Compact course notes

Pure Math 764, Winter 2014

Introduction to Algebraic Geometry

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Contents

1 Basic geometric objects

Algebraic geometry is the study of zero sets of polynomials.

1.1 Definitions and notation

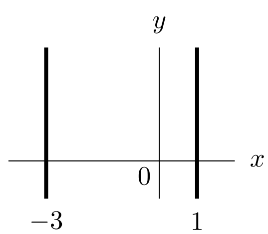
Definition 1.1.1. We introduce the following notation:

$$\begin{aligned} \mathbf{K}: &\quad \text{a field (not necessarily algebraically closed)} \\ \mathbb{A}^n(\mathbf{K}) \text{ or } \mathbb{A}^n: &\quad \text{affine } n\text{-space, i.e. the set of } n\text{-tuples } \{(a_1,\dots,a_n) \;:\; a_1,\dots,a_n \in \mathbf{K}\} \\ \mathbf{K}[x_1,\dots,x_n]: &\quad \text{the polynomial ring in } n \text{ variables } x_1,\dots,x_n \text{ over } \mathbf{K} \end{aligned}$$

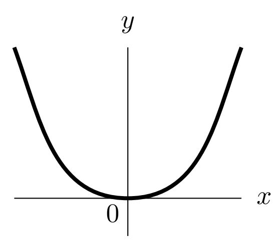
Note that $\mathbb{A}^1$ is called the *affine line* and $\mathbb{A}^2$ is the *affine plane*. Further, for $f \in \mathbf{K}[x_1,\dots,x_n]$ non-constant, a point $p \in \mathbf{A}^n$ is termed a *zero* of f is $f(p) = 0$. We write $V(f) = \{p \in \mathbf{A}^n \;:\; f(p) = 0\}$ for the set of zeros of f in $\mathbf{A}^n$, also the *hypersurface defined by* f.

Example 1.1.2. A hypersurface in $\mathbf{A}^1$ is a finite set of points or $\emptyset$. For example,
- in $\mathbf{R}^1$, $V((x-1)(x+3)) = \{1, 3\}$ and $V(x^2+1) = \emptyset$.
- in $\mathbf{C}$, $V(x^2+1) = \{i, -i\}$.

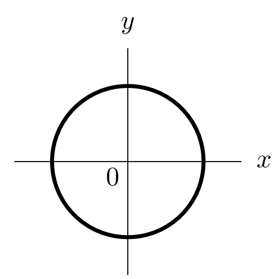
A hypersurface in $\mathbf{A}^2$ is called a (*affine plane*) *curve*. For example,
- in $\mathbf{R}^2$, $V((x-1)(x+3)) = V(x-1) \cup V(x+3)$, which is a union of two lines.
- in $\mathbf{R}^2$, $V(y-x^2)$ is a parabola and $V(x^2-y^2-1)$ is the unit circle.
- in $\mathbf{Q}^2$, $V(x^2+y^2-1)$ is the set of all rational points on the unit circle.



$V((x-1)(x+3))$



$V(y-x^2)$



$V(x^2+y^2-1)$

A point is called *rational* if its coordinates are in $\mathbf{Q}$. Note that the unit circle has as infinite number of rational points, since it can be parametrized using rational functions, by

$$(x, y) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right), \; t \in \mathbf{R}.$$

We get a rational point for all $t \in \mathbf{Z}$. Note that the unit circle is an example of a rational curve (i.e. it can be parametrized by rational functions). Not all curves are rational. We will see that elliptic curves are not rational.

A hypersurgace in $\mathbf{A}^3$ is called an *affine surface*. For example,
- in $\mathbf{A}^3$, $V(xyz) = V(x) \cup V(y) \cup V(z) = \{x = 0\} \cup \{y = 0\} \cup \{z = 0\}$, a union of planes in $\mathbf{A}^3$.

More generally, if S is any set of polynomials in $\mathbf{K}[x_1,\dots,x_n]$, we define $V(S) = \{p \in \mathbf{A}^n \;:\; f(p) = 0 \; \forall \; f \in S\} = \bigcap_{f\in S} V(f)$. Further, if $S = \{f_1,\dots,f_m\}$ is a finite set of polymonials, we write $V(f_1,\dots,f_m)$ instead of $V(\{f_1,\dots,f_m\})$.

1.2 Affine algebraic sets

Definition 1.2.1. A subset $X \subset \mathbf{A}^n$ is an (*affine*) algebraic set if $X = V(S)$ for some $S \subset \mathbf{K}[x_1,\dots,x_n]$.

Example 1.2.2. The sets $\emptyset = V(1)$, $\mathbf{A}^n = V(0)$ and $V(y-x^2)$ are all algebraic. But not all sets are algebraic. For example,

· in $\mathbf{R}^1$, $X = [0,1]$ is not algebraic. If X were algebraic, then $X \subset V(S)$ for some $S \subset \mathbf{R}[x]$. Since $X \ncong \mathbf{R}$, at least one of the polynomials in S, say f, is non-zero. Then $X = V(S) = \bigcap_{g\in S} V(g) \subset V(f)$, but $V(f)$ is at most a finite set of points since f is a polynomial in 1 variable.

· in $\mathbf{R}^2$, the curve $C = \{(x,y) \ : \ y = \sin(x)\}$ is not algebraic. Suppose that C is algebraic, so $C = V(S)$ for some $S \subset \mathbf{R}[x,y]$. Then S must contain at least one non-zero polynomial (else $C \cong \mathbf{R}^2$). So $C = \bigcap_{g\in S} V(g) \subset V(f)$ with $f = f(x,y)$. Then there exists at least one real number $-1 \leqslant y_0 \leqslant 1$ such that $h(x) = f(x,y_0)$ is not the zero polynomial. Note we have $f(x,y) = a_0(y) + a_1(y)x + \cdots + a_m(y)x^m$, so if $f(x,y_0) = 0$ for all $y_0 \in [-1,1]$, then $a_i = 0$ for all i. But each a_i is a polynomial in one variable and must therefore have at most a finite number of roots (if it is non-zero). So if $a_i = 0$, then $f = 0$, which is a contradiction. So, in summary, we start with $V(h(x)) =$ (at most a finite number of points), implying

$$(C \cap V(y-y_0)) \subset (v(f(x,y)) \cap V(y-y_0)) = V(h(x)) = (\text{at most a finite number of points}).$$

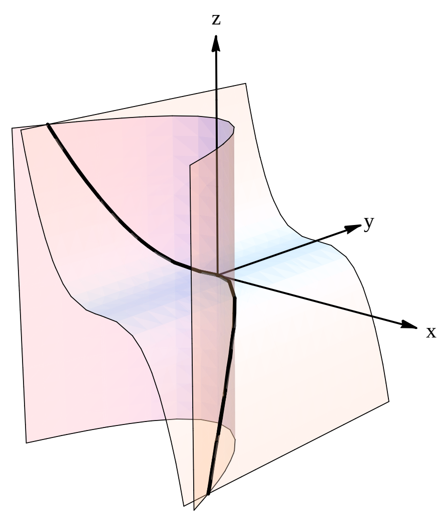
But $C \cap V(y-y_0) = \{(\arcsin(y_0) + 2\pi n - \pi m, y_0) \ : \ m,n \in \mathbf{Z}\}$, which is infinite. Hence C is not algebraic.

Remark 1.2.3. In general, one can show that in $\mathbf{A}^n$ a line must intersect any algebraic curve in a finite set of points. This gives us a test for determining whether or not a set is algebraic: if a set X intersects a line in an infinite number of points, it cannot be algebraic (by a line, we mean a set determined by a point $(a_1,\dots,a_n) \in \mathbf{A}^n$, and a direction vector $(b_1,\dots,b_n) \in \mathbf{A}^n$. That is, $L = \{a_1 + tb_1, \dots, a_n + tb_n \ : \ t \in k\}$).

Example 1.2.4. Note that the intersection of 2 algebraic sets may be infinite. For example, consider the twisted cubic, given by

$$C = \{(t,t^2,t^3) \in \mathbf{R}^3 \ : \ t \in \mathbf{R}\} = V(y-x^2, z-x^3) = V(y-x^2) \cap V(z-x^3).$$

So C is an algebraic set that is the intersection of the surfaces $V(y-x^2)$ and $V(z-x^3)$, visualized below.



Theorem 1.2.5. The only algebraic sets in $\mathbf{A}^1$ are $\mathbf{A}^1, \emptyset$, and finite sets of points.

<u>*Proof:*</u> Clearly $\emptyset = V(1)$ and $\mathbf{A}^1 = V(0)$ are algebraic. Further, if $\{a_1,\dots,a_m\}$ is a finite set of points in $\mathbf{A}^1$, then $\{a_1 \dots, a_m\} = V((x-a_1)(x-a_2)\cdots(x-a_m))$, so it is algebraic. It remains to show that these are the only algebraic sets in $\mathbf{A}^1$. So let $X \subset \mathbf{A}^1$ be any algebraic set, so $X = V(S)$ for some $S \subset \mathbf{K}[x]$.

· if $S = \emptyset$ or $\{0\}$, then $X = \mathbf{A}^1$

· if $X \neq \emptyset$ nor $\{0\}$, then there exists a non-zero $f \in S$ with $X = V(S) \subset V(f)$, which is at most a finite set of points. Hence $X = \emptyset$ or a finite set of points. ■

Proposition 1.2.6. The following are properties of algebraic sets:

1. if $S \subset T \subset \mathbf{K}[x_1, \ldots, x_n]$, then $V(T) \subset V(S)$

2. if $I = \langle S \rangle$ for $S \subset \mathbf{K}[x_1, \ldots, x_n]$, then $V(I) = V(S)$

Proof: **1.** Let $p \in V(T)$. Then $f(p) = 0$ for all $f \in T \supset S$. Hence $f(p) = 0$ for all $f \in S$, so $p \in V(S)$.

2. Since $S \subset \langle S \rangle = I$, by **1.** we have that $V(I) \subset V(S)$. We check the other inclusion. So let $p \in V(S)$. Then $f(p) = 0$ for all $f \in S$. Consider $g \in I = \langle S \rangle$, Then $g = \sum a_i f_i$ with $a_i \in \mathbf{K}[x_1, \ldots, x_n]$ and $f_i \in S$. Hence $g(p) = \sum a_i(p) f_i(p) = 0$, so $g \in V(I)$. ■

Recall that a commutative ring R is *Noetherian* iff every ideal in R is finitely generated. In particular, fields are Noetherian (as $\langle 0 \rangle$ and $k = \langle 1 \rangle$ are the only ideals).

Theorem 1.2.7. [Hilbert basis theorem]
If R is a Noetherian ring, then $R[x_1, \ldots, x_n]$ is Noetherian.

The above implies that $\mathbf{K}[x_1, \ldots, x_n]$ is Noetherian, giving the following corollary.

Corollary 1.2.8. Every algebraic set over $\mathbf{A}^n(\mathbf{K})$ is the zero set of a finite set of polynomials.

Proof: If X is algebraic, then $X = V(S) = V(\langle S \rangle)$ for some $S \subset \mathbf{K}[x_1, \ldots, x_n]$. But $S = \langle g_1, \ldots, g_m \rangle$ for some $g_1, \ldots, g_m \in \mathbf{K}[x_1, \ldots, x_n]$ (not necessarily in S), by Hilbert. So $X = V(g_1, \ldots, g_m)$. ■

Remark 1.2.9. This implies that any algebraic set in $\mathbf{A}^n$ is the intersection of a finite number of hypersurfaces. If $X = V(g_1, \ldots, g_m)$, then $X = \bigcap_{i=1}^m V(g_i)$ and each $V(g_i)$ is a hypersurface.

Proposition 1.2.10. The following are properties of algebraic sets:

1. If $\{I_\alpha\}$ is a collection of ideals in $\mathbf{K}[x_1, \ldots, x_n]$, then $V(\bigcup_\alpha I_\alpha) = \bigcap_\alpha V(I_\alpha)$

2. If $I, J \subset \mathbf{K}[x_1, \ldots, x_n]$ are two ideals, define $IJ = \sum_k a_k b_k \ : \ a_k \in I, b_k \in J\}$. Then $V(IJ) = V(I) \cup V(J)$.

3. $\emptyset = V(1)$ and $\mathbf{A}^n = V(0)$ are algebraic, and $\{(a_1, \ldots, a_n)\}$ is algebraic by $V(x_1 - a_1, \ldots, x_n - a_n)$ for all such n-tuples

Proof: **1.** This follows from a sequence of equivalence statements:

$$\begin{aligned} p \in V\left(\bigcup_\alpha I_\alpha\right) & \text{ iff } f(p) = 0 \ \forall \ f \in I_\alpha \ \forall \ \alpha \\ & \text{ iff } p \in V(I_\alpha) \ \forall \ \alpha \\ & \text{ iff } p \in \bigcap_\alpha V(I_\alpha) \end{aligned}$$

2. Let $p \in V(I) \cup V(J)$, WLOG $p \in V(I)$. Then $f(p) = 0$ for all $f \in I$, which implies that for all $h \in IJ$, we have $h = \sum_k a_k b_k$ with $a_k \in I$, $b_k \in J$. So $h(p) = \sum_k a_k(p) b_k(p) = 0$. For the other inclusion, suppose that $p \notin V(I)$ (we will show that $p \in V(J)$). Since $p \notin V(I)$, there exists an $f \in I$ such that $f(p) \neq 0$. But for any polynomial $g \in J$, $fg \in IJ$, and $f(p)g(p) = 0$. But $f(p) \neq 0$, and k has no zero divisors, so $g(p) = 0$ for all $g \in J$. Hence $V(IJ) \subset (V(I) \cup V(J))$.

