Compact course notes

Pure Math 745, Fall 2013

Representation Theory

Lecturer: D. McKinnon
transcribed by: J. Lazovskis
University of Waterloo
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1 Introduction

1.1 Definitions

Definition 1.1.1. Let G be a group. A *representation* of G in a vector space V is a homomorphism $\rho : G \to GL(V)$, where V is taken to be a finite-dimensional complex vector space ($\mathbb{C}^n$).

$GL(V)$ is the group of invertible linear maps $V \to V$, also viewed as $n \times n$ matrices.

Example 1.1.2. For ρ the trivial homomorphism, ρ is termed the trivial or unit representation.

- Consider $G = D_4$, the group of symetries of a square. There is a representation of G in $\mathbb{C}^2$ corresponding to this description.

- Consider $G = S_n$ and $V = \mathbb{C}^n$. The permutation representation of S_n is the homomorphism that takes a permutation $\sigma \in S_n$ to its corresponding permutation matrix.

Definition 1.1.3. An injective representation is termed *faithful*.

Example 1.1.4. Let G be a finite group acting on a finite set X. There is a permutation representation corresponding to this action on the vector space $V = \bigoplus_{x\in X} \mathbb{C}\vec{v}_x$.

Note that every group G acts on itself by the left multiplication action. If G is finite, the corresponding permutation representation is termed the left-regular representation.

1.2 Subrepresentations and morphisms

Definition 1.2.1. Let $\rho : G \to GL(V)$ be a representation of G. A subspace $W \subset V$ is termed *G-stable*, *G-invariant*, or a *subrepresentation* of V iff one of the following equivalent conditions hold:

1. $\rho(g)(\vec{w}) \in W$ for all $g \in G$ and $\vec{w} \in W$
2. $\rho(g)(W) \subset W$ for all $g \in G$
3. $\rho(g)|_W \in GL(W)$ for all $g \in G$
4. $\rho(G)|_W \subset GL(W)$

Example 1.2.2. Let $G = \mathbb{Z}_2$, $V = \mathbb{C}^2$, and $\rho : G \to GL_2(\mathbb{C})$ be given by $\rho(0) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\rho(1) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Let $W = \text{span}\{(1,1)\}$. Then W is G-invariant. Further, the spaces 0 and V are always G-invariant. We also have that $W' = \text{span}\{(1,-1)\}$ is G-invariant.

Note that to find the G-invariants of W, it suffices to check that $\rho(g_i)(\vec{w}_j) \in W$ for generators g_i of G and basis vectors $\vec{w}_j$ of W.

Definition 1.2.3. Let $\rho : G \to GL(V)$ and $\tau : G \to GL(W)$ be representations of G. A *morphism* from ρ to τ (or from V to W) is a linear transformation $T : V \to W$ such that $\tau(g)(T(\vec{v})) = T(\rho(g)(\vec{v}))$ for all $g \in G$ and $\vec{v} \in V$.

Example 1.2.4. Let $G = \mathbb{Z}_4$. Let $\rho : G \to GL(\mathbb{C})$ and $\tau : G \to GL(\mathbb{C}^2)$ be representations of G, defined by $\rho(n) = i^n$ and $\tau(n) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)^n$. Define a map $T : \mathbb{C} \to \mathbb{C}^2$ by $T(z) = (z, z)$. Then T is a morphism from ρ to τ.

Theorem 1.2.5. Let $\rho : G \to GL(V)$ and $\tau : G \to GL(W)$ be representations of a group G, and let $T : V \to W$ be a morphism from ρ to τ. Let $V_1 \subset V$ and $W_1 \subset W$ be subrepresentations of V and W, respectively. Then $T(V_1) \subset W$ is a subrepresentation of W and $T^{-1}(W_1) \subset V$ is a subrepresentation of V.

Definition 1.2.6. Let $\rho : G \to GL(V)$ be a representation. Then ρ is *irreducible* iff the only subrepresentations of ρ are ρ and the 0-representation.

