text{, and} \\ T_a^{0,1} &= \operatorname{span}_{\mathbf{C}}\{d_a \overline{z}\} = \text{ cotangent vectors of type } (0,1), \end{aligned}$$

with $T_a^{(1)} = T_a^{1,0} \oplus T_a^{0,1}$. Note that this definition is independent of the local chart (U, φ). Indeed, let $(U', \varphi' = z')$ be another chart with $a \in U'$. Then $d_a z' = (\partial z'/\partial z)(a) d_a z + (\partial z'/\partial \overline{z})(a) d_a \overline{z}$, but

$$\frac{\partial z'}{\partial \overline{z}} = \varphi^* \left(\frac{\partial (z' \circ \varphi^{-1})}{\partial z}\right) = \varphi^* \left(\frac{\partial (\varphi' \circ \varphi^{-1})}{\partial z}\right) = \varphi^*(0),$$

since $\varphi' \circ \varphi^{-1}$ is holomorphic. So $\partial z'/\partial \overline{z} = \varphi^*(z) = 0$. Therefore $\alpha = (\partial z'/\partial z)(a) \in \mathbf{C}$, implying that $T_a^{1,0} = \operatorname{span}_{\mathbf{C}}\{d_a z'\}$. Similarly, $T_a^{0,1} = \operatorname{span}_{\mathbf{C}}\{d_a \overline{z'}\}$.

Definition 5.1.10. Suppose that Y is an open subset of a Riemann surface X. A *differential form* of degree 1 (or a *1-form*) on Y is a mapping $\omega : Y \to \bigcup_{a \in Y} T_a^{(1)}$, with $\omega(a) \in T_a^{(1)}$ for all $a \in Y$. If $\omega(a) \in T_a^{1,0}$ for all $y \in Y$, then ω is a 1-form of type $(1,0)$ or a $(1,0)$-form. Similarly if $\omega(a) \in T_a^{0,1}$ for all $y \in Y$.

Remark 5.1.11. Note that any 1-form ω on Y may be written as $\omega = f dx + g dy$, where $f, g : Y \to \mathbf{C}$ are defined as $f(a) = c_1$ and $g(a) = c_2$, if $\omega(a) = c_1 d_a x + c_2 d_a y$. However, f, g may not ever be continuous.

Definition 5.1.12. A 1-form on Y is called *differentiable* (resp. holomorphic) if, with respect to any chart $(U, \varphi = z)$, ω may be written as $\omega / f dz + g d\overline{z}$ on $U \cap Y$, with $f, g \in \mathcal{E}(U \cap Y)$ (resp. $\omega = f dz$ on $U \cap Y$ with $f \in \mathcal{O}(U \cap Y)$). We introduce the following notation:

$$\begin{aligned} \mathcal{E}^{(1)}(Y) &= \{\text{differentiable 1-forms on } Y\}, \\ \mathcal{E}^{1,0}(Y) &= \{\text{differentiable } (1,0)\text{-forms on } Y\}, \\ \mathcal{E}^{0,1}(Y) &= \{\text{differentiable } (0,1)\text{-forms on } Y\}, \\ \Omega(Y) &= \{\text{holomorphic 1-forms on } Y\}. \end{aligned}$$

These sets, with the natural restriction of functions, gives sheaves $\mathcal{E}^{(1)}$, $\mathcal{E}^{1,0}$, $\mathcal{E}^{0,1}$, Ω.

Example 5.1.13. Consider the folowing examples of forms on $\mathbf{C}$:

$$\begin{aligned} \omega &= z\overline{z} dz - 3 d\overline{z} \in \mathcal{E}^{(1)}(\mathbf{C}), \\ \omega &= z\overline{z} dz \in \mathcal{E}^{1,0}(\mathbf{C}) \setminus \Omega(\mathbf{C}), \\ \omega &= z dz \in \Omega(\mathbf{C}). \end{aligned}$$

If $f \in \mathcal{E}(Y)$, then for $df(a) = d_a f$,

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \overline{z}} \in \mathcal{E}^{(1)}(Y).$$

5.2 Exterior differentiation

Definition 5.2.1. Let V be a vector space over $\mathbf{C}$. Then $\bigwedge^2 V$ is a vector space over $\mathbf{C}$ whose elements are finito sums of elements of the forms $v_1 \wedge v_2$, for $v_1, v_2 \in V$ satisfying the following rules, for all $v_1, v_2, v_3 \in V$ and $\lambda \in \mathbf{C}$:

$$\begin{aligned}(v_1 + v_2) \wedge v_3 &= v_1 \wedge v_2 + v_3 \wedge v_3 \\ (\lambda v_1) \wedge v_2 &= \lambda(v_1 \wedge v_2) = v_1 \wedge (\lambda v_2) \\ v_1 \wedge v_2 &= -v_2 \wedge v_1\end{aligned}$$