3. This follows directly from the previous parts. ■

Remark 1.2.11. Property **1.** above tells us that intersections of algebraic sets are algebraic. Property **2.** tells us that finite unions of algebraic sets are algebraic. However, infinite unions of algebraic sets need not be algebraic.

Example 1.2.12. The sets $\mathbf{Z} \subset \mathbf{R}$ and $\mathbf{Q} \subset \mathbf{R}$ are not algebraic, because $\mathbf{R}$ is an infinite field.

Note that if $\mathbf{K}$ is finite, any set is algebraic, because $\mathbf{A}^n(\mathbf{K})$ is finite, and any subset of it is a finite union of points, whcih are algebraic.

1.3 Topologies

Definition 1.3.1. Given a set X, a *topology* on X is a set τ in the power set of X such that

1. $X, \emptyset \in \tau$
2. if $\{U_\alpha\}_{\alpha \in I} \subset \tau$, then $\bigcup_{\alpha \in I} U_\alpha \in \tau$
3. if $\{U_1, \ldots, U_n\} \subset \tau$, then $\bigcap_{i=1}^n U_i \in \tau$

The pair (X, τ) is termed a *topological space*, with elements of τ termed τ-open, or simply open sets. The complement of an open set is a closed set.

Example 1.3.2. A starndard example of a topology is the metric topology on $\mathbf{R}^n$. In $\mathbf{R}$, the open sets are the unions of open intervals.

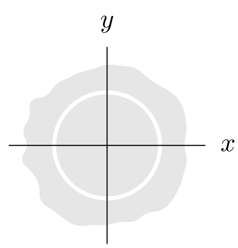
Remark 1.3.3. Note that the closed sets of a topology on X are given by the properties

1. $X, \emptyset$ are closed
2. if $\{U_\alpha\}_{\alpha \in I}$ are closed, then $\bigcap_{\alpha \in I} U_\alpha$ is closed
3. if $\{U_1, \ldots, U_n\}$ are closed, then $\bigcup_{i=1}^n U_i$ is closed

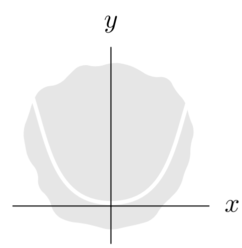
Definition 1.3.4. The *Zariski topology* on $\mathbf{A}^n$ is defined by taking open sets to be the complements of algebraic sets. Moreover, given any algebraic set $X \subset \mathbf{A}^n$, we endow it with the induced topology, where open sets are the intersection of X with an open set in $\mathbf{A}^n$.

Example 1.3.5. Consider the Zariski topology on the affine line $\mathbf{A}^1$. The closed sets are the algebraic sets $\emptyset, \mathbf{A}^1, \{a_1, \ldots, a_m\}$, so the open sets are of the form $\emptyset, \mathbf{A}^1, \mathbf{A}^1 \setminus \{a_1, \ldots, a_m\}$.

Example 1.3.6. In $\mathbf{R}^2$, here are some examples of open sets:



$U = \mathbf{R}^2 \setminus V(x^2 + y^2 - 1)$



$U = \mathbf{R}^2 \setminus V(y - x^2)$

We will see that in $\mathbf{A}^2$, then algebraic sets are $\emptyset, \mathbf{A}^2$, and finite unions of algebraic curves. Hence the open sets are $\emptyset, \mathbf{A}^2$, and $\mathbf{A}^2 - \bigcup$(a finite number of algebraic curves).

Definition 1.3.7. A topology is called *Hausdorff* if it separates points. That is, if for all $p, q \in X$, there exist open neighborhoods $V_p \ni p$, $V_q \ni q$ such that $V_p \cap V_q = \emptyset$.

Example 1.3.8. The metric topology on $\mathbf{R}^n$ is Hausdorff. The Zariski topology on $\mathbf{R}^n$ is not Hausdorff.

1.4 Ideals

Definition 1.4.1. Any algebraic set is of the form $X = V(I)$ for some ideal $I \lhd \mathbf{K}[x_1, \ldots, x_n]$. However, not every subset of $\mathbf{A}^n$ is algebraic. Given any $X \subset \mathbf{A}^n$, we define $I(X) = \{f \in \mathbf{K}[x_1, \ldots, x_n] \ : \ f(p) = 0$ for all $p \in X\}$ to be the *ideal* of X. It is easy to check that $I(X)$ is indeed an ideal of $\mathbf{K}[x_1, \ldots, x_n]$.

We will see that not every ideal in $\mathbf{K}[x_1, \ldots, x_n]$ is the ideal of a set of points $X \subset \mathbf{A}^n$. Nonetheless, if the ideal $I \subset \mathbf{K}[x_1, \ldots, x_n]$ is such that $I = I(X)$ for some $X \subset \mathbf{A}^n$, we say that I is *closed*.

Example 1.4.2. Consider the affine line $\mathbf{A}^1$, whose algebraic sets are $\mathbf{A}^1, \emptyset$, and $\{a_1, \ldots, a_m\}$ for all $a_i \in \mathbf{K}$. Their ideals are

$$I(\{a_1, \ldots, a_m\}) = \langle (x - a_1) \cdots (x - a_m) \rangle ,$$
$$I(\mathbf{A}^1) = \begin{cases} \{0\} & \text{if } \mathbf{K} \text{ is infinite} \\ \left\langle x^{p^n} - x \right\rangle & \text{if } \mathbf{K} \text{ has } p^n \text{ elements} \end{cases} .$$

Next consider $\mathbf{R}^1$, ets that are not algebraic in it, and the associated ideals:

$$X = [0, 1],\ I(X) = \{0\},$$
$$|X| = \infty,\ I(X) = \{0\}.$$

Proposition 1.4.3. For $X = \{(a, b)\} \subset \mathbf{A}^2$, the ideal $I(X) = \langle x - a, y - b \rangle$.

Note we do not need both to occur simultaneously, so we do not multiply $x - 1$ with $y - b$.

Proof: Let us first show that $\langle x - a, y - b \rangle$ is maximal in $\mathbf{K}[x, y]$. Note $\mathbf{K}[x, y]/ \langle x - a, y - b \rangle = \mathbf{K}[\overline{x}, \overline{y}]$, where $\overline{x}$ and $\overline{y}$ are the residues of x, y, respectively, in the quotient. Letting $\overline{x} = a$ and $\overline{y} = b$, $\mathbf{K}[\overline{x}, \overline{y}]\mathbf{K}[a, b] = \mathbf{K}$, so $\mathbf{K}[x, y]/ \langle x - a, y - b \rangle$ is a field, so $\langle x - a, y - b \rangle$ is maximal. But, $\langle x - a, y - b \rangle \subset I(\{(a, b)\}) \subsetneq \mathbf{K}[x, y]$, as $1 \notin I(\{(a, b)\})$. Hence $\langle x - a, y - b \rangle = I(\{(a, b)\})$ by the maximality of $\langle x - a, y - b \rangle$. ■

We will also do this proof in a different manner.

Proof: Clearly, $\langle x - a, y - b \rangle \subset I(\{(a, b)\})$. Let us now show that $I(\{(a, b)\}) \subset \langle x - a, y - b \rangle$. Let $f \in I(\{(a, b)\})$ so that $f(a, b) = 0$. Divide f by $x - a$ to eliminate all the x's from its expression, thus getting $f(x, y) = (x - a)g(x, y) + (y - b)h(y)$ for some $h \in \mathbf{K}[x, y]$. So $f \in \langle x - a, y - b \rangle$, proving that $I(\{(a, b)\}) \subset \langle x - a, y - b \rangle$. ■

Proposition 1.4.4. The following are properties of ideals in $\mathbf{K}[x_1, \ldots, x_n]$:

1. If $X \subset Y \subset \mathbf{A}^n$, then $I(Y) \subset I(X)$.

2.

$$I(\emptyset) = \mathbf{K}[x_1, \ldots, x_n]$$
$$I(\{(a_1, \ldots, a_n)\}) = \langle x_1 - a_1, \ldots, x_n - a_n \rangle \ \forall\ (a_1, \ldots, a_n) \in \mathbf{A}^n$$
$$I(\mathbf{A}^n) = \{0\} \ \text{ if } \mathbf{K} \text{ is infinite}$$

3.

$$S \subset I(V(S)) \ \text{ for all } S \subset \mathbf{K}[x_1, \ldots, x_n]$$
$$X \subset V(I(X)) \ \text{ for all } X \subset \mathbf{A}^n$$

4.

$$I(V(I(X))) = I(X) \ \text{ for all } X \subset \mathbf{A}^n$$
$$V(I(S)) = V(S) \ \text{ for all } S \subset \mathbf{K}[x_1, \ldots, x_n]$$

Proof: Let us show that $V(I(V(S))) = V(S)$ for al $S \subset \mathbf{K}[x_1, \ldots, x_n]$. By **3.** we have that $S \subset I(V(S))$, so that $V(I(V(S))) \subset V(S)$. We also get the other inclusion from the same part. The first identity is identical. ■

Example 1.4.5. Note that equality for **3.** does not always hold. For example, if $S = \left\langle x^2 + 1 \right\rangle \subset \mathbf{R}[x]$, then $V(S) = \emptyset$ and $I(V(S)) = I(\emptyset) = \mathbf{R}[x]$. But $S = \left\langle x^2 + 1 \right\rangle \subsetneq \mathbf{R}[x] = I(V(S))$.

Another example is with $X = [0, 1] \subset \mathbf{R}^1$. Then $I(X) = \{0\}$ and $V(I(X)) = V(\{0\}) = \mathbf{R}^1$, but $X = [0, 1] \subsetneq \mathbf{R}^1 = V(I(X))$.

Definition 1.4.6. Let $X \subset \mathbf{A}^n$ and $I \lhd \mathbf{K}[x_1, \ldots, x_n]$. Define $\overline{X}$ to be the smallest algebraic set containing X, or the closure of X in the Zariski topology. Similarly, define $\overline{I}$ to be the smallest closed ideal containing I, or the closure of I in $\mathbf{K}[x_1, \ldots, x_n]$.

Remark 1.4.7. Let $X \subset \mathbf{A}^n$ and $I \lhd \mathbf{K}[x_1, \ldots, x_n]$. Then

$$\begin{aligned} X = V(I(X)) &\text{ iff } X \text{ is algebraic, and} \\ I = I(V(I)) &\text{ iff } I \text{ is closed.} \end{aligned}$$

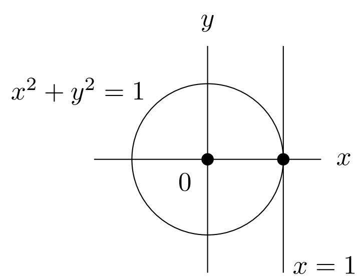
Proposition 1.4.8. Let $X \subset \mathbf{A}^n$ and $I \lhd \mathbf{K}[x_1, \ldots, x_n]$. Then $\overline{X} = V(I(X))$ and $\overline{I} = I(V(I))$.

Proof: Let us show that $\overline{X} = V(I(X))$. First note that the set $V(I(X))$ is algebraic and $X \subset V(I(X))$. It remains to show that if $Y \subset \mathbf{A}^n$ is an algebraic set such that $X \subset Y \subset V(I(X))$, then $Y = V(I(X))$. Let Y be such an algebraic set. By assumption, $Y \subset V(I(X))$, so the only thing to check is that $V(I(X)) \subset Y$. But $X \subset Y$, so $I(Y) \subset I(X)$ and $V(I(X)) \subset V(I(Y)) = Y$ since Y is algebraic. ■

Example 1.4.9. Let $X = [0, 1]$. Then X is not closed in $\mathbf{R}$ since it is infinite but not all of $\mathbf{R}$. Further, $\overline{X} = V(I(X)) = V(I([0, 1])) = V(0) = \mathbf{R}$. Hence X is *dense* in $\mathbf{R}$.

In general, a subset $Y \subset X$ of a topological space X is called *dense* if $\overline{Y} = X$. In fact, any $X \subset \mathbf{A}^1(\mathbf{K})$ that is infinite is dense in $\mathbf{A}^1(\mathbf{K})$ as long as $\mathbf{K}$ is infinite.

Next consider the ideal $I = \left\langle x^2 + y^2 - 1, x - 1 \right\rangle \subset \mathbf{R}[x, y]$. Then $\overline{I} = I(V(I))$.



As $V(I) = V(x^2 + y^2 - 1, x - 1) = V(x^2 + y^2 - 1) \cap V(x - 1) = \{(1, 0)\}$, it follows that

$$\begin{aligned} \overline{I} &= I(V(I)) \\ &= I(\{(1, 0)\}) \\ &= \langle x - 1, y \rangle \\ &\supsetneq \left\langle x^2 + y^2 - 1, x - 1 \right\rangle \\ &= I. \end{aligned}$$

The second-last line follows as $y \notin I$.

1.5 Propreties of ideals

Definition 1.5.1. Let R be a ring. Then $I \lhd R$ is called *radical* if

$$I = \mathrm{Rad}(I) = \sqrt{I} := \{a \in R \ : \ a^n \in I \text{ for some } n > 0\}.$$

Remark 1.5.2. Note that $I \subset \sqrt{I}$. Further, the definition of a radical ideal is equivalent to the following:

$$I = \sqrt{I} \ \text{ iff } \ \Big(a^n r \in I \text{ for some } n > 0 \implies a \in I\Big). \tag{1}$$

This is easier to use as a defining property of radical ideals in examples.