Example 1.2.7. Every 1-dimensional representation is irreducible, as its only subspaces are 0 or improper.

Definition 1.2.8. The *direct sum* of vector spaces V and W is defined to be the vector space below. Addition and scalar multiplication work as indicated.

$$V \oplus W = \{\vec{v} \oplus \vec{w} \ : \ \vec{v} \in V, \vec{w} \in W\}$$
$$(\vec{v}_1 \oplus \vec{w}_1) + (\vec{v}_2 \oplus \vec{w}_2) = (\vec{v}_1 + \vec{v}_2) \oplus (\vec{w}_1 + \vec{w}_2)$$
$$\lambda(\vec{v} \oplus \vec{w}) = \lambda\vec{v} \oplus \lambda\vec{w}$$

The vector space $V \oplus W$ is the smallest vector space that contains both V and W.

Definition 1.2.9. Let $\rho : G \to GL(V)$ and $\tau : G \to GL(W)$ be representations of G. Define $\rho \oplus \tau : G \to GL(W)$ by

$$[(\rho \oplus \tau)(g)](\vec{v} \oplus \vec{w}) = [\rho(g)](\vec{v}) \oplus [\tau(g)](\vec{w})$$

Example 1.2.10. Let $G = \mathbb{Z}_2$, and representations $\rho : G \to \mathbb{C}$ and $\tau : G \to \mathbb{C}$ be defined by the trivial representation and $\tau(0) = 1,\ \tau(1) = -1$, respectively. The direct sum $\sigma : \rho \oplus \tau$ is given by

$$\sigma : G \to GL_2(\mathbb{C}) = GL(\mathbb{C} \oplus \mathbb{C})$$
$$\sigma(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma(1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

1.3 Inner products

Definition 1.3.1. A *complex inner product space* is a complex vector space V with a pairing $\langle\ ,\ \rangle : V \times V \to \mathbb{C}$ satisfying:

1. $\langle v_1 + v_2, w\rangle = \langle v_1, w\rangle + \langle v_2, w\rangle$
2. $\langle c\vec{v}, \vec{w}\rangle = c\,\langle \vec{v}, \vec{w}\rangle$
3. $\langle \vec{v}, \vec{w}\rangle = \overline{\langle \vec{w}, \vec{v}\rangle}$
4. $\langle \vec{v}, \vec{v}\rangle \in \mathbb{R}$ and $\langle \vec{v}, \vec{v}\rangle \geqslant 0$, $\langle \vec{v}, \vec{v}\rangle = 0$ iff $\vec{v} = 0$

Example 1.3.2. This is an example of an inner product:

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n)\rangle = x_1\overline{y_1} + \cdots + x_n\overline{y_n}$$

Definition 1.3.3. A linear transformation $T : V \to V$ is termed *unitary* iff for all $\vec{w}, \vec{v} \in V$, $\langle T\vec{v}, T\vec{w}\rangle = \langle \vec{v}, \vec{w}\rangle$.

Definition 1.3.4. Let W be a subspace of a complex inner product space. The *orthogonal complement* if W in V is

$$W^\perp = \{\vec{x} \in V \ : \ \langle \vec{w}, \vec{x}\rangle = 0 \ \forall\ \vec{w} \in W\}$$

Note that $W \cap W^\perp = \{0\}$, and $\dim(W) + \dim(W^\perp) = \dim(V)$.

Definition 1.3.5. A morphism $T : \rho \to \tau$ is an *isomorphism* iff there is another morphism $T' : \tau \to \rho$ such that $T \circ T' = \text{id}$ and $T' \circ T = \text{id}$.

Theorem 1.3.6. Let V be a complex inner product space and G a group. Let $\rho : G \to GL(V)$ be a representation such that $\rho(g)$ is unitary for all $g \in G$. Then there are irreducible representations $\rho_1, \ldots, \rho_n$ of G such that $\rho \cong \rho_1 \oplus \cdots \oplus \rho_n$.