Remark 5.2.2. By the 3rd property above, $v \wedge v = -v \wedge v$, so $v \wedge v = 0$ for all $v \in V$. Next, suppose that $\dim(V) = 2$ and $V = \operatorname{span}_{\mathbf{C}}\{e_1, e_2\}$. Then for all $v, v' \in V$, $v = a_1 e_1 + a_2 e_2$ and $v' = a_1' e_1 + a_2' e_2$, with $a_i, a_i' \in \mathbf{C}$, meaning that

$$v \wedge v' = (a_1 e_1 + a_2 e_2) \wedge (a_1' e_1 + a_2' e_2) = (a_1 a_2' - a_2 a_1') e_1 \wedge e_2.$$

Therefore $\bigwedge^2 V = \operatorname{span}_{\mathbf{C}}\{e_1 \wedge e_2\}$ and $\dim_{\mathbf{C}}(\bigwedge^2 V) = 1$.

Now let us consider $V = T_a^{(1)}$, the cotangent space of a Riemann surface X at $a \in X$. We set $T_a^{(2)} = \bigwedge^2 T_a^{(1)}$. Let $(U, \varphi = z = x + iy)$ be a chart with $a \in U$. Then

$$T_a^{(2)} = \operatorname{span}_{\mathbf{C}}\{d_a x \wedge d_a y\} = \operatorname{span}_{\mathbf{C}}\{d_a z \wedge d_a \overline{z}\}.$$

Note that $d_a z \wedge d_a \overline{z} = -2i d_a x \wedge d_a y$, since $d_a z = d_a(x + iy) = d_a x + i d_a y$ and $d_a \overline{z} = d_a(x - iy) = d_a x - i d_a y$.

Definition 5.2.3. Suppose that Y is an open subset of a Riemann surface X. A *differential of degree 2*, or a 2-*form* on Y, is a map $\omega : Y \to \bigcup_{a \in Y} T_a^{(2)}$, with $\omega(a) \in T_a^{(2)}$ for all $a \in Y$. Further, if $(U, \varphi = z)$ is a local chart on X, then ω may be written as $\omega = f dz \wedge d\overline{z}$, where $\omega(a) = f(a) d_a z \wedge d_a \overline{z}$ for all $a \in U \cap Y$ (so $f : U \cap Y \to \mathbf{C}$). If $f \in \mathcal{E}(U \cap Y)$, then ω is called a *differentiable 2-form* on $U \cap Y$. We set

$$\mathcal{E}^{(2)}(Y) = \{\text{differentiable 2-forms on } Y\},$$

and get a corresponding sheaf $\mathcal{E}^{(2)}$.

Example 5.2.4. Consider the following examples of 2-forms.

· On $\mathbf{C}$, $\omega = 2z\overline{z} dz \wedge d\overline{z} \in \mathcal{E}^{(2)}(\mathbf{C})$.

· If $\omega_1, \omega_2 \in \mathcal{E}^{(1)}(Y)$, then $\omega_1 \wedge \omega_2$ is a differentiable 2-form on Y, defined as $(\omega_1 \wedge \omega_2)(a) = \omega_1(a) \wedge \omega_2(a)$, for all $a \in Y$. For example, on $\mathbf{C}$, if $\omega_1 = 2\overline{z} dz$ and $\omega_2 = \sin(\overline{z}) dz - 3e^z d\overline{z}$, then

$$\omega_1 \wedge \omega_2 = -(2\overline{z})(3e^z) dz \wedge d\overline{z}.$$

Similarly, for $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we have that

$$\begin{aligned}dx &= dr(\cos(\theta)) + r(-\sin(\theta)) d\theta = \cos(\theta) dr - r\sin(\theta) d\theta \\ dy &= dr(\sin(\theta)) + r\cos(\theta) = \sin(\theta) dr + r\cos(\theta) d\theta.\end{aligned}$$

So $dx \wedge dy = r dr \wedge d\theta$.

Definition 5.2.5. Let $Y \subset X$ be an open subset of a Riemann surface X. We have seen the following maps:

$$\begin{array}{ccc} d : \mathcal{E}(Y) & \to & \mathcal{E}^{(1)}(Y) \\ f & \mapsto & df \end{array}, \quad \begin{array}{ccc} \partial : \mathcal{E}(Y) & \to & \mathcal{E}^{1,0}(Y) \\ f & \mapsto & \partial f \end{array}, \quad \begin{array}{ccc} \overline{\partial} : \mathcal{E}(Y) & \to & \mathcal{E}^{0,1}(Y) \\ f & \mapsto & \overline{\partial} f \end{array}.$$