Proposition 1.5.3. If $I \lhd \mathbf{K}[x_1, \ldots, x_n]$ is closed (i.e. there exists $X \subset \mathbf{A}^n$ such that $I = I(X)$), then I is radical.

Proof: Suppose that $I = \sqrt{I}$. Let us verify that I satisfies the condition in the remark above. Let $a \in \mathbf{R}$ be such that $a^n \in I$ for some $n > 0$. Then by the definition of $\sqrt{I}$, we have $a \in \sqrt{I}$. But $I = \sqrt{I}$ implies $a \in I$, so the condition is satisfied.

Conversely, suppose that I satisfies the condition. We need to verify that $\sqrt{I} \subset I$. By definition, $a^n \in I$ for some $n > 0$. The condition then tells us that $a \in I$. ■

Example 1.5.4. The ring R is a radical ideal, as are prime ideals. This follows as for $a^n \in P \lhd R$ for $n > 0$ and P prime, $a^{n-1} \in P$ or $a \in P$. If $a^{n-1} \in P$, then a^{n-2} or $a \in P$, and so on. We finally get that $a \in P$, so P is radical.

The ideal $I = \langle x^2 + 1 \rangle \lhd \mathbf{R}[x]$ is prime since $x^2 + 1$ is irreducible over $\mathbf{R}$, hence I is radical.

The ideal $\langle x - a, y - b \rangle \lhd \mathbf{K}[x, y]$ is maximal, hence prime, so it is radical.

However, not all ideals are radical. For example, for $I = \langle x^2 + y^2 - 1, x - 1 \rangle$, $y^2 = (x^2 + y^2 - 1) - (x - 1)(y - 1) \in I$, but $y \notin I$, so I is not radical. But note that $y \in \sqrt{I}$, since $y^2 \in I$. Also, $x - 1 \in \sqrt{I}$, since $x - 1 \in I$. Then $\langle x - 1, y \rangle \subset \sqrt{I}$ and $\langle x - 1, y \rangle$ is maximal, but $I \neq \mathbf{K}[x, y]$, as $1 \notin \sqrt{I}$, so $\sqrt{I} = \langle x - 1, y \rangle$.

Proposition 1.5.5. If the ideal $I \subset \mathbf{K}[x_1, \ldots, x_n]$ is closed, then I is radical.

Proof: Suppose that I is closed, so that $I = I(X)$ for some $X \subset \mathbf{A}^n$. Let us show that I satisfies (1). Let $f \in \mathbf{K}[x_1, \ldots, x_n]$ be such that $f^n \in I = I(X)$. Then $f^n(p) = f(p) \cdots f(p) = 0$, but $f(p) \in \mathbf{K}$, which is a field, so $f(p) = 0$ for all p. This implies that $f \in I(X) = I$, so (1) is satisfied. ■

Note that the converse of the above claim is not necessarily true. For example, $\langle x^2 + 1 \rangle \subsetneq \mathbf{R}[x]$ is radical, but not closed, as $\overline{\langle x^2 + 1 \rangle} = \mathbf{R}$.

Proposition 1.5.6. For $X \subset \mathbf{A}^n$ any set, $I(X)$ is radical.

Proposition 1.5.7. If $I \lhd \mathbf{K}[x_1, \ldots, x_n]$, then $I \subset \sqrt{I} \subset \overline{I} = I(V(I))$.

Proof: We have already seen that $I \subset \sqrt{I}$. Let us show that $\sqrt{I} \subset I(V(I))$. Let $f \in \sqrt{I}$, so that $f^n \in I$ for some $n > 0$. This means, in particular, that

$$\begin{aligned} & f^n(p) = 0 \ \forall\, p \in V(I) \\ \Longrightarrow\ & f(p) = 0 \ \forall\, p \in V(I) \\ \Longrightarrow\ & f \in I(V(I)) = \overline{I}. \end{aligned}$$

The second line follows as $f(p) \in \mathbf{K}$. ■

If $\mathbf{K}$ is algebraically closed (i.e. $\mathbf{K} = \overline{\mathbf{K}}$), we have a stronger statement.

Theorem 1.5.8. [Hilbert's Nullstellensatz] If $\mathbf{K} = \overline{\mathbf{K}}$ and $I \lhd \mathbf{K}[x_1, \ldots, x_n]$, then $I(V(I)) = \sqrt{I}$.

Remark 1.5.9. The above implies that $I = \overline{I}$ iff $I = \sqrt{I}$, or equivalently, there is a 1-1 correspondence between closed and radical ideals. This gives us the following correspondences:

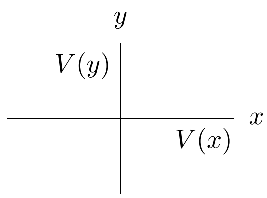
$$\begin{pmatrix} \text{algebraic} \\ \text{set in } \mathbf{A}^n \end{pmatrix} \overset{1\,:\,1}{\longleftrightarrow} \begin{pmatrix} \text{closed ideals} \\ \text{in } \mathbf{K}[x_1, \ldots, x_n] \end{pmatrix} \qquad \begin{array}{ccc} X & \mapsto & I(X) \\ V(J) & \mapsto & J \end{array} \qquad \text{because} \qquad \begin{array}{ccccc} X & \mapsto & I(X) & \mapsto & V(I(X)) = X \\ J & \mapsto & V(J) & \mapsto & I(V(J)) = J \end{array},$$

if X is algebraic and J is closed.

Definition 1.5.10. An algebraic set $X \subset \mathbf{A}^n$ is *irreducible* if $X \neq \emptyset$ and X cannot be expressed as $X = X_1 \cup X_2$, where X_1, X_2 are algebraic sets not equal to X. Otherwise, X is *reducible*.

Example 1.5.11. The set $\mathbf{A}^1$ is irreducible if $\mathbf{K}$ is infinite, since the only proper algebraic subsets of $\mathbf{A}^1$ are finite sets of points. Moreover, $I(\mathbf{A}^1) = (0)$ if $\mathbf{K}$ is infinite, which is a prime ideal.

Consider the example of $V(xy) = V(x) \cup V(y) \subset \mathbf{A}^2$, which is reducible.



We claim that $I(V(xy)) = \langle xy \rangle \subset \mathbf{K}[x, y]$, which is not prime, since $xy \in \langle xy \rangle$, but $x, y \notin \langle xy \rangle$. Clearly, $\langle xy \rangle \subset I(V(xy))$, so we just have to show that $I(V(xy)) \subset \langle xy \rangle$. Let $f \in I(V(xy))$, for which

$$\begin{aligned} & f(p) = 0 \ \forall \ p \in V(xy) = V(x) \cup V(y) \\ \Longrightarrow \ & f(p) = 0 \ \forall \ p \in V(x) \text{ and } \forall \ p \in V(y) \\ \Longrightarrow \ & f \in I(V(x)) \text{ and } f \in I(V(y)). \end{aligned}$$

But $I(V(x)) = \langle x \rangle$. Indeed, $\langle x \rangle \subset I(V(x)) \subset \mathbf{K}[x, y]$. Also, if $g \in I(V(x)) \subset \mathbf{K}[x, y]$, then $g(0, y) = 0$ for all y. Now, $g(x, y)$ can be written as $g(x, y) = a_0(x) + a_1(x)y + \cdots + a_m(x)y^m$, so

$$\begin{aligned} g(0, y) = 0 \ \forall \ y \iff & \ a_i(0) = 0 \ \forall \ i \\ \Longrightarrow & \ a_i \in \langle x \rangle \subset \mathbf{K}[x] \ \forall \ i \\ \Longrightarrow & \ g \in \langle x \rangle \subset \mathbf{K}[x, y] \\ \Longrightarrow & \ I(V(x)) \subset \langle x \rangle \\ \Longrightarrow & \ I(V(x)) = \langle x \rangle . \end{aligned}$$

Similarly, $I(V(y)) = \langle y \rangle$, so $f \in \langle x \rangle \cap \langle y \rangle = \langle xy \rangle$, and we have proved the claim.

Proposition 1.5.12. An algebraic set $X \subset \mathbf{A}^n$ is irreducible iff $I(X)$ is prime.

Note that Fulton also considers $\emptyset$ to be irreducible, but then $I(\emptyset) = \mathbf{K}[x_1, \ldots, x_n]$ is not prime. However, most authors assume irreducible algebraic sets are non-empty.

Proof: Let $X \subset \mathbf{A}^n$ be irreducible algebraic, and $f, g \in \mathbf{K}[x_1, \ldots, x_n]$ such that $fg \in I(X)$. Let us show that $f \in I(x)$ or $g \in I(X)$. Note that $\langle fg \rangle \subset I(X)$, so that

$$\begin{aligned} & X = V(I(X)) \subset V(\langle fg \rangle) = V(fg) = V(f) \cup V(g) \\ \Longrightarrow \quad & X = \underbrace{(X \cap V(f))}_{\text{algebraic}} \cup \underbrace{(X \cap V(g))}_{\text{algebraic}} . \end{aligned}$$

Hence $X = X \cap V(f)$ or $X = X \cap V(g)$ by the irreducibility of X. This implies that $X \subset V(f)$ or $X \subset V(g)$, further implying that $f \in I(X)$ or $g \in I(X)$. Hence $I(X)$ is prime.

Conversely, let's assume that $I(X)$ is prime. Suppose that $X = X_1 \cup X_2$ with $X_1, X_2 \subset \mathbf{A}^n$ algebraic. Then, since X, X_1, X_2 are algebraic, we have that $X = V(I(X))$, $X_1 = V(I(X_1))$, $X_2 = V(I(X_2))$. Also, $I(X) = I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$. If $I(X) = I(X_1)$, then $X = V(I(X)) = V(I(X_1)) = X_1$. Otherwise, there exists $f \in I(X_1)$ such that $f \notin I(X)$. But since $I(X_1)$ and $I(X_2)$ are ideals, and $f \in I(X_1)$, it follows that $fg \in I(X_1) \cap I(X_2)$ for all $g \in I(X_2)$. But $I(X_1) \cap I(X_2) = I(X)$, which is prime. This forces $g \in I(X)$ for all $f \in I(X_2)$, since $f \notin I(X)$. Hence $I(X_2) = I(X)$, and $X_2 = X$. ■

2 Affine varieties

2.1 Classification of algebraic sets

Definition 2.1.1. An (*affine*) *variety* is an irreducible algebraic set in $\mathbf{A}^n$.

Example 2.1.2. Consider the following examples of affine varieties.

a. The space $\mathbf{A}^n(\mathbf{K})$ with $\mathbf{K}$ infinite is a variety since $I(\mathbf{A}^n(\mathbf{K})) = (0)$, which is prime.

b. For all $p = (a_1, \ldots, a_n \in \mathbf{A}^n$, we have seen that $I(\{p\}) = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$, which is maximal, therefore prime. Hence $\{p\}$ is a variety.

c. If $\mathbf{K}$ is finite, then $\mathbf{A}^n(\mathbf{K})$ is not a variety, since it can be written as a union of points (and fields have at least 2 points, 1 and 0).

d. Suppose that $\mathbf{K} = \overline{\mathbf{K}}$, and consider an irreducible polynomial $f \in \mathbf{K}[x_1, \ldots, x_n]$. Then $\langle f \rangle$ is prime and therefore also radical. So $I(V(\langle f \rangle)) = \sqrt{\langle f \rangle} = \langle f \rangle$, by the Nullstellensatz and the fact that $\langle f \rangle$ is radical. Then $V(f)$ is irreducible and therefore a variety.

Lemma 2.1.3. If $\mathbf{K} = \overline{\mathbf{K}}$ and $f \in \mathbf{K}[x_1, \ldots, x_n]$ is irreducible, then $V(f)$ is irreducible and $I(V(f)) = \langle f \rangle$.

Remark 2.1.4. So when $\mathbf{K} = \overline{\mathbf{K}}$, we have the following 1-1 correspondence:

Geometric in $\mathbf{A}^n$	Algebraic in $\mathbf{K}[x_1, \ldots, x_n]$
$\mathbf{A}^n$	(0)
algebraic set	radical ideal
variety	prime ideal
point	maximal ideal
$\emptyset$	$\mathbf{K}[x_1, \ldots, x_n]$

Note that if $\mathbf{K} \neq \overline{\mathbf{K}}$, then prime ideals may not correspond to algebraic sets. For example, for $f(x, y) = x^2 + y^2(y-1)^2 \subset \mathbf{R}[x, y]$, we have that $V(f) = \{(0,0), (0,1)\}$, which is reducible. But f is irreducible over $\mathbf{R}$, as $f = (x + iy(y-1))(x - iy(y-1))$, and $\mathbf{R}[x, y] \subset \mathbf{C}[x, y]$. So if f would be reducible in $\mathbf{R}[x, y]$, then we would gen a different factorization of f in $\mathbf{C}[x, y]$, which is impossible, since $\mathbf{C}[x, y]$ is a UFD (unique factorization domain).

Example 2.1.5. If $\mathbf{K} \neq \overline{\mathbf{K}}$, then two prime ideals may have the same zero set. For example, in $\mathbf{R}[x, y]$,

$$\begin{aligned} &\langle x^2 + y^2 \rangle \text{ is prime and } V(\langle x^2 + y^2 \rangle) = \{(0,0)\}, \\ &\langle x, y \rangle \text{ is maximal, and so prime, and } V(\langle x, y \rangle) = \{(0,0)\}. \end{aligned}$$

Hence there is not a 1-1 correspondence between prime ideals and varieties, of $\mathbf{K} \neq \overline{\mathbf{K}}$.