Proof: By a simple induction, it suffices to prove that if V is irreducible, then there are 2 proper subrepresentations W, W' such that $V \cong W \oplus W'$. Thus assume that V is reducible. Then there is a proper non-trivial subrepresentation W of V. Let $W' = W^\perp$. It remains to show that $W' = W^\perp$ is G-invariant. Thus, choose any $\vec{w} \in W^\perp$ and any $g \in G$ (from now on, ρ will be omitted). We will show that $g(\vec{w}) \in W^\perp$. To do this, we must take any $\vec{v} \in W$ and show that $\langle \vec{v}, g(\vec{w})\rangle = 0$.

Since ρ is unitary, g^{-1} is unitary, so $\langle \vec{v}, g(\vec{w})\rangle = \langle g^{-1}(\vec{v}), \vec{w}\rangle = 0$. This means that $g(\vec{w}) \in W^\perp$, so $W^\perp$ is G-invariant, so W, $W^\perp$ are both representations, and $V \cong W \oplus W^\perp$ as desired. ■

Next we are going to find an inner product such that ρ will always be unitary.

Theorem 1.3.7. Let G be a finite group with $\rho : G \to GL(V)$ a finite-dimensional representation of G. Then there is some complex inner product $\langle \cdot\ ,\ \cdot \rangle$ such that $\rho(g)$ is unitary for all $g \in G$.

Proof: Let $\langle \cdot\ ,\ \cdot \rangle_1$ be any inner product on V. Define a new pairing on V by

$$\langle \vec{v}, \vec{v} \rangle_2 = \sum_{g \in G} \langle [\rho(g)](\vec{v}), [\rho(g)](\vec{v}) \rangle_1 .$$

The pairing easily satisfies $\langle \vec{v}_1 + \vec{v}_2, \vec{v} \rangle_2 = \langle \vec{v}_1, \vec{v} \rangle_2 + \langle \vec{v}_2, \vec{v} \rangle_2$ and $\langle c\vec{v}, \vec{v} \rangle_2 = c \langle \vec{v}, \vec{v} \rangle_2$ and $\langle \vec{u}, \vec{v} \rangle_2 = \overline{\langle \vec{v}, \vec{u} \rangle_2}$. If $\vec{u} = \vec{v}$, then $\langle \vec{u}, \vec{v} \rangle_2 \in \mathbb{R}_{\geqslant 0}$, because each summand is, and the sum is 0 iff $g(\vec{v}) = 0$, which happens iff $\vec{v} = 0$, as desired. Now, $\rho(g)$ is unitary with respect to $\langle \cdot\ ,\ \cdot \rangle_2$ for all $g \in G$, as

$$\begin{aligned} \langle g(\vec{u}), g(\vec{v}) \rangle_2 &= \sum_{h \in G} \langle h(g(\vec{u})), h(g(\vec{v})) \rangle_1 \\ &= \sum_{h \in G} \langle (hg)(\vec{u}), (hg)(\vec{v}) \rangle_1 \\ &= \sum_{h \in G} \langle h(\vec{u}), h(\vec{v}) \rangle_1 \\ &= \langle \vec{u}, \vec{v} \rangle_2 \end{aligned}$$

■

Note that the previous two theorems immediately imply that every finite-dimensional representation of a finite group G is isomorphic to a direct sum of irreducible representations.

Theorem 1.3.8. Let $\rho : G \to GL(V)$ and $\tau : G \to GL(W)$ be irreducible representations of G. Let $T : V \to W$ be a morphism. Then T is either an isomorphism, or 0.