We can extend these differential operators to $\mathcal{E}^{(1)}(Y)$ as follows. Let $(U, \varphi = z = x + iy)$ be a chart. Set $\omega = f dx + g dy = a dz + b d\overline{z}$, so then

$$\begin{aligned}d\omega &= df \wedge dx + dg \wedge dy = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy = \left(\frac{\partial a}{\partial x} - \frac{\partial b}{\partial y}\right) dz \wedge d\overline{z}, \\ \partial\omega &= \partial a \wedge dz + \partial b \wedge d\overline{z} = \frac{\partial b}{\partial z} dz \wedge d\overline{z}, \\ \overline{\partial}\omega &= \overline{\partial} a \wedge dz + \overline{\partial} b \wedge d\overline{z} = -\frac{\partial a}{\partial z} dz \wedge d\overline{z}.\end{aligned}$$

Proposition 5.2.6. For $f \in \mathcal{E}(Y)$ and $\omega \in \mathcal{E}^{(1)}(Y)$, we have that

$$\begin{aligned} d(df) &= \partial(\partial f) = \overline{\partial}(\overline{\partial} f) = 0, \\ d\omega &= \partial\omega + \overline{\partial}\omega, \\ d(f\omega) &= df \wedge \omega + f d\omega. \end{aligned}$$

Definition 5.2.7. A differentiable function $\omega \in \mathcal{E}^{(1)}(Y)$ is called *d-closed* (resp. *$\overline{\partial}$-closed*) if $d\omega = 0$ (resp. $\overline{\partial}\omega = 0$). Further, ω is called *d-exact* (resp. *$\overline{\partial}$-exact*) if $\omega = df$ (resp. $\omega = \overline{\partial} f$) for some $f \in \mathcal{E}(Y)$.

Note that if ω is d-exact, then $d\omega = 0$ since $\omega = df$ for some $f \in \mathcal{E}(Y)$, and $d\omega = d(df) = 0$. Similarly, if ω is $\overline{\partial}$-exact, then $\overline{\partial}\omega = 0$.

5.3 de Rham and Dolbeault cohomology

Definition 5.3.1. Set $Z^k_{dR} = \{d\text{-closed } k\text{-forms}\}$ and $B^k_{dR} = \{d\text{-exact } k\text{-forms}\}$, noting that $B^k_{dR} \subset Z^k_{dR}$ for all k. Define the *kth de Rham cohomology group* of X to be

$$\begin{aligned} H^k_{dR}(X) &= Z^k_{dR}/B^k_{dR} \\ &= \ker(d : \mathcal{E}^{(k)}(X) \to \mathcal{E}^{(k+1)}(X))/\mathrm{Im}(d : \mathcal{E}^{(k-1)}(X) \to \mathcal{E}^{(k)}(X)). \end{aligned}$$

Note that H^i_{dR} is a $\mathbf{C}$-vector space uvder scalar multiplication and addition of forms. Further, $H^1_{dR}(X)$ measures the extent to which an i-form ω with $d\omega = 0$ fails to be of the form $\omega = d\alpha$ with α an $(i-1)$-form (where $\mathcal{E}(X)$ is the set of 0-forms). Finally, note that

$$H^0_{dR}(X) = \left\{ f \in \mathcal{E}(X) \ : \ \frac{\partial f}{\partial z} = \frac{\partial f}{\partial \overline{z}} = 0 \text{ everywhere} \right\} \cong \mathbf{C}.$$

Also, note that H^i_{dR} is a homotopy invariant, so X homotopic to Y implies $H^i_{dR}(X) \cong H^i_{dR}(Y)$ for all i.

Proposition 5.3.2. [Poincare lemma]
$H^1_{dR}(\mathbf{C}) \cong H^2_{dR}(\mathbf{C})$.

Proposition 5.3.3. $H^1_{dR}(\mathbf{C}^*) \neq 0$.

Proof: Consider $\alpha = \frac{dz}{z} \in \mathcal{E}^{(1)}(\mathbf{C}^*)$. Then

$$d\alpha = \frac{\partial}{\partial z}\left(\frac{1}{z}\right) dz \wedge dz + \frac{\partial}{\partial \overline{z}}\left(\frac{1}{z}\right) d\overline{z} \wedge dz = 0.$$

If α is exact, then there exists $f \in \mathcal{E}(\mathbf{C}^*)$ such that $\alpha = df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \overline{z}} d\overline{z}$, hence $\frac{\partial f}{\partial z} = \frac{1}{z}$ and $\frac{\partial f}{\partial \overline{z}} = 0$. Therefore $f'(z) = \frac{1}{z}$ and $f \in \mathcal{O}(\mathbf{C}^*)$, which is a contradiction, since $\frac{1}{z}$ does not have an antiderivative on all of $\mathbf{C}^*$. So α is not exact, meaning than $[\alpha] \neq 0$ in $H^1_{dR}(\mathbf{C})$, so $H^1_{dR}(\mathbf{C}^*) = 0$. ■

One can even show that $H^1_{dR}(\mathbf{C}^*) \cong \mathbf{C}$ and $H^2_{dR}(\mathbf{C}^*) = 0$.