Proposition 2.1.6. Every algebraic set $X \subset \mathbf{A}^n$ is a finite union of irreducible algebraic sets.

Proof: Let $X \subset \mathbf{A}^n$ be algebraic, and suppose that X is not the finite union of irreducible algebraic sets. This means, in particular, that X is irreducible, so that it can be written as $X = X_1 \cup X_2$, with one of X_1, X_2 an algebraic set that cannot be written as a finite non-trivial union of irreducible algebraic sets. Suppose that, WLOG, it is X_1. Thus, X_1 is also reducible, and can be written as $X_1 = X_3 \cup X_4$, with X_3 an algebraic set that is not a finite non-trivial union of irreducible algebraic sets. Continue this process to get an infinite strict descending chain of algebraic sets

$$X \supsetneq X_1 \supsetneq X_3 \supsetneq x_5 \supsetneq \cdots .$$

Take ideals of these algebraic sets to reverse the inclusion as

$$I(X) \subsetneq I(X_1) \subsetneq I(X_3) \subsetneq I(X_5) \subsetneq \cdots .$$

The strict inclusion follows because if $I(X) = I(X_1)$, then $X = V(I(X)) = V(I(X_1)) = X_1$, as X, X_1 are algebraic. But $\mathbf{K}[x_1, \ldots, x_n]$ is Noetherian, so every strict ascending chain of ideals must terminate, implying that there is $m \in \mathbf{Z}$ such that $I(X_m) = I(X_{m+1}) = I(X_{m+2}) = \cdots$. This implies that $X_m = X_n$ for all $n \geqslant m$, a contradiction. This proves the proposition. ■

Definition 2.1.7. Now consider an algebraic set $X \subset \mathbf{A}^n$, and suppose that it can be written as $X = X_1 \cup \cdots \cup X_m$ with each X_i an irreducible algebraic set. Then, if $X_i \subset X_j$ with $i \neq j$, we get rid of X_i. By repeating this procedure enough times, we can write X as $X = X_{i_1} \cup \cdots \cup X_{i_k}$, where each X_{i_j} is an irreducible algebraic set, and $X_{i_j} \not\subset X_{i_\ell}$ for all $j \neq \ell$, and $\{i_1, \ldots, i_\ell\} \subset \{1, \ldots, m\}$. This expression is called the (*irredundant*) *decomposition* of X into irreducible algebraic sets.

Theorem 2.1.8. Every algebraic set $X \subset \mathbf{A}^n$ has a unique decomposition as a finite union of irreducible algebraic sets.

Proof: Suppose that $X = X_1 \cup \cdots \cup X_k = Y_1 \cup \cdots \cup Y_{k'}$, where each X_i, Y_j is an irreducible algebraic set, with $X_i \not\subset X_\ell$ if $i \neq \ell$ and $Y_j \not\subset Y_m$ if $j \neq m$. Then for all i,

$$X_i = X_i \cap X = X_i \cap (Y_1 \cup \cdots \cup Y_{k'}) = \bigcup_j X_i \cap Y_j.$$

But X_i is irreducible, so we must have that $X_i = X_i \cap Y_{j_0}$ for some $j_0 \in \{1, \ldots, k'\}$. In particular, it means that $X_i \subset Y_{j_0}$. Similarly, $Y_{j_0} \subset X_{i_0}$ for some $i_0 \in \{1, \ldots, k\}$. So $X_i \subset Y_{j_0} \subset X_{i_0}$, meaning that $X_i = Y_{j_0} = X_{i_0}$. This can be repeated for all i and j, showing that each x_i corresponds to a Y_j, and vice versa. ■

Example 2.1.9. Consider $X = V(y^4 - x^3, y^4 - x^3y^2 + xy^2 - x^3) \subset \mathbf{C}^2$. We generate factors by noting that

$$y^4 - x^2 = (y^2 - x)(y^2 + x),$$
$$y^4 - x^2y^2 + xy^2 - x^3 = (y - x)(y + x)(y^2 + x),$$

where all of the factors on the right are irreducible by Eisenstein. So we may write

$$X = V(y^2 + x) \cup V(y^2 - x, (y - x)(y + x)) = V(y^2 + x) \cup \{(0, 0), (1, 1), (1, -1)\}.$$

Here $V(y^2 + x)$ is irreducible since $y^2 + x$ is irreducible and $\mathbf{C} = \overline{\mathbf{C}}$, and $\{(0, 0), (1, 1), (1, -1)\} = \{(0, 0)\} \cup \{(1, 1)\} \cup \{(1, -1)\}$ is irreducible because points are irreducible. We found these points by solving the system of equations given by $y^2 - x = 0$ and $(y - x)(y + x) = 0$. However, we see that $(0, 0) \in V(y^2 + x)$, whereas $(1, 1), (1, -1) \notin V(y^2 + x)$. Thus the decomposition of X is

$$X = V(y^2 + x) \cup \{(1, 1)\} \cup \{1, -1\}.$$

Remark 2.1.10. So far we have see that the algebraic sets in $\mathbf{A}^1$ consist of $\emptyset, \mathbf{A}^1$, and finite sets of points. Since any algebraic set admits a decomposition as a finite union of irreducible algebraic sets, which is unique, it is enough to classify the irreducible algebraic sets in $\mathbf{A}^2$. Potential candidates are $\mathbf{A}^2$, $V(f)$ with f irreducible and $V(f)$ infinite, and $\{pt\}$. We will see that these are the only ones. But first we need a technical lemma.

Lemma 2.1.11. If $f, g \in \mathbf{K}[x, y]$ with no common factors, then $V(f, g) = V(f) \cap V(g)$ is at most a finite set of points.

Proof: First note that f, g can be considered as polynomials in $\mathbf{K}[x][y] \subset \mathbf{K}(x)[y]$, which is a PID (principal ideal domain), since $\mathbf{K}(x)$ is a field. Recall Gauss's lemma, which says that an integral domain D with a fraction field F having $f \in D[y]$ irreducible in $D[y]$ implies f is irreducible in $F[y]$.

Then, if f, g have no common factors in $\mathbf{K}[x][y]$, then they have no common factors in $\mathbf{K}(x)[y]$, because the irreducible factors of f, g in $\mathbf{K}[x][y]$ are the same as the irreducible factors in $\mathbf{K}(x)[y]$, since it is a UFD. Now, since f and g don't have common factors in $\mathbf{K}(x)[y]$, which is a PID, there exists $s, t \in \mathbf{K}(x)[y]$ such that $sf + tg = 1$. But, there exists $d \in \mathbf{K}[x]$ such that $ds = a$, $dt = b \in \mathbf{K}[x][y]$, implying that $aF = bg \in \mathbf{K}[x]$. Let $(x_0, y_0) \in V(f, g)$. Then $0 = a(x_0, y_0)f(x_0, y_0) + b(x_0, y_0)g(x_0, y_0) = d(x_0)$, so x_0 is a root of $d \in \mathbf{K}[x]$. Hence there are only a finite number of possibilites for x_0. Similarly, one finds there are only a finite number of possibilities for y_0. So $V(f, g)$ is at most a finite set of points. ■

Proposition 2.1.12. If f is an irreducible polynomial in $\mathbf{K}[x, y]$ and $V(f)$ is infinite, then $I(V(f) = \langle f \rangle$. In particular, $V(f)$ is an irreducible algebraic set.

Proof: Clearly $\langle f \rangle \subset I(V(f))$, so we just need to show that $I(V(f)) \subset \langle f \rangle$. Let $g \in I(V(f))$, so then $V(f) \subset V(f, g)$. But $V(f)$ is infinite, meaning that f and g have a common factor by the Lemma above. Hence $f \mid g$ since f is irreducible. Then $g \in \langle f \rangle$, so $I(V(f)) \subset \langle f \rangle$.

Theorem 2.1.13. [CLASSIFICATION OF IRREDUCIBLE ALGEBRAIC SETS IN $\mathbf{A}^2(\mathbf{K})$ FOR $|\mathbf{K}| = \infty$]
The irreducible algebraic sets in $\mathbf{A}^2$ are $\mathbf{A}^2$, $\{pt\}$, and $V(f)$ with $f \in \mathbf{K}[x, y]$ irreducible and $|V(f)| = \infty$.

Proof: Let $X \subset \mathbf{A}^n$ be algebraic, and assume that $X \neq \mathbf{A}^2$, $X \neq \{pt\}$. By ireducibility, X is infinite and $I(X)$ is prime. Note that $I(X) \neq \{0\}$, otherwise $X = \mathbf{A}^2$. So there exists a non-zero $f \in I(X)$. Moreover, we can assume that f is ireducible, since an ireducible factor of f is in $I(X)$, because $I(X)$ is prime. We now claim that $I(X) = \langle f \rangle$. Certainly $\langle f \rangle \subset I(X)$. Let $g \in I(X)$ and suppose that $g \notin \langle f \rangle$. Then f and g do not have a common factor (because f is irreducible), forcing $V(f, g)$ to be finite. But, $X \subset V(f, g)$ with X infinite. Hence $g \in \langle f \rangle$ implies $I(X) = \langle x \rangle$, so $X = V(I(X)) = V(f)$. ■

2.2 Coordinate rings and polynomial maps

Recall that an affine variety is an irreducible algebraic subset of $\mathbf{A}^n$ endowed with the induced Zariski topology. Since the only irreducible subset of $\mathbf{A}^n(\mathbf{K})$ with $\mathbf{K}$ finite are points, we will ossume from now on that $\mathbf{K}$ is infinite.

Definition 2.2.1. Suppose that X is a variety. Then $I(X)$ is prime, and $\Gamma(X) = \mathbf{K}[x_1, \ldots, x_n]/I(X)$ is called the *coordinate ring* of X. Note that since $I(X)$ is prime, $\Gamma(X)$ is a domain. In fact, $\mathbf{K}[x_1, \ldots, x_n]/I(X)$ is a domain iff $I(X)$ is prime iff X is irreducible.

Remark 2.2.2. Given any polynomial $f \in \mathbf{K}[x_1, \ldots, x_n]$, one may think of f as a polynomial function on X by restricting f to X. But if we choose $f, g \in \mathbf{K}[x-1, \ldots, x_n]$, they may define the same polynomial function on X if $f|_X = g|_X$. In fact

$$f|_X = g|_X \iff f = g \text{ on } X \iff f - g \in I(X).$$

Therefore $\Gamma(X) = \{\text{polynomial functions on } X\}$.

Example 2.2.3. Consider the following examples of sets and their coordinate rings.

a. $X = \mathbf{A}^n$, $I(X) = (0)$. Then $\Gamma(X) = \mathbf{K}[x_1, \ldots, x_n]/(0) = \mathbf{K}[x_1, \ldots, x_n]$.

b. $X = \{pt\} = \{(a_1, \ldots, a_n)\}$, $I(X) = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$. Then $\Gamma(X) = \mathbf{K}[x_1, \ldots, x_n]/\langle x_1 - a_1, \ldots, x_n - a_n \rangle = \mathbf{K}$. Note that this is consistent with the fact that any function on a singleton is constant.

c. $X = V(y - x^2) \subset \mathbf{A}^2$, $I(X) = \langle y - x^2 \rangle$. Since $X = V(f)$ with $f = y - x^2$ irreducible and X infinite, $\Gamma(X) = \mathbf{K}[x, y]/\langle y - x^2 \rangle = \mathbf{K}[\overline{x}, \overline{y}]$ with $\overline{y} = \overline{x}^2$. Then $\Gamma(X) = \mathbf{K}[\overline{x}] = \mathbf{K}[t]$ for $t = \overline{x}$. So this is a polynomial ring in one variable.

Theorem 2.2.4. Let X be an affine variety. Then $\Gamma(X)$ is Noetherian.

Proof: Consider the projection map $\pi : \mathbf{K}[x_1, \dots, x_n] \to \mathbf{K}[x_1, \dots, x_n]/I(X)$. Let us show that $J \lhd \Gamma(X)$ is finitely generated. First note that the inverse image $\pi^{-1}(J)$ is an ideal in $\mathbf{K}[x_1, \dots, x_n]$ that contains $I(X)$. But $\mathbf{K}[x_1, \dots, x_n]$ is Noetherian, so $\pi^{-1}(J)$ is generated by $f_1, \dots, f_k$, i.e. $\pi^{-1}(J) = \langle f_1, \dots, f_k \rangle$ for $f_i \in \mathbf{K}[x_1, \dots, x_n]$. Then $J = \pi(\pi^{-1}(J)) = \langle \overline{f}_1, \dots, \overline{f}_k \rangle$, so it is finitely generated (where $\overline{f}_i$ represents the residue class of f_i). ■

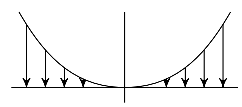
Remark 2.2.5. The coordinate ring $\Gamma(X)$ has additional structure to its ring structure. It is also a vector space over $\mathbf{K}$, where the vector space addition is the usual addition in the ring, and scalar multiplication coincides with multiplication in the ring. Such a ring is called a $\mathbf{K}$-*algebra*.

Example 2.2.6. Consider the following examples of $\mathbf{K}$-algebras.
- $\mathbf{K}[x_1, \dots, x_n]$ is a $\mathbf{K}$-algebra.
- If A is a $\mathbf{K}$-algebra and $I \lhd A$, then A/I is a $\mathbf{K}$-algebra.