Proposition 5.3.4. Let X be a compact Riemann surface. Then $H^1_{dR}(X) \cong \mathbf{C}^{2g}$, where g is the genus of X, and $H^2_{dR}(X) = 0$.

Proof: Consider $\mathbf{P}^1 = \mathbf{C} \cup (\mathbf{C}^* \cup \{\infty\}) = U \cup U'$, and let $\alpha \in \mathcal{E}^{(1)}(\mathbf{P}^1)$ be such that $d\alpha = 0$, so $d(\alpha|_U) = 0$ and $d(\alpha|_{U'}) = 0$. Since $U \cong \mathbf{C}$ and $U' \cong \mathbf{C}$, by Poincare there exists $f \in \mathcal{E}(U)$ and $f' \in \mathcal{E}(U')$ with $\alpha|_U = df$ and $\alpha|_{U'} = df'$. So, on $U \cap U' \cong \mathbf{C}^*$, we have that

$$df|_{U \cap U'} = \alpha|_{U \cap U'} = df'|_{U \cap U'}.$$

Hence $d(f - f') = 0$ on $U \cap U'$, so $f - f' \in H^0_{dR}(U \cap U') \cong \mathbf{C}$, so $f - f'$ is constant. We may assume that the constant is 0 (else replace f' by $f' + c$). So $f = f'$ on $U \cap U'$, and $h = \left\{ \begin{smallmatrix} f & \text{on } U \\ f' & \text{on } U' \end{smallmatrix} \right.$ is a differential form on X such that $\alpha = dh$. So α is d-exact. meaning that $H^1_{dR}(\mathbf{P}^1) = 0$. ■

One can also show that $H^1_{dR}(\mathbf{C}/\Gamma) \cong \mathbf{C}^2$, so the torus $\mathbf{C}/\Gamma$ has genus 1.

Definition 5.3.5. Recall that we have maps $\mathcal{E}(X) \xrightarrow{\overline{\partial}} \mathcal{E}^{0,1}(X)$ and $\mathcal{E}^{1,0}(X) \xrightarrow{\overline{\partial}} \mathcal{E}^{1,1}(X)$. Note that $\overline{\partial}(\mathcal{E}^{0,1}(X)) = 0$ since for all $\alpha \in \mathcal{E}^{0,1}(X)$, $\alpha = f d\overline{z}$, so $\overline{\partial}\alpha = \overline{\partial} f \wedge d\overline{z} = \frac{\partial f}{\partial z} d\overline{z} \wedge d\overline{z} = 0$. Define $H^{p,q}_{\overline{\partial}}(X)$ to be the (p,q)*th Dolbeault cohomology group* of X, with

$$\begin{aligned}
H^{0,0}_{\overline{\partial}}(X) &= \ker(\overline{\partial} : \mathcal{E}(X) \to \mathcal{E}^{0,1}(X)),\\
H^{1,0}_{\overline{\partial}}(X) &= \ker(\overline{\partial} : \mathcal{E}^{1,0}(X) \to \mathcal{E}^{1,1}(X)),\\
H^{0,1}_{\overline{\partial}}(X) &= \mathcal{E}^{0,1}(X)/\mathrm{Im}(\overline{\partial} : \mathcal{E}(X) \to \mathcal{E}^{0,1}(X)),\\
H^{1,1}_{\overline{\partial}}(X) &= \mathcal{E}^{1,1}(X)/\mathrm{Im}(\overline{\partial} : \mathcal{E}^{1,0}(X) \to \mathcal{E}^{1,1}(X)).
\end{aligned}$$

Note that these cohomology groups are vector spaces, for example

$$\begin{aligned}
H^{0,0}_{\overline{\partial}}(X) &= \left\{ f \in \mathcal{E}(X) \ : \ \overline{\partial} f = \frac{\partial f}{\partial \overline{z}} d\overline{z} = 0 \text{ everywhere} \right\}\\
&= \left\{ f \in \mathcal{E}(X) \ : \ \frac{\partial f}{\partial z} = 0 \text{ everywhere} \right\}\\
&= \mathcal{O}(X).
\end{aligned}$$

Remark 5.3.6. If X is compact, then $H^{0,0}_{\overline{\partial}}(X) \cong \mathbf{C}$. Otherwise $H^{0,0}_{\overline{\partial}}(X)$ is very big, for example, if $X = \mathbf{C}$, then

$$H^{0,0}_{\overline{\partial}}(\mathbf{C}) \cong \mathcal{O}(\mathbf{C}) \supset \mathbf{C}[z].$$

Note that $H^{1,0}_{\overline{\partial}}(X) = \{\alpha \in \mathcal{E}^{1,0}(X) \ : \ \overline{\partial}\alpha = 0\}$. As $\alpha \in \mathcal{E}^{1,0}(X)$, it follows that $\alpha = f dz$, so

$$\overline{\partial}\alpha = \overline{\partial} f \wedge dz = \frac{\partial f}{\partial \overline{z}} d\overline{z} \wedge dz,$$

so if $\overline{\partial}\alpha = 0$, then $\frac{\partial f}{\partial \overline{z}} = 0$. So $f \in \mathcal{O}(U)$ and $\alpha = f dz \in \Omega(X)$, so $H^{1,0}_{\overline{\partial}}(X) \cong \Omega(X)$.