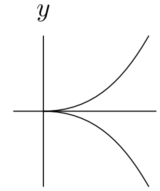
Definition 2.2.7. Let $X \subset \mathbf{A}^n$ and $Y \subset \mathbf{A}^m$ be varieties. A function $\varphi : X \to Y$ is called a *polynomial map* if there exist polynomials $f - 1, \dots, f_m \in \mathbf{K}[x_1, \dots, x_n]$ such that $\varphi(x) = (f_1(x), \dots, f_m(x))$ for all $x \in X$. Note that the f_i are uniquely determined by φ up to elements in $I(X)$. So we can think of the components of φ as being elements of $\Gamma(X)$.

Example 2.2.8. Consider the following examples of polynomial maps.
- Polynomial functions $f : X \to \mathbf{K} = \mathbf{A}^1$
- Any linear map $\mathbf{A}^n \to \mathbf{A}^m$
- Any affine map $A^n \to \mathbf{A}^m$ given by $x \mapsto Ax + b$ for $A \in M_{m \times n}(\mathbf{K})$ and $b \in \mathbf{A}^m$
- Compositions of polynomial maps
- The map as given below:

$$\begin{array}{rcl} X = V(y - x^2) \subset \mathbf{A}^2 & \to & \mathbf{A}^1 \\ (x, y) & \mapsto & x \end{array}$$



$$\begin{array}{rcl} \mathbf{A}^1 & \to & Y = V(y^2 - x^3) \subset \mathbf{A}^2 \\ t & \mapsto & (t^2, t^3) \end{array}$$



Proposition 2.2.9. Let $X \subset \mathbf{A}^n$ and $Y \subset \mathbf{A}^m$ be two varieties and $\varphi : X \to Y$ a polynomial map. Then
1. for any algebraic $Z \subset Y$, $\varphi^{-1}(Z) \subset X$ is algebraic (i.e. φ is continuous in the Zariski topology), and
2. $\overline{\varphi(X)}$ is irreducible in $\mathbf{A}^m$.

Proof: **1.** Suppose that $\mathbf{A}^n$ has ambient coordinates $x_1, \dots, x_n$ and $\mathbf{A}^m$ has ambient coordinates $y_1, \dots, y_m$. Then the map given by

$$\begin{array}{rrcl} \varphi : & X \subset \mathbf{A}^n & \to & Y \subset \mathbf{A}^m \\ & (x_1, \dots, x_n) & \mapsto & (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)) \end{array}$$

with $f_i \in \mathbf{K}[x_1, \dots, x_n]$, since φ is a polynomial. Let $Z \subset Y$ be algebraic. Then $Z = V(g_1, \dots, g_k)$ for $g_i \in \mathbf{K}[y_1, \dots, y_m]$, with

$$\begin{aligned} \varphi^{-1}(Z) &= \{p \in X \ : \ \varphi(p) \in Z\} \\ &= \{p \in X \ : \ g_i(\varphi(p)) = 0 \ \forall \ i\} && \text{since } Z = V(g_1, \dots, g_k) \\ &= \{p \in X \ : \ g_i(f_1(p), \dots, f_m(p)) = 0 \ \forall \ i\} \\ &= V(g_1(f_1, \dots, f_m), \dots, g_k(f_1, \dots, f_m)), \end{aligned}$$

so $\varphi^{-1}(Z)$ is algebraic in $\mathbf{A}^n$.

2. Suppose $\overline{\varphi(X)} = Z_1 \cup Z_2$ with Z_1, Z_2 algebraic. Let us show that $\overline{\varphi(X)} = Z_1$ or Z_2, implying that $\overline{\varphi(X)}$ is irreducible. First note that $X = \varphi^{-1}(\overline{\varphi(X)}) = \varphi^{-1}(Z_1) \cup \varphi^{-1}(Z_2)$, where $\varphi^{-1}(Z_1)$, $\varphi^{-1}(Z_2)$ are algebraic by **1.**, since Z_1, Z_2 are algebraic. This implies that

$$\begin{aligned} X = \varphi^{-1}(Z_1) \text{ or } X = \varphi^{-1}(Z_2) \quad &\Longrightarrow \quad \varphi(X) \subset Z_1 \text{ or } \varphi(X) \subset Z_2 \\ &\Longrightarrow \quad \overline{\varphi(X)} \subset \overline{Z_1} = Z_1 \text{ or } \overline{\varphi(X)} \subset \overline{Z_2} = Z_2. \end{aligned}$$

Since $Z_1, Z_2 \subset \overline{\varphi(X)}$, this means that $\overline{\varphi(X)} = Z_1$ or Z_2. ■

Example 2.2.10. The proposition above can be used to determine whether an algebraic subset of $\mathbf{A}^n$ is irreducible. For example, consider $SL(n,k) = \{A \in gl(n,k) \;:\; \det(A) = 1\}$. Note that $gl(n,k) = \{n \times n$ matrices over $\mathbf{K}\} \cong \mathbf{K}^{n^2} \cong= \mathbf{A}^{n^2}$. Then $SL(n,k) = \det^{-1}(\{1\})$, which is an algebraic set, since $\det : \mathbf{A}^{n^2} \to \mathbf{K} = \mathbf{A}^1$ is a polynomial map.

Remark 2.2.11. We have 3 tests for determining the irreducibility of an algebraic set $Z \subset \mathbf{A}^m$: Z is irreducible iff

1. $I(Z)$ is prime, or
2. $\Gamma(Z) = \mathbf{K}[y_1, \ldots, y_m]/I(Z)$ is a domain, or
3. $Z = \overline{\varphi(X)}$ for some polynomial map $\varphi : X \to \mathbf{A}^m$ with $X \subset \mathbf{A}^n$ a variety.

Example 2.2.12. Consider the twisted cubic $X = V(y - x^2, z - x^3) \subset \mathbf{A}^3$ and $I(X) = \left\langle y - x^2, z - x^3 \right\rangle$. Observe that

$$\begin{aligned} \Gamma(X) &= \mathbf{K}[x,y,z]/\left\langle y - x^2, z - x^3 \right\rangle \\ &= \mathbf{K}[\overline{x}, \overline{y}, \overline{z}] && \text{with } \overline{y} = \overline{x}^2, \overline{z} = \overline{x}^3 \\ &= \mathbf{K}[\overline{x}] \\ &= \mathbf{K}[t], && \text{with } t = \overline{x} \end{aligned}$$

which is a domain. Hence X is irreducible. Also, $X = \varphi(\mathbf{A}^1)$, with $\varphi : \mathbf{A}^1 \to X \subset \mathbf{A}^3$ given by $t \mapsto (t, t^2, t^3)$.

Definition 2.2.13. Two varieties $X \subset \mathbf{A}^n$ and $Y \subset \mathbf{A}^m$ are said to be *isomorphic* if there exists an invertible polynomial map $\varphi : X \to Y$ whose inverse $\varphi^{-1} : X \to Y$ is also a polynomial map. We then write $X \cong Y$.

Example 2.2.14. Consider the following examples of isomorphic varieties.

· $\varphi : X = V(y - x^2) \subset \mathbf{A}^2 \to \mathbf{A}^1$ given by $(x,y) \mapsto x$. The inverse $\varphi^{-1} : \mathbf{A}^1 \to X \subset \mathbf{A}^2$ is given by $t \mapsto (t, t^2)$. Hence $X \cong \mathbf{A}^1$.

· $\varphi : X = V(xy - 1) \subset \mathbf{A}^2 \to \mathbf{A}^1$ given by $(x,y) \mapsto x$. This polynomial map is not surjective, since no point in X gets mapped to 0. Hence φ is not an isomorphism. Note we can show that there does not exist an isomorphism between X and $\mathbf{A}^1$. Here, $X = V(f)$ with $f = xy - 1$ is irreducible, implying that $I(X) = \langle f \rangle$, because we are in $\mathbf{A}^2$ and X is irreducible. So then we find that

$$\Gamma(X) = \mathbf{K}[x,y]/\left\langle xy - 1 \right\rangle = \mathbf{K}[\overline{x}, \overline{y}]$$

with $\overline{xy} = 1$. We will see that $\Gamma(X) \ncong \Gamma(\mathbf{A}^1)$, so $X \ncong \mathbf{A}^1$.

· $\varphi : \mathbf{A}^1 \to V(y^2 - x^3) \subset \mathbf{A}^2$ given by $t \mapsto (t^2, t^3)$ is a bijection, with inverse $\varphi^{-1}(x,y) = y^{1/3}$. But, φ^{-1} cannot be a polynomial, map, because if $\varphi^{-1}(x,y) = p(x,y)$ was a polynomial, then $t = \varphi^{-1}(\varphi(t)) = p(t^2, t^3)$, which is an expression whose powers of t are strictly greater than 1. Also note that

$$\Gamma(X) = \mathbf{K}[x,y]/\left\langle y^2 - x^3 \right\rangle = \mathbf{K}[\overline{x}, \overline{y}],$$

for $\overline{y}^2 = \overline{x}^3$.

Remark 2.2.15. Isomorphisms that are affine coordinate changes are called *affine equivalences*. It is possible to show that any irreducible conic in $\mathbf{R}^2$ is affinely equivalent to

$$\begin{array}{c} y^2 = x \\ \text{parabola} \end{array} \quad \text{or} \quad \begin{array}{c} x^2+y^2=1 \\ \text{circle} \end{array} \quad \text{or} \quad \begin{array}{c} x^2-y^2=1 \\ \text{hyperbola} \end{array}.$$

Definition 2.2.16. Let $\varphi : X \to Y$ be a polynomial map between two varieties X, Y. Define the *pullback* along φ by

$$\begin{array}{rccc} \varphi^* : & \Gamma(Y) & \to & \Gamma(X) \\ & \overline{g} & \mapsto & \overline{g \circ \varphi} \end{array}.$$

Let us check that φ^* is well-defined. Let $X \subset \mathbf{A}^n$ with ambient coordinates $x_1, \ldots, x_n$ and $Y \subset \mathbf{A}^m$ with ambient coordinates $y_1, \ldots, y_m$. Suppose that $\overline{g} = \overline{g'}$ in $\Gamma(Y) = \mathbf{K}[y_1, \ldots, y_m]/I(Y)$. Then $g' = g + h$ for some $h \in I(Y)$, and

$$g' \circ \varphi = g \circ \varphi + h \circ \varphi = g \circ \varphi,$$

because for all $p \in X$, $\varphi(p) \in Y$, so $h(\varphi(p)) = 0$. Hence $\overline{g' \circ \varphi} = \overline{g \circ \varphi}$ in $\Gamma(X) = \mathbf{K}[x_1, \ldots, x_n]/I(X)$, and φ^* is well-defined.

Remark 2.2.17. Note that the pullback is *functiorial*. Moreover,

- $(\mathrm{id}_X)^* = \mathrm{id}_{\Gamma(X)}$
- $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$
- φ^* is a **K**-algebra homomorphism, i.e. a **K**-linear ring homomorphism.

The last follows as $\Gamma(X)$ is a **K**-algebra because it is a ring that admits a **K**-vector space structure.

Example 2.2.18. Since the pullback φ^* is a **K**-algebra homomorphism, it is enough to specify it on the generators $\overline{y_i}$ of $\Gamma(Y) = \mathbf{K}[y_1, \ldots, y_m]/I(Y) = \mathbf{K}[\overline{y_1}, \ldots, \overline{y_m}]$. For example, $\varphi : \mathbf{A}^1 \to X = V(y^2 - x^3) \subset \mathbf{A}^2$ is given by $t \mapsto (t^2, t^3)$. Then the map φ^* is completely defined by

$$\begin{array}{rrcl} \varphi^* : & \Gamma(X) = \mathbf{K}[\overline{x}, \overline{y}] & \to & \Gamma(\mathbf{A}^1) = \mathbf{K}[t] \\ & \overline{x} & \mapsto & \overline{x \circ \varphi} = t^2 \\ & \overline{y} & \mapsto & \overline{y \circ \varphi} = t^3 \end{array}.$$

Proposition 2.2.19. [Faithfulness]
If $\varphi : X \to Y$ and $\psi : X \to Y$ are polynomial maps and $\varphi^* = \psi^*$, then $\varphi = \psi$.

Proof: Let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_m)$ be ambient coordinates for $\mathbf{A}^n$, $\mathbf{A}^m$, respectively. Then $\varphi = (f_1, \ldots, f_m)$ and $\psi = (g_1, \ldots, g_m)$ for $f_i, g_i \in \mathbf{K}[x_1, \ldots, x_n]$. Note that $f_i = y_i \circ \varphi$ and $g_i = y_i \circ \psi$. So if $\varphi^* = \psi^*$, then

$$\overline{f_i} = \overline{y_i \circ \varphi} = \varphi^*(\overline{y_i}) = \overline{y_i \circ \psi} = \overline{g_i}.$$

Hence f_i and g_i agree up to an element of $I(X)$ for all i, so $\varphi = \psi$. ■

Proposition 2.2.20. Let $\varphi : X \to Y$ be a polynomial map. Then φ is an isomorphism if and only if φ^* is an isomorphism of **K**-algebras, in which case $(\varphi^*)^{-1} = (\varphi^{-1})^*$.

Proof: Suppose that φ has a polynomial inverse $\varphi^{-1} : Y \to X$. Then $\varphi \circ \varphi^{-1} = \mathrm{id}_Y$ and $\varphi^{-1} \circ \varphi = \mathrm{id}_X$, so $(\varphi^{-1})^* \circ \varphi^* = (\varphi \circ \varphi^{-1})^* = (\mathrm{id}_Y)^* = \mathrm{id}_{\Gamma(Y)}$. Similarly, $\varphi^* \circ (\varphi^{-1})^* = \mathrm{id}_{\Gamma(X)}$, so φ^* is isomorphic with inverse $(\varphi^{-1})^*$. Note that $(\varphi^{-1})^*$ is a **K**-algebra homomorphism, since it is the pullback of a polynomial map.