Proposition 5.3.7. If X is a compact Riemann surface, then $H^{1,0}_{\overline{\partial}}(X) \cong \mathbf{C}^g$, where g is the genus of X.

Example 5.3.8. Consider $X = \mathbf{P}^1$, for which $H^{1,0}_{\overline{\partial}}(\mathbf{P}^1) = 0$. Let $\alpha \in H^{1,0}_{\overline{\partial}}(\mathbf{P}^1) \cong \Omega(\mathbf{P}^1)$ and set $\mathbf{P}^1 = U \cup U'$ as before, for $w = \frac{1}{z} \in U'$. Then $\alpha|_U = f(z)dz$ with $f \in \mathcal{O}(U)$ and $\alpha|_{U'} = g(w)dw$ with $g \in \mathcal{O}(U')$. On $U \cap U'$,

$$\begin{aligned}
f(z)dz = \alpha|_{U \cap U'} = g(w)dw \quad &\text{iff} \quad f(1/w)d(1/w) = g(w)dw\\
&\text{iff} \quad f(1/w)(-1/w^2)dw = g(w)dw\\
&\text{iff} \quad f(1/w)(-1/w^2) = g(w)\\
&\text{iff} \quad f(1/w) = -w^2 g(w).
\end{aligned}$$

Since $f \in \mathcal{O}(U)$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and since $g \in \mathcal{O}(U')$, $g(w) = \sum_{n=0}^{\infty} b_n w^n$, so the last statement above becames

$$\sum_{n=0}^{\infty} a_n w^{-n} = \sum_{n=0}^{\infty} (-b_m) w^{n+2}.$$

Hence $a_n = 0$ and $b_n = 0$ for all n. So $f = g = 0$, meaning that $\alpha = 0$, so $H^{1,0}_{\overline{\partial}}(\mathbf{P}^1) = 0$.

Definition 5.3.9. Let $F : X \to Y$ be a holomorphic map between Riemann surfaces. For every open set $V \subset Y$, we have F^*, the *pullback of differentiable functions*, given by

$$\begin{array}{rccc}
F^* : & \mathcal{E}(V) & \to & \mathcal{E}(F^{-1}(V)) \\
& f & \mapsto & f \circ F
\end{array}.$$

This may be generalized to differential forms. If $(V, \psi = \omega)$ is a chart on Y, then

$$\begin{array}{rcl} F^* : \mathcal{E}^{(1)}(V) & \to & \mathcal{E}^{(1)}(F^{-1}(V)) \\ \omega = f dw + g d\overline{w} & \mapsto & F^*(f) d(F^*(w)) + F^*(g) d(F^*(\overline{w})) \end{array}$$

and

$$\begin{array}{rcl} F^* : \mathcal{E}^{(2)}(V) & \to & \mathcal{E}^{(2)}(F^{-1}(V)) \\ \omega = f dw \wedge d\overline{w} & \mapsto & F^*(f) d(F^*(w)) \wedge d(F^*(\overline{w})) \end{array}.$$

This also generalizes to n-forms.

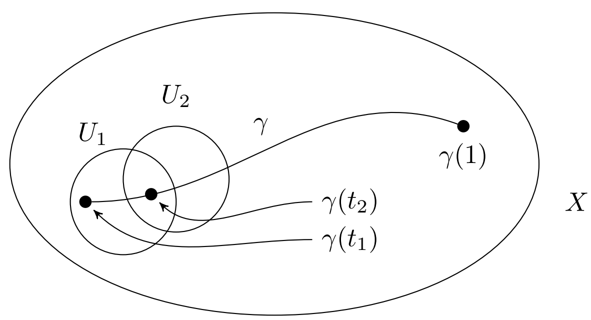
Example 5.3.10. Consider the space $X = Y = \mathbf{C}$ and $F : \mathbf{C} \to \mathbf{C}$ given by $z \mapsto 2\overline{z} = w$. Take $\omega = 1dw - 3wd\overline{w}$, for which

$$\begin{aligned} F^*(\omega) &= F^*(1) d(F^*(w)) + F^*(-3w) d(F^*(\overline{w})) \\ &= 1d(2\overline{z}) + (-3(2\overline{z})) d(\overline{2\overline{z}}) \\ &= 2d\overline{z} - 6\overline{z}(2dz). \end{aligned}$$

Proposition 5.3.11. For all $f \in \mathcal{E}(V)$ and $\omega \in \mathcal{E}^{(1)}(W)$,
i. $F^*(df) = d(F^*(f))$, and
ii. $F^*(d\omega) = d(F^*(\omega))$.

5.4 Integration of 1-forms and primitives

Definition 5.4.1. Let X be a Riemann surface and $\omega \in \mathcal{E}^{(1)}(X)$. Let $\gamma : [0,1] \to X$ be a piecewise-continuously differentiable curve in X, i.e. γ is cntinuous and there exists a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0,1]$ and charts $(U_k, \varphi_k = z_k)$, where $z_k = x_k + iy_k$ and $x_k = \mathrm{Re}(z_k)$ and $y_k = \mathrm{Im}(z_k)$ for all k, such that $\gamma([t_{k-1}, t_k]) \subset U_k$ and the forms $\{x_k, y_k\} \circ \gamma : [t_{k-1}, t_k] \to \mathbf{R}$ are C^1.



Define the integral

$$\int_\gamma \omega = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left(f_k(\gamma(t)) \frac{dx_k(\gamma(t))}{dt} + g_k(\gamma(t)) \frac{dy_k(\gamma(t))}{dt} \right) dt,$$

whenever $\omega|_{U_k} = f_k dx_k + g_k dy_k$ for all k. Note that this is independent of the charts chosen (check this).

Theorem 5.4.2. Let $F \in \mathcal{E}(X)$. Then $\int_\gamma dF = F(\gamma(1)) - F(\gamma(0))$.

Proof: In the local charts (U_k, z_k), $dF = \frac{\partial F}{\partial x_k} dx_k + \frac{\partial F}{\partial y_k} dy_k$, so

$$\int_{t_{k-1}}^{t_k} \left(\frac{\partial F}{\partial x_k}(\gamma(t)) \frac{dx_k(\gamma(t))}{dt} + \frac{\partial F}{\partial y_k}(\gamma(t)) \frac{dy_k(\gamma(t))}{dt} \right) dt = \int_{t_{k-1}}^{t_k} \frac{d(F(\gamma(t)))}{dt} dt = F(\gamma(t_k)) - F(\gamma(t_{k-1})),$$

by the fundametal theorem of calculus and as $F(\gamma(t)) : [0,1] \to \mathbf{C}$. ■

Corollary 5.4.3. If γ is a closed curve (i.e. $\gamma(0) = \gamma(1)$, then $\int_\gamma dF = 0$.

Definition 5.4.4. Let $\omega \in \mathcal{E}^{(1)}(X)$. A function $F \in \mathcal{E}(X)$ is called a *primitive* of ω if $dF = \omega$.

Remark 5.4.5. Note the following:

· The element $\omega \in \mathcal{E}^{(1)}(X)$ has a primitive iff ω is d-exact (in which case it is also d-closed.

· Primitives are unique up to a constant. For example, if $F \in \mathcal{E}(X)$ is a primitive of $\omega \in \mathcal{E}^{(1)}(X)$, then so is $F + c$ for all $c \in \mathbf{C}$. Moreover, if $F, G \in \mathcal{E}(X)$ are primitives of $\omega \in \mathcal{E}^{(1)}(X)$, then $dF = dG = \omega$, so $d(F - G) = 0$, meaning that $F - G = c$ for some c.

· If $\omega \in \mathcal{E}^{(1)}(X)$ has a primitive F, then $\int_\gamma \omega$ is path independent, because it is completely determined by the value of F at the endpoint of γ.

· If $\omega \in \mathcal{E}^{(1)}(X)$ is d-clsode and $\int_\gamma \omega \neq 0$ for some closed curve γ, then ω is not exact. For example, with $X = \mathbf{C}^*$ and γ the unit circle with $\omega = \frac{dz}{z}$, we have that $\int_\gamma \omega = 2\pi \neq 0$, so ω is not d-exact.

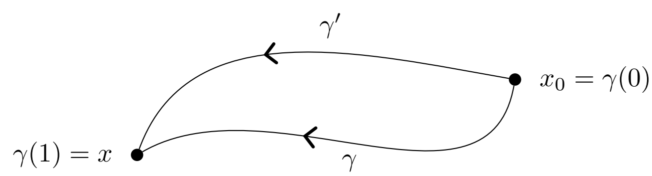
Remark 5.4.6. Let $\omega \in \mathcal{E}^{(1)}(X)$ be d-closed. When does ω have a primitive?

· If $H^1_{dR}(X) = 0$, then d-closed 1-forms are always d-exact, and therefore admit primitives. For example, if $X = \mathbf{P}^1$, since $H^1_{dR}(\mathbf{P}^1) = 0$, there exist primitives. Similarly, if $X = \mathbf{C}$ or an open disk, and as $H^1_{dR}(\mathbf{C}) = 0$, by Poincare there exist primitives. Since every Riemann surface is locally diffeomorphic to $\mathbf{C}$ on a disk in $\mathbf{C}$, d-closed 1-forms are locally d-exact.