Conversely, suppose that φ^* is an isomorphism of **K**-algebras with inverse Ψ. Then by the next proposition, $\Phi = \varphi^*$ for some unique polynomial map $\psi : Y \to X$. To see that $\psi = \varphi^{-1}$, note that $(\psi \circ \varphi)^* = \varphi^* \circ \psi^* = \varphi^* \circ (\varphi^*)^{-1} = \mathrm{id}_{\Gamma(Y)} = (\mathrm{id}_Y)^*$. Thus $\psi \circ \varphi = \mathrm{id}_Y$, and similarly, $\varphi \circ \psi = \mathrm{id}_X$. ■

Proposition 2.2.21. [Fullness]
If $\Phi : \Gamma(X) \to \Gamma(Y)$ is a **K**-algebra homomorphism, then there exists a unique polynomial map $\varphi : X \to Y$ with $\varphi^* = \Phi$.

Proof: Let $\Phi : \Gamma(Y) \to \Gamma(X)$ be a $\mathbf{K}$-algebra homomorphism. Here $X \subset A^n$ and $Y \subset \mathbf{A}^m$. Suppose that the ambient coordinates in $\mathbf{A}^n$ are $x_1, \dots, x_n$ and in $\mathbf{A}^m$ are $y_1, \dots, y_m$. Assume that there exists a polynomial map $\varphi : X \to Y$ such that $\varphi^* = \Phi$. Then $\varphi = (f_1, \dots, f_m)$ with $f_i \in \mathbf{K}[x_1, \dots, x_n]$ and

$$\underbrace{\varphi^*(\overline{y_j})}_{= \overline{y_j \circ f} \,=\, \overline{f_j}} = \Psi(\overline{y_j}) \quad \text{iff} \quad \overline{f_j} = \Phi(\overline{y_j}).$$

So for all $j = 1, \dots, m$, pick a representative f_j of the residue class $\Phi(\overline{y_j})$, and set $\varphi = (f_1, \dots, f_m)$. Then certainly $\varphi : \mathbf{A}^n \to \mathbf{A}^m$ is a polynomial. But we still need to check that **(i.)** $\varphi(X) \subset \varphi(Y)$ so that we get $\varphi : X \to Y$, and **(ii.)** $\varphi^* = \Phi$.

(i.) It is enough to check that $I(Y) \subset I(\varphi(X))$ because then $\varphi(X) \subset V(I(\varphi(X)) \subset V(I(Y)) = Y$, as Y is algebraic. Next. let $g \in I(Y)$. Then $\overline{g} = 0$ in $\Gamma(Y)$ and $\Phi(\overline{g}) = 0$. To show that $g \in I(\varphi(X))$, we need to verify that

$$\begin{aligned} g(\varphi(p)) = 0 \ \forall\, p \in X \ &\text{iff} \ (g \circ \varphi)(p) = 0 \ \forall\, p \in X \\ &\text{iff} \ g \circ \varphi \in I(X) \\ &\text{iff} \ \overline{g \circ \varphi} = 0 \in \Gamma(X). \end{aligned}$$

But we see that

$$\begin{aligned} \overline{g \circ \varphi} = \overline{g(f_1, \dots, f_m)} & \\ &= g(\overline{f_1}, \dots, \overline{f_m}) \\ &= g(\Phi(\overline{y_1}), \dots, \Phi(\overline{y_m})) && \text{for } \overline{g} = \sum_I a_i \overline{y_{i_1}} \cdots \overline{y_{i_d}} \\ &= \Phi(g(\overline{y_1}, \dots, \overline{y_m})) && \text{since } \Phi \text{ is a } \mathbf{K}\text{-algebra hom.} \\ &= \Phi(\overline{g}) \\ &= 0 \end{aligned}$$

in $\Gamma(X)$. Hence $g \in I(\varphi(X))$, so $\varphi(X) \subset Y$.

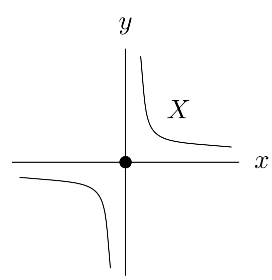
(ii.) Since $\mathbf{K}$-algebra homomorphisms are completely determined by their image on the generators of the $\mathbf{K}$-algebra, and by construction, $\varphi^*(\overline{y_j}) = \Phi(\overline{y_j})$, we haze $\varphi^* = \Phi$. Finally, the choice of f_js was unique up to elements of $I(X)$, implying that φ is the unique polynomial such that $\varphi^* = \Phi$. ■

Corollary 2.2.22. For X, Y varieties, $X \cong Y$ iff $\Gamma(X) \cong \Gamma(Y)$.

Proof: If there exists an isomorphism $\varphi : X \to Y$, then $\varphi^* : \Gamma(X) \to \Gamma(Y)$ is an isomorphism. Conversely, if there exists a $\mathbf{K}$-algebra homomorphism $\Phi : \Gamma(Y) \to \Gamma(X)$, then $\Phi = \varphi^*$ for some isomorphism $\varphi : X \to Y$. ■

Example 2.2.23. Consider $X = V(xy - 1) \subset \mathbf{A}^2$. Is $X \cong \mathbf{A}^1$? We have already seen that

$$\begin{aligned} \Gamma(X) &= \mathbf{K}[x, y] / \langle xy - 1 \rangle \\ &= \mathbf{K}[\overline{x}, \overline{y}] \text{ with } \overline{xy} = 1 \\ &= \mathbf{K}[\overline{x}, \overline{x}^{-1}] \\ &= \text{(ring of Laurent polynomials).} \end{aligned}$$



And we also know that $\Gamma(\mathbf{A}^1) = \mathbf{K}[t]$. By the theorem, we know that $X \cong \mathbf{A}^1$ iff $\mathbf{K}[\overline{x}, \overline{x}^{-1}] \cong \mathbf{K}[t]$. So assume that $\mathbf{K}[\overline{x}, \overline{x}^{-1}] \cong \mathbf{K}[t]$, so there exists a $\mathbf{K}$-algebra homomorphism $\Phi : \mathbf{K}[\overline{x}, \overline{x}^{-1}] \to \mathbf{K}[t]$. In particular, Φ is a surjective ring homomorphism, implying that $\Phi(1) = 1$. Then $\Phi(\overline{x}) \cdot \Phi(\overline{x}^{-1}) = \Phi(\overline{x} \cdot \overline{x}^{-1}) = \Phi(1) = 1$. Hence $\Phi(\overline{x})$ and $\Phi(\overline{x}^{-1})$ are units in $\mathbf{K}[t]$. Therefore $\Phi(\overline{x}), \Phi(\overline{x}^{-1}) \in \mathbf{K}$, so $\Phi(\mathbf{K}[\overline{x}, \overline{x}^{-1}]) \in \mathbf{K}$, contradicting surjectivity. Hence $\mathbf{K}[\overline{x}, \overline{x}^{-1}] \cong \mathbf{K}[t]$, so $X \cong \mathbf{A}^1$.

Definition 2.2.24. A $\mathbf{K}$-algebra A is *finitely generated* if there exist $a_1, \ldots, a_n \in A$ such that $A = \mathbf{K}[a_1, \ldots, a_n]$. Equivalently, there exists a surjective $\mathbf{K}$-algebra homomorphism $\varphi : \mathbf{K}[x_1, \ldots, x_n] \to A$ for some $n \in \mathbf{N}$ (so that if $a_i = \varphi(x_i)$, then $A = \mathbf{K}[a_1, \ldots, a_n]$).

Example 2.2.25. Consider the following examples of $\mathbf{K}$-algebras:
· $\mathbf{K}[x_1, \ldots, x_n]$ is a finitely-generated $\mathbf{K}$-algebra.
· Any quotient of a finitely-generated $\mathbf{K}$-algebra is finitely-generated, because if $A = \mathbf{K}[a_1, \ldots, a_n]$ with $a_i \in A$ and $I \lhd A$, then $A/I = \mathbf{K}[\overline{a_1}, \ldots, \overline{a_n}]$ with $\overline{a_i} \in A/I$. So $\Gamma(X)$ is a finitely-generated $\mathbf{K}$-algebra for all varieties X.

Proposition 2.2.26. Suppose that $\mathbf{K} = \overline{\mathbf{K}}¿$ and A is a finitely-generated $\mathbf{A}$-algebra that is an integral domain. Then there exists a variety X such that $A \cong \Gamma(X)$ as $\mathbf{K}$-algebras.

Proof: Since A is finitely-generated, there exists a surjective $\mathbf{K}$-algebra homomorphism $\varphi : \mathbf{K}[x_1, \ldots, x_n] \to A$. Set $I = \ker(\varphi)$. Then $A \cong \mathbf{K}[x_1, \ldots, x_n]/I$, so set $X = V(I)$. But $I(X) = I(V(I)) = \sqrt{I} = I$, by the Nullstellensatz and as I is prime and A is an integral domain. ■

Remark 2.2.27. This gives us a nice correspondence between objects:

Geometric	Algebraic
affine variety X	finitely-generated $\mathbf{K}$-algebra and integral domain $\Gamma(X)$
algebraic set X	radical ideal $I(X)$
algebraic subset of X	radical ideal in $\Gamma(X)$
subvariety of X	prime ideal in $\Gamma(X)$
point in X	maximal ideal in $\Gamma(X)$
polynomial maps $\varphi : X \to Y$	$\mathbf{K}$-algebra homomorpisms $\varphi^* : \Gamma(Y) \to \Gamma(X)$

2.3 Rational functions and local rings

Let $X \subset \mathbf{A}^n$ be a variety. Then $\Gamma(X)$ is an integral domain, and we may consider its quotient field, i.e. field of fractions.

Definition 2.3.1. Given a variety $X \subset \mathbf{A}^n$, the quotient field of $\Gamma(X)$ is called the *field of rational functions* on X, or the *function field* of X, and is denoted by $\mathbf{K}(X)$.

Example 2.3.2. Unlike polynomial functions, rational functions may not be defined at every point in X.
· Let $X = \mathbf{A}^n$. Then $\mathbf{K}(X) = \mathbf{K}(x)$ and $1/x$ is not defined at $x = 0$.
· Let $X = V(y - x^2) \subset \mathbf{A}^2$. Then $\Gamma(X) = \mathbf{K}[\overline{x}, \overline{y}] = \mathbf{K}[\overline{x}]$ for $\overline{y} = \overline{x}^2$, so $\mathbf{K}(X) = \mathbf{K}(\overline{x})$, and $1/\overline{x} \in \mathbf{K}(X)$ is not defined when $\overline{x} = 0 \iff (x, y) = (0, 0) \in X$.

Definition 2.3.3. A rational function f on X is said to be *defined*, or *regular* at $p \in X$ if it may be written as $f = \frac{\overline{a}}{\overline{b}}$ for some $\overline{a}, \overline{b} \in \Gamma(X)$, and $b(p) \neq 0$. In this case, we say that $a(p)/b(p) \in \mathbf{K}$ is the value of f at p, and denote it by $f(p)$. Moreover, the set of points where f is not defined is called the *pole set* of f. Points where f is not defined are called *poles*.

Remark 2.3.4. Suppose that $f = \overline{a}/\overline{b} = \overline{a}'/\overline{b}'$ is $\mathbf{K}(X)$. This means that

$$\begin{aligned}\overline{a}\overline{b}' = \overline{a}'\overline{b} \text{ in } \Gamma(X) &\text{ iff } \overline{a}\overline{b}' - \overline{a}'\overline{b} = 0 \text{ in } \Gamma(X)\\ &\text{ iff } ab' - a'b = 0 \text{ in } X.\end{aligned}$$

So if $p \in X$ is such that $b(p) = b'(p) \neq 0$, then $a(p)/b(p) = a'(p)/b'(p)$. That is, the value of f at p is well-defined, i.e. does not depend on the choice of $\overline{a}, \overline{b} \in \Gamma(X)$, with $f = \overline{a}/\overline{b}$ and $b(p) \neq 0$.

Example 2.3.5. Consider the following examples in function fields.

· Let $X = \mathbf{A}^1$ and $f = 1/x \in \mathbf{K}(X)$. Then f is defined everywhere except at $x = 0$. However, $f(x) = x^2/x$ is defined everywhere on X.

· Let $X = V(x^2 + y^2 - 1) \subset \mathbf{A}^2$. Then $I(X) = \langle x^2 + y^2 - 1 \rangle$, so $\Gamma(X) = \mathbf{K}[\overline{x}, \overline{y}]$, with $\overline{x}^2 = 1 - \overline{y}^2$. Take $f = \overline{y}^3/(1 - \overline{x}^2) \in \mathbf{K}(X)$. The potential poles of f are points where $1 - x^2 = 0$, or $x = \pm 1$ on X, or $(x, y) = (\pm 1, 0)$ on X. However,

$$f = \frac{\overline{y}^2}{1 - \overline{x}^2} = \frac{\overline{y}^2 \cdot \overline{y}}{1 - \overline{x}^2} = \overline{y},$$

and since $\overline{y}$ is defined ot $(\pm 1, 0)$, we have that f is defined at $(\pm 1, 0)$, and so f is defined everywhere. Now, take $f = (1 - \overline{y})/\overline{x} \in \mathbf{K}(X)$. Then potential poles occur where $\overline{x} = 0$, or $x = 0$ on X, or $(x, y) = (0, \pm 1)$. Let us check if these points are indeed poles. We assume that $\mathrm{char}(\mathbf{K}) \neq 2$, and check first at $(0, 1)$. Observe that

$$f = \frac{1 - \overline{y}}{\overline{x}} = \frac{(1 - \overline{y})(1 + \overline{y})}{\overline{x}(1 + \overline{y})} = \frac{1 - \overline{y}^2}{\overline{x}(1 + \overline{y})} = \frac{\overline{x}}{1 + \overline{y}},$$

and since $\overline{x}/(1 + \overline{y})$ is defined at $(0, 1)$, so is f and $f(0, 1) = 0/(1 + 1) = 0$, so this is not a pole. Let us now check for the point $(0, -1)$. Suppose that this is not a pole, so there exist $\overline{a}, \overline{b} \in \Gamma(X)$ such that $f = \overline{a}/\overline{b}$, and $b(0, -1) \neq 0$. Then

$$\frac{1 - \overline{y}}{\overline{x}} = \frac{\overline{a}}{\overline{b}} \text{ in } \mathbf{K}(X) \quad \Longleftrightarrow \quad (1 - y)b = ax \text{ on } X.$$

Hence at $(0, -1)$, we have that

$$(1 - (-1))b(0, -1) = a(0, -1) \cdot 0 \quad \Longleftrightarrow \quad 2b(0, -1) = 0,$$

which is a contradiction, since $\mathrm{char}(\mathbf{K}) \neq 2$ and $b(0, -1) \neq 0$. Hence f is not defined at $(0, -1)$, and $(0, -1)$ is a pole of f.

Proposition 2.3.6. The pole set of a rational function on X is an algebraic subset of X.

Proof: Let $f \in \mathbf{K}(X)$. If $\overline{a}/\overline{b}$ is any representation of f (i.e. $f = \overline{a}/\overline{b}$ and $\overline{a}, \overline{b} \in \Gamma(X)$), then $V(b)$ is the pole set of a/b. Further, the pole set of V is given by $\bigcap_{f = \overline{a}/\overline{b}} V(b)$, which is algebraic. ■

Remark 2.3.7. Note the following facts.

· The set of all points where $f \in \mathbf{K}(X)$ is defined is called the *domain* of f, which we denote by D_f. Note that D_f is an open subset of X since $D_f = X \setminus (\text{pole set of } f)$, and the pole set of f is closed. Therefore if D_f is closed, then $D_f = X$.

· Rational functions are continuous with respect to the Zariski topology.

· If $f \in \mathbf{K}(X)$ is such that $f = 0$ on an open subset $U \subset X$, thon $f = 0$ on X. This implies the identity theorem.

Theorem 2.3.8. [Identity theorem]
If $f, g \in \mathbf{K}(X)$ are such that $f = g$ on some open subset $U \subset X$, then $f = g$ on X.

Proof: Suppose that $f = g$ on $U \subset X$ open. Then $h = f - g = 0$ on $U \subset X$ open, so $h = 0$ on X, meaning that $f = g$ on X. Tho enly thing left to prove is that if $f = 0$ on U, then $f = 0$ on X. So let $p \in U$, and since $f = 0$ on U, the rational function f must be defined at p. So there exist $\overline{a}, \overline{b} \in \Gamma(X)$ such that

$f = \overline{a}/\overline{b}$ and $b(p) \neq 0$. Let $V = X \setminus V(b)$. Then $b \neq 0$ on V, implying that the quotint $\overline{a}/\overline{b}$ makes sense on V. Moreover, $f = \overline{a}/\overline{b}$ on $U \cap V \subset U$. But $f = 0$ on $U \cap V$, so $\overline{a}/\overline{0}$ on $U \cap V$, meaning that $\overline{a} = 0$ on $U \cap V$. Therefore $a = 0$ (since $b \neq 0$ on $U \cap V$), so $U \cap V \subset V(a)$. Hence $X = \overline{U \cap V} \subset \overline{V(a)} = V(a) \subset X$, as $V(a)$ is algebraic. Hence $f = 0$ on X. ■

Remark 2.3.9. Some authors define rational functions formally as equivalence classes of pairs (U, f), where f is a rational function defined on U, with $U \subset X$ open. The equivalence relation is given by

$$(f, U) \sim (g, V) \iff (\text{there exists } W \subset U \cap V \text{ open with } f|_W = g|_W).$$

In this case, we call (f, U) a *germ* of rational functions.

Definition 2.3.10. Let $X \subset \mathbf{A}^n$ and $Y \subset \mathbf{A}^m$ be two varieties. A map $\varphi : X \to Y$ such that $\varphi(x) = (f_1(x), \ldots, f_n(x)) \in Y$ for all $x \in X$ whenever the f_is are defined is called a *rational map*. We say that φ is *defined* at $x \in X$ if each f_i is defined at x and $\varphi(x) \in Y$. Moreover, the *domain* of φ is the set of all points where φ is defined.

Example 2.3.11.

2.4 A proof of the Nullstellensatz

Theorem 2.4.1. If $\mathbf{K} = \overline{\mathbf{K}}$ and $I \lhd \mathbf{K}[x_1, \ldots, x_n]$, then $I(V(I)) = \sqrt{I}$.

We will need the following fact: let $\overline{\mathbf{K}} = \mathbf{K}$ and let $K = \mathbf{K}[a_1, \ldots, a_r]$ be a finitely-generated $\mathbf{K}$-algebra. Note that there may be relations among the generators $a_1, \ldots, a_r$. If K is a field, the $K = \mathbf{K}$.

Theorem 2.4.2. [Weak Nullstellensatz]
Let $\mathbf{K} = \overline{\mathbf{K}}$. Then every maximal ideal in $R = \mathbf{K}[x_1, \ldots, x_n]$ is of the form $\langle x_1 - a_1, \ldots, x_n - a_n \rangle$ for $a_i \in \mathbf{K}$

3 Dimension

Corollary 3.0.3. If $Y \subset X \subset \mathbf{A}^m$ has codimension r in X, then there exist subvarieties $Y_0, \ldots, Y_r$ of X of codimension $0, \ldots, r$, respectively, such that $Y = Y_r \subsetneq \cdots \subsetneq Y_0 = X$, with $\dim(Y_i) = \dim(X) - i$.

Proof: This will be done by induction on r. For $r = 1$, let $Y_1 = Y$ and $Y_0 = X$. For $r > 1$, suppose that is is true for all r up to $r-1$. Then $\dim(Y) = \dim(X) - r$. Since $Y \subsetneq X$, $I(X) \subsetneq I(Y)$, meaning that there exists $f \in I(Y)$ (which we assume to be irreducible, since $I(Y)$ is prime) such that $f \notin I(X)$. Hence $f \neq 0$ on X, so $V(f) \cap X \neq X$. So every irreducible component of $V(f) \cap X$ has codimension 1 in X. Since $Y \subset V(f) \cap X$, we may pick Y_1 to be the irreducible component of $V(f) \cap X$ containing Y. Set $Y = Y_r \subsetneq Y_1 \subsetneq Y_0 = X$, so now Y has codimension $r-1$ in Y_1. Then induction gives the rest of the sets Y_i. ■

3.1 Multiple points and tangent lines

3.2 Intersection multiplicity

Proposition 3.2.1. [Properties of intersection multiplicity]
Let $C : f = 0$ be smooth and $D : g = 0$. Then:

1. $I(p, C \cap D)$ is invariant under affine coordinate changes

2. $I(p, C \cap D) = \infty$ iff C and D have a common component passing through p

3. If C, D intersect properly, than $I(p, C \cap D) < \infty$, and $I(p, C \cap D) = 0$ iff $p \notin C \cap D$

4. $I(p, C \cap D) = 1$ iff C, D intersect transversally at p. Otherwise, $I(p, C \cap D) \leqslant m_p(C) m_p(D)$, with equality holding iff C, D have no common tangent directions at p

5. [Additivity] If $g = g_1 g_2$, then $I(p, C \cap D) = P(p, C \cap V(g_1)) + I(p, C \cap V(g_2))$

6. If $E =: h = 0$ with $\overline{h} = \overline{g}$ in $\Gamma(C)$, then $(p, C \cap D) = I(p, C \cap E)$

7. [Symmetry] If C, D are smooth at p, then $I(p, C \cap D) = I(p, D \cap C)$ (i.e. $\mathrm{ord}_p^C(\overline{g}) = \mathrm{ord}_p^D(\overline{f})$)

Proof:

4 Projective varieties

4.1 Projective space and algebraic sets

Definition 4.1.1. Let $\mathbf{K}$ be any field. Consider $\mathbf{A}^{n+1}(\mathbf{K})$. The set of all lines through tho erigin $0 = (0, \ldots, 0)$ is called the n-dimensional *projective space*, and is denoted $\mathbf{P}^n(\mathbf{K})$, or just $\mathbf{P}^n$, if $\mathbf{K}$ is understood. Then

$$\mathbf{P}^n = (\mathbf{A}^{n+1} - 0)/\mathbf{K}^*,$$

where $(x_1, \ldots, x_{n+1}) \sim (\lambda x_1, \ldots, \lambda x_{n+1})$ for all $\lambda \in \mathbf{K}^*$. The equivalence class $\{(\lambda x_1, \ldots, \lambda x_{n+1}) \ : \ \lambda \in \mathbf{K}^*\}$ is the set of all points on the line L joining 0 and $(x_1, \ldots, x_{n+1})$.

If p is a point in $\mathbf{P}^n$, then any $(n-1)$-tuple $(a_1, \ldots, a_{n+1})$ in the equivalence class of p is called a set of *homogeneous coordinates* for p. Equivalence classes are denoted $p = [a_1 : \cdots : a_{n+1}]$ to distinguish them from the affine coordinates. Note that $[a_1 : \cdots : a_{n+1}] = [\lambda a_1 : \cdots : \lambda a_{n+1}]$ for all $\lambda \in \mathbf{K}^*$.

Remark 4.1.2. Projective n-space can be covered with $n+1$ copies of affine n-space. For all i, let $U_i = \{[x : \cdots : x_{n+1}] \ : \ x_i \neq 0\}$. Then for any $[x_1 : \cdots : x_{n+1}] \in U$, we have $[x_1 : \cdots : x_{n+1}] = [\frac{1}{x_i} x_1 : \cdots : 1 : \cdots : \frac{1}{x_i} x_{n+1}]$. Thus

$$[x_1 : \cdots : x_{n+1}] \overset{1:1}{\longleftrightarrow} \left(u_1 = \frac{x_1}{x_i}, \ldots, \widehat{u_i}, \ldots, u_{n+1} = \frac{x_{n+1}}{x_i} \right).$$

Hence $U_i \cong \mathbf{A}^n$. For example, we may cover $\mathbf{P}^2 = (\mathbf{A}^3 - 0)/\mathbf{K}^*$, given by $[x : y : z]$ in homogeneous coordinates, by

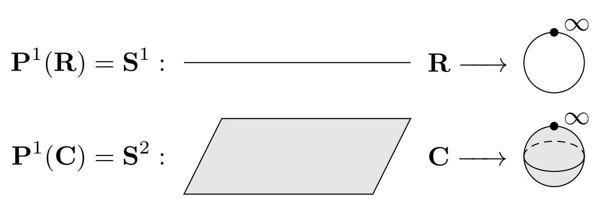
$$U_x = \{x \neq 0\} = \{[1 : u : v] \ : \ u, v \in \mathbf{K}\} \quad , \quad U_y = \{[\tfrac{x}{y} : 1 : \tfrac{z}{y}]\} \quad , \quad U_z = \{[\tfrac{x}{z} : \tfrac{y}{z} : 1]\}.$$

Conversely, affine n-space may be considered as a subspace of $\mathbf{P}^n$, through the injection $\mathbf{A}^n \hookrightarrow \mathbf{P}^n$. Hence for all i, $H_i = \mathbf{P}^n - U_i = \{x_i = 0\} = \{[x_1 : \cdots : 0 : \cdots : x_{n+1}]\}$ is called a *hyperplane*, which can be identified with $\mathbf{P}^{n-1}$ by the correspondence

$$H_i \ni [x_1 : \cdots : 0 : \cdots : x_{n+1}] \leftrightarrow [x_1 : \cdots : \widehat{x_i} : \cdots : x_{n+1}] \in \mathbf{P}^{n-1}.$$

Note that we cannot have $x_1 = \cdots = x_{n+1} = 0$, otherwise the original point is not defined. In particular, $H_\infty = H_{n+1}$ is called the *hyperplane at infinity*, with $\mathbf{P}^n = U_{n+1} \cup H_\infty = \mathbf{A}^n \cup \mathbf{P}^{n+1}$.

Example 4.1.3. Consider the following examples of projective space.
· $\mathbf{P}^0(\mathbf{K}) = \{pt\}$.
· $\mathbf{P}^1(\mathbf{K}) = \mathbf{A}^1 \cup \mathbf{P}^1 = \mathbf{A}^1 \cup \{pt\}$. For example,



· $\mathbf{P}^2(\mathbf{K}) = \mathbf{A}^2 \cup \ell_\infty = H_\infty = \{[x : y : 1]\} \cup \{[x : y : 0]\}$.

Definition 4.1.4. Let $f \in \mathbf{K}[x_1, \ldots, x_{n+1}]$. Then $p = [a_1 : \cdots : a_{n+1}] \in \mathbf{P}^n$ is a zero of f if and only if $f(\lambda a_1, \ldots, \lambda a_{n+1}) = 0$ for all $\lambda \in \mathbf{K}^*$, in which case we write $f(p) = 0$.