· If $H^1_{dR}(X) \neq 0$, then globally the primitives of d-closed 1-forms will be multivalued functions. For example, if $X = \mathbf{C}^*$, then $H^1_{dR}(\mathbf{C}^*) \neq 0$, and for $\omega = \frac{dz}{z}$, we write $\omega = d(\log(z))$. Note however, that not every d-closed 1-form on $\mathbf{C}^*$ has a multivalued global primitive, i.e. for $\omega = dz = df$, we have that $f(z) = z \in \mathcal{E}^{(1)}(\mathbf{C}^*)$.

Proposition 5.4.7. Let $\omega \in \mathcal{E}^{(1)}(X)$ be d-closed. If $\int_\gamma \omega = 0$ for any closed loop γ, then ω is d-exact. That is, then there exists $F \in \mathcal{E}(X)$ such that $dF = \omega$.

Proof: Let $x_0 \in X$. Then for all $x \in X$, let γ be a curve joining x and x_0. First note that it $\tilde{\gamma} : [0,1] \to X$ is given by $t \mapsto \gamma(1-t)$, then $\int_{\tilde{\gamma}} \omega = -\int_\gamma \omega$. Next, if γ' is any other curve joining x_0 to x, we have that $\gamma + \tilde{\gamma}'$ is a closed curve, so $\int_{\gamma+\tilde{\gamma}'} \omega = 0$.



Moreover, $\int_\gamma \omega + \int_{\tilde{\gamma}'} \omega = \int_\gamma \omega - \int_{\gamma'} \omega$, so $\int_\gamma \omega = \int_{\gamma'} \omega$ for any two such curves. We thus get a well-defined function $F : X \to \mathbf{C}$ given by $x \mapsto \int_x^{x_0} \omega$, slightly abusing notation. We now claim that $dF = \omega$. It is enough to prove that $d_x F = \omega(x)$ for all $x \in X$. So let $x \in X$, and note that locally, ω has a primitive on a neighborhood of x by the Poincare lemma. Suppose that this primitive is f. Then $\omega(x) = d_x f = [f - f(x)]$ mod m_x^2, and $df = \omega$ around x. So for all y near x,

$$F(y) - F(x) = \int_x^y \omega = f(y) - f(x),$$

implying that

$$d_x F = [F - F(x)] \mod m_x^2 = [f - f(x)] \mod m_x^2 = d_x f = \omega(x).$$

■

Theorem 5.4.8. Let X be a Riemann surface and $\omega \in \mathcal{E}^{(1)}(X)$ be d-closed. Then there exists a Riemann surface $\hat{X}$ and an unbranched holomorphic map $p : \hat{X} \to X$ such that $p^*\omega = dF$ for some $F \in \mathcal{E}(X)$.

Proof: Let $\mathcal{F}$ be the sheaf of primitives of ω on X. For all $U \subset X$ open, we then have $\mathcal{F}(U) = \{f \in \mathcal{E}(U) : df = \omega|_U\}$ and natural restriction functions. Let $\hat{X}$ be a connected component of $|\mathcal{F}|$ and $p = p|_{\hat{X}}$, where $p : |\mathcal{F}| = \bigcup_{x \in X} \mathcal{F}_x \to X$ is given by $\varphi \in \mathcal{F}_x \mapsto x$. We have see that there exists a unique complex structure on $\hat{X}$ such that $\hat{X}$ is a Riemann surface and $p : \hat{X} \to X$ is an unramified holomorphic map. Let $F : \hat{X} \to \mathbf{C}$ be given by $\varphi \mapsto \varphi(p(\varphi))$. Then $F \in \mathcal{E}(X)$ (i.e. F is differentiable), and if $\varphi = [f]$ with $f \in \mathcal{F}(U)$ for U a neighborhood of $p(\varphi)$, we have that $df = \omega|_U$, implying that

$$F(\varphi) = f(p(\varphi)) = f \circ p(\varphi) = p^*(f)(\varphi),$$

meaning that $F = p^*(f)$, so $dF = d(p^*f) = p^*(df) = p^*(\omega)$. ■

Definition 5.4.9. Let X be a Riemann surface. The space $\widetilde{X}$ is termed a *universal cover* of X if:
· $\widetilde{X}$ is simply connected
· there exists a covering map $\pi : \widetilde{X} \to X$
· if $p : Y \to X$ is any other covering map, then there exists a holomorphic fiber-preserving map $\tau : \widetilde{X} \to Y$ with $\tau \circ p = \pi$
Then $\widetilde{X}$ is a Riemann surface, and is unique. Also note that if X is simply connected, then $\widetilde{X} = X$ and $\pi = \text{id}$.

Let X be a Riemann surface and $\omega \in \mathcal{E}^{(1)}(X)$ be d-closed. Then $p^*\omega = dF$ for some $F \in \mathcal{E}(\dot{X})$, where $p\dot{X} \to X$ is an unbranched holomorphic map. But, p is a covering map (since it is unbranched and holomorphic), so there exists $\tau : \widetilde{X} \to \dot{X}$ with $\tau \circ p = \pi$. Thus, if $f = \tau^*F$, then

$$dF = d(\tau^*F) = \tau^*(dF) = \tau^*(p^*\omega) = (p \circ \tau)^*\omega = \pi^*\omega.$$

Corollary 5.4.10. If $\omega \in \mathcal{E}^{(1)}(X)$ is d-closed, then $\pi^*\omega$ has a primitive on $\widetilde{X}$.

Corollary 5.4.11. If X is simply connected, any d-closed $\omega \in \mathcal{E}^{(1)}(X)$ has a primitive on X. Therefore $H^1_{dR}(X) = 0$.

Example 5.4.12. Since $\mathbf{C}$, $\mathbf{P}^1$ are simply connected, $H^1_{dR}(\mathbf{C}) = H^1_{dR}(\mathbf{P}^1) = 0$.

5.5 Integration of 2-forms

Definition 5.5.1. Let $U \subset \mathbf{C}$ be open and $\omega \in \mathcal{E}^{(2)}(U)$. Then $\omega = f dx \wedge dy = g dz \wedge d\overline{z}$, and for $D \subset U$, define

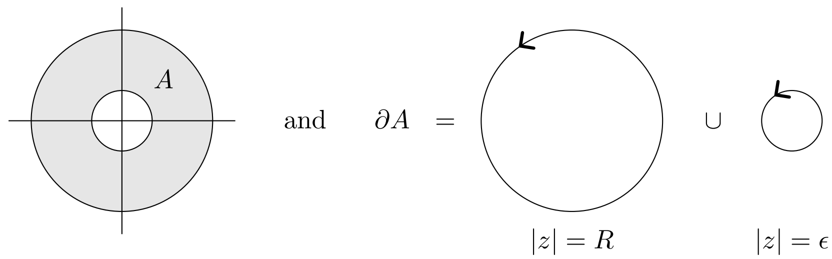
$$\iint_D \omega = \iint_D f(x,y) dx dy = \iint_D f(z, \overline{z}) dz d\overline{z},$$

where the right side is usual integration in $\mathbf{R}^2$.

Theorem 5.5.2. [Stokes]
Let $U \subset \mathbf{C}$ be open and $A \subset U$ compact, connected with smooth boundary ∂A (i.e. ∂A is a smooth curve). Then for all $\omega \in \mathcal{E}^{(1)}(U)$,

$$\iint_A d\omega = \oint_{dA} \omega.$$

Proof: Due to time constraints, we will simplify. Suppose that $\omega = g dy$ and $A = \{z \in \mathbf{C} \ : \ \epsilon \leqslant |z| \leqslant R\}$, with $0 < \epsilon < R$. This gives the situation below:



Then

$$d\omega = dg \wedge dy = \frac{\partial g}{\partial x} dx \wedge dy = \left(\cos(\theta) \frac{\partial g}{\partial r} - \frac{\sin(\theta)}{r} \frac{\partial g}{\partial \theta} \right) dr \wedge d\theta,$$

so

$$\begin{aligned} \iint_A &= \int_0^{2\pi} \int_\epsilon^R B(r, \theta) dr d\theta \\ &= \int_0^{2\pi} g(R, \theta) \cos(\theta) d\theta - \int_0^{2\pi} g(\epsilon, \theta) \epsilon \cos(\theta) d\theta \\ &= \int_{|z|=R} \omega - \int_{|z|=\epsilon} \omega \\ &= \int_{\partial A} \omega. \end{aligned}$$

■

5.6 Cauchy integral formula

5.7 The exact cohomology sequence

Definition 5.7.1. Let X be a topological space, and $\mathcal{F}, \mathcal{G}$ sheaves on X. Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a sheaf homomorphism. That is, for $U \subset X$ open, there exists a group homomorphism $\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that for all $V \subset U$ open, the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\alpha_U} & \mathcal{G}(U) \\ \rho \downarrow & & \downarrow \rho \\ \mathcal{F}(V) & \xrightarrow[\alpha_V]{} & \mathcal{G}(V) \end{array}$$

commutes, i.e. for all $f \in \mathcal{F}(U)$, $\alpha_V(f|_U) = \alpha_U(f|_V)$. Let $U \subset X$ be open. Define

$$\begin{aligned} \ker(\alpha)(U) &= \ker(\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)), \\ \operatorname{Im}(\alpha)(U) &= \operatorname{Im}(\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)), \\ \operatorname{coker}(\alpha)(U) &= \mathcal{G}(U) / \operatorname{Im}(\alpha)(U). \end{aligned}$$

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References

[For99] Otto Forster. *Lectures on Riemann Surfaces*. Springer, 1999.