Let $S \subset \mathbf{K}[x_1, \ldots, x_{n+1}]$. Then $V_p(S) = \{p \in \mathbf{P}^n \ : \ f(p) = 0 \text{ for all } p \in S\}$, called the *zero set* of S in $\mathbf{P}^n$. Moreover, if $Y \subset \mathbf{P}^n$ is such that $Y = V_p(S)$ for some $S \subset \mathbf{K}[x_1, \ldots, x_n]$, then Y is called a *projective algebraic set*.

Finally, for $Y \subset \mathbf{P}^n$, define $I_p(Y) = \{f \in \mathbf{K}[x_1, \ldots, x_{n+1}] \ : \ f(p) = 0 \text{ for all } p \in Y\}$ to be the *projective ideal* of Y.

Lemma 4.1.5. Let $f \in \mathbf{K}[x_1, \dots, x_{n+1}]$ and write $f = f_m + \cdots + f_d$, where f_i is an i-form for all i. Then if $p \in \mathbf{P}^n$, we have $f(p) = 0$ iff $f_i(p) = 0$ for all i.

Proof: Suppose that $p = [a_1 : \cdots : a_{n+1}]$. Then

$$\begin{aligned}
f(p) = 0 \quad &\iff \quad f(\lambda a_1, \dots, \lambda a_{n+1}) = 0 \ \forall \ \lambda \in \mathbf{K}^* \\
&\iff \quad f_m(\lambda a_1, \dots, \lambda a_{n+1}) + \cdots + f_d(\lambda a_1, \dots, \lambda a_{n+1}) = 0 \\
&\iff \quad \lambda^m f_m(a_1, \dots, a_{n+1}) + \cdots + \lambda^d f_d(a_1, \dots, a_{n+1}) = 0 \\
&\iff \quad f_m(a_1, \dots, a_{n+1}) = \cdots = f_d(a_1, \dots, a_{n+1}) = 0 \\
&\iff \quad f_m(\lambda a_1, \dots, \lambda a_{n+1}) = \cdots = f_d(\lambda a_1, \dots, \lambda a_{n+1}) = 0 \\
&\iff \quad f_i(p) = 0 \ \forall \ i.
\end{aligned}$$

■

Thus, if $f = f_m + \cdots + f_d$ with f_i an i-form, then $V_p(f) = V_p(f_m, \dots, f_d)$. Also, if $f \in I_p(Y)$ for some $Y \subset \mathbf{P}^n$, then $f_i \in I_p(Y)$ for all i. Therefore we have the following:

Proposition 4.1.6.
i. Every algebraic set in $\mathbf{P}^n$ is the zero set of a finite set of forms.
ii. If $Y \subset \mathbf{P}^n$, then $I_p(Y)$ is generated by forms.

Definition 4.1.7. An ideal $I \lhd K[x_1, \dots, x_{n+1}]$ is called *homogeneous* if $f \in I$ and $f = f_m + \cdots + f_d$, with f_i an i-form, then $f_i \in I$ for all i. Note that $I_p(Y)$ is homogeneous for all $Y \subset \mathbf{P}^n$.

Remark 4.1.8. The proof of the above lemma, for **i.** in the affine case, follows as $Y \subset \mathbf{P}^n$ implies $I_p(Y)$ is radical. Moreover, $I_p(Y)$ is homogeneous. We thus have a correspondence:

$$\begin{array}{ccc}
\mathbf{P}^n & & \mathbf{K}[x_1, \dots, x_n] \\
(\text{algebraic set } Y) & \longleftrightarrow & \left(\begin{array}{c} \text{homogeneous} \\ \text{radical ideal} \end{array} \right)
\end{array}$$

However, we will see that this correspondence is not 1 : 1, since there is more than one homogeneous radical idal corresponding to the empty set $\emptyset$. For example, since $V_a(\langle x_1, \dots, x_{n+1} \rangle) = (0, \dots, 0)$, we have that

$$\emptyset = V_p(a) = V_p(\langle x_1, \dots, x_{n+1} \rangle).$$

Proposition 4.1.9. Let $I, J \lhd \mathbf{K}[x_1, \dots, x_n]$. Then
i. I is homogeneous iff I can be generated by forms,
ii. if I, J are homogeneous, then $I + J$, IJ, $I \cap J$, $\sqrt{I}$ are homogeneous, and
iii. I is a prime homogeneous ideal iff for forms $f, g \in \mathbf{K}[x_1, \dots, x_n]$ with $fg \in I$, it follows that $f \in I$ and $g \in I$.

Proof: **iii.** The direction $\Rightarrow$ is clear, so let us prove the $\Leftarrow$ direction. Suppose that I is homogeneous oand satisfies the described property. Let us show that I is prime. Let $f, g \in \mathbf{K}[x_1, \dots, x_{n+1}]$ and suppose that $fg \in I$. Write $f = f_m + \cdots + f_d$ and $g = g_{m'} + \cdots + g_{d'}$, where f_i, g_i are i-forms. Then

$$fg = f_m g_{m'} + \sum_{k > m+m'}^{d+d'} \sum_{i+j=k} f_i g_j,$$

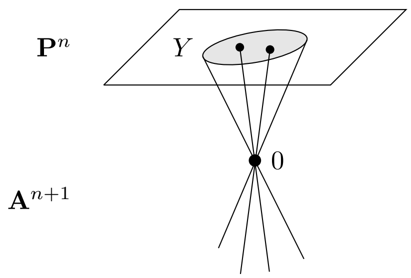
and $f_m g_{m'} \in I$ since I is homogeneous. If $f_m \notin I$, then $g_{m'} \in I$ by the condition. So $g - g_{m'} = g_{m'+1} + \cdots + g_{d'} \in I$, and $f(g - g_{m'}) \in I$. Repeating the process,

$$g(g - g_{m'}) = f_m g_{m'+1} + \sum_{k > m+m'+1}^{d+d'} \sum_{k=i-j} f_i g_j,$$

so $f_m g_{m'+1} \in I$ with $f_m \notin I$, so $g_{m'+1} \in I$ by the condition. Repeating several times this process, we get that $g_i \in I$ for all i, so $g \in I$. Note that if $g_{m'} \notin I$, then $f \in I$. And if $f_m, g_{m'} \notin I$, then repeat the process with $(f - f_m)(g - g_{m'})$. ■

Example 4.1.10. Consider the following examples.
· $I = \langle x^2 \rangle$ and $I = \langle x^2, y \rangle$ in $\mathbf{K}[x, y]$ are homogeneous ideals.
· $I = \langle x^2 + x \rangle$ is not homogeneous since $x^2 + x$ is not a form.

Definition 4.1.11. Let $\theta : \mathbf{A}^{n+1} \setminus \{0\} \to \mathbf{P}^n$ be the standard projection $(x_1, \ldots, x_{n+1} \mapsto [x_1 : \cdots : x_{n+1}]$. If $Y \subset \mathbf{P}^n$, the *affine cone* over Y is $C(Y) = \theta^{-1}(Y) \cup \{0\}$, and looks as in the diagram below.



For example, if $P = \{p\}$ for some $p \in \mathbf{P}^n$, then $C(\{p\})$ is the line in $\mathbf{A}^{n+1}$ defined by p. So for all $Y \subset \mathbf{P}^n$, $C(Y)$ is the union of all lines in $\mathbf{A}^{n+1}$ befined by the points in Y.

Remark 4.1.12. These are some properties of the affine cone:
· $C(\emptyset) = \{0\}$
· $C(Y_1 \cup Y_2) = C(Y_1) \cup C(Y_2)$
· $C(Y_1) = C(Y_2)$ iff $Y_1 = Y_2$
· if $\emptyset \neq Y \subset \mathbf{P}^n$, then $I_p(Y) = I_a(C(Y))$
· if $I \lhd \mathbf{K}[x_1, \ldots, x_{n+1}]$ is a homogeneous ideal such that $V_p(I) \neq \emptyset$, then $C(V_p(I)) = V_a(I)$. In particular, $C(Y) = V_a(I)$ for some non-empty $Y \subset \mathbf{P}^n$ iff $Y = V_p(I)$.

Example 4.1.13. Consider the following examples.
· $\mathbf{P}^n = V_p(0)$
· Let $p = [a : b] \in \mathbf{P}^1$. Then $C(\{p\})$ is the line in $\mathbf{A}^2$ through 0 and (a, b), or $V_a(bx - ay)$. Hence $\{p\} = V_p(bx - ay)$, so points are projective algebraic sets. In general, if $p = [a_1 : \cdots : a_{n+1}] \in \mathbf{P}^n$ with $a_i \neq 0$ for some i, then $\{p\} = V_p(a_i x_1 - a_1 x_i, \ldots, aix_{n+1} - a_{n+1} x_i)$, so points in $\mathbf{P}^n$ are projective algebraic sets.
· Let $Y = V_p(x - y, x^2 - yz) \subset \mathbf{P}^2$. Then

$$C(Y) = V_a(x - y, x^2 - yz) = v_a(x, y) \cup V_a(x - y, x - z) = \{(0, 0, t) \ : \ t \in \mathbf{K}\} \cup \{(s, s, s) \ : \ s \in \mathbf{K}\},$$

hence $Y = \{[0 : 0 : 1]\} \cup \{[1 : 1 : 1]\}$.

Example 4.1.14. Consider the following examples of projective ideals:
· $I_p(\mathbf{P}^n) = \langle 0 \rangle$, since $I_p(\mathbf{P}^n) = I_a(C(P^n)) = I_a(\mathbf{A}^{n+1}) = \langle 0 \rangle$.
· $I_p(\emptyset) = \langle 1 \rangle$
· for $p = [a_1 : \cdots : a_{n+1}]$ with $a_i \neq 0$ for some i, then

$$I_p(\{p\}) = \langle a_i x_1 - a_1 x_i, \ldots, a_i x_{n+1} - a_{n+1} x_i \rangle .$$

Proposition 4.1.15. Let $\{U_i\}_{i \in I}$ be a family of projective algebraic sets. Then $U_i \cup U_j$ is projective algebraic for any $i, j \in I$, and $\bigcap_{i \in I} U_i$ is projective algebraic. Moreover, $\emptyset$ and $\mathbf{P}^n$ are projective algebraic.

Proposition 4.1.16. [Projective Nullstellensatz]
Let $\mathbf{K} = \overline{\mathbf{K}}$ and $I \lhd \mathbf{K}[x_1, \ldots, x_{n+1}]$. Then

1. $V_p(I) = \emptyset$ iff there exists $N \in \mathbf{N}$ such that I contains all forms of degree $\geqslant N$, and

2. $V_p(I) \neq \emptyset$ implies $I_p(V_p(I)) = \sqrt{I}$.

Proof: For **1.** we have that

$$\begin{aligned} V_p(I) = \emptyset \quad &\iff \quad V_a(I) = \emptyset \text{ or } \{(0, \ldots, 0\} \\ &\iff \quad V_a(I) \subset \{(0, \ldots, 0)\} \\ &\iff \quad I_a(\{(0, \ldots, 0)\}) \subset I_a(V_a(I)). \end{aligned}$$

However, $\langle x_1, \ldots, x_{n+1} \rangle = I_a(\{(0, \ldots, 0)\})$ and $I_a(V_a(I)) = \sqrt{I}$, so $V_p(I) = \emptyset$ iff $x_i^{m_i} \in I$ for all i, so $x_i^m \in I$ for all i, for $m = \max_i\{m_i\}$. Then $V_p(I) = \emptyset$ iff $\langle x_1, \ldots, x_{n+1} \rangle^N \subset I$ for some $N \geqslant m$, but that holds iff any form of degree at least N is contained in I.

For **2.** the affine Nullstellensatz gives that $I_p(V_p(I)) = I_a(C(V_p(I)) = I_a(V_a(I)) = \sqrt{I}$. ■

4.2 Rational functions

4.3 Projective plane curves

Proposition 4.3.1. Let C be an irreducible plane curve of degree 2. Then C is smooth.

Proof: Suppose that C is not smooth, so there is some $p \in C$ at which C is singular. Then for $C = V_p(f)$, it would be that $m_p(C) \geqslant 2$. Let $q \in C \setminus \{p\}$ and $L = V_p(h)$ the line through p and q. By Bezout, $C \cap L = \{p, q\}$, and assuming that $L \not\subset C$,

$$2 = \deg(L)\deg(C) = \deg(h)\deg(f) = I(p, L \cap C) + I(q, L \cap C) \geqslant m_p(L)m_p(C) + m_q(L)m_q(C) \geqslant 2 + 1 = 3,$$

which is a contradiction. Hence L is a component of C, so C is reducible, a contradiction. Hence C has no singularities, and is smooth. ■

4.4 Divisors

Definition 4.4.1. Let C be a smooth projective plane curve and $\mathrm{Div}^0(C)$ the subgroup of $\mathrm{Div}(C)$ consisting of all degree 0 divisors on C. If $D \in \mathrm{Div}^0(C)$ is such that $D = \div(f)$ for some $f \in \mathbf{K}(C)$, we say that D is *principal*. If $D, D' \in \mathrm{Div}^0(C)$ are such that $D - D'$ is principal, then D and D' are called *linearly equivalent*, and we write $D \equiv D'$. Finally, let $P(C)$ denote the subgroup of $\mathrm{Div}^0(C)$ consisting of all principal divisors. Let

$$Cl^0(C) = \mathrm{Div}^0(C)/P(C)$$

be the *divisor class group* of degree zero of C.

Index of notation

Index

Index of mathematicians

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