Proof: The kernel of T is a subrepresentation of V, which, by irreducibility of ρ, must be 0 or V. If $\ker(T) = V$, then $T = 0$. If $\ker(T) = 0$, then T is injective, so since $\mathrm{Im}(T) = 0$ or W, we must have that either $T = 0$ or T is surjective. So either $T = 0$ or T is an isomorphism. ■

Theorem 1.3.9. [Schur's lemma]
Let $T : V \to V$ be a morphism of irreducible representations. Then $T = \lambda \mathrm{id}$ for some scalar $\lambda \in \mathbb{C}$.

Proof: Let λ be an eigenvalue of T. Then $T = \lambda I$ is a morphism (this is easy to check). Since $T - \lambda I$ is not an isomorphism, by the previous theorem, it must be 0. So $T = \lambda I$. ■

Theorem 1.3.10. Let G be a finite abelian group and $\rho : G \to GL(V)$ an irreducible representation. Then $\dim(V) = 1$.

Proof: It turns out (you should check this) that if G is abelian, then for every $g\ \in$G$, \rho(g) : V \to V$ is a morphism, so by Schur's lemma, $\rho(g) = \lambda_g I$ for some $\lambda_g \in \mathbb{C}$. But this means that every subspace is G-invariant, so since ρ is irreducible, V can't have any non-trivial spaces, and so $\dim(V) = 1$. ■

2 Character theory

2.1 Tensor products

Definition 2.1.1. Let V, W be complex vector spaces. Let H be the vector space whose basis is $\{\vec{v} \otimes \vec{w}\ :\ \vec{v} \in V, \vec{w} \in W\}$. Note that H is very large. Define a subspace R of H to be the span of all vectors in H of the following forms:

$$\vec{v}_1 \otimes \vec{w} + \vec{v}_2 \otimes \vec{w} - (\vec{v}_1 + \vec{v}_2) \otimes \vec{w}$$
$$\vec{v} \otimes \vec{w}_1 + \vec{v} \otimes \vec{w}_2 - \vec{v} \otimes (\vec{w}_1 + \vec{w}_2)$$
$$(\lambda \vec{v}) \otimes \vec{w} - \lambda(\vec{v} \otimes \vec{w})$$
$$\vec{v} \otimes (\lambda \vec{w}) - \lambda(\vec{v} \otimes \vec{w})$$

Define $V \otimes W = H/R$

Example 2.1.2. For $V = W = \{0\}$, $V \otimes W = \{0\}$, as $H = \text{span}\{0 \otimes 0\}$ and $\lambda(0 \otimes 0) = (\lambda 0) \otimes 0 = 0 \otimes 0$ for all λ.

Example 2.1.3. For $V = \{0\}$ and W any vector space, $V \otimes W = \{0\}$, because for any $w \in W$,

$$0 \otimes w + 0 \otimes w = (0 + 0) \otimes w = 0 \otimes w \implies 0 \otimes w = 0.$$

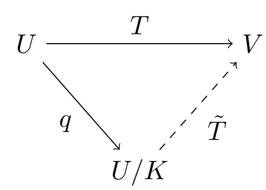
Example 2.1.4. Let $V = \text{span}\{\vec{v}\}$, $W = \text{span}\{\vec{w}\}$ for $\vec{v}, \vec{w}$ non-zero. Then $V \otimes W$ is spanned by by elements of the form $(\lambda \vec{v}) \otimes (\mu \vec{w})$. But $(\lambda \vec{v}) \otimes (\mu \vec{w}) = (\lambda\mu)(\vec{v} \otimes \vec{w})$, so $V \otimes W$ is spanned by $\{\vec{v} \otimes \vec{w}\}$. To see that $\vec{v} \otimes \vec{w}$ is non-zero, we use a very useful trick, described below.

Remark 2.1.5. Let $q : H \to V \otimes W$ be the "reduction mod R" linear transformation. Then q is surjective and H is non-zero, so it suffices to show that q is not identically zero. Define $T : H \to \mathbb{C}$ by

$$T\left(\sum_i a_i(\lambda_i \vec{v}) \otimes (\mu_i w)\right) = \sum_i a_i \lambda_i \mu_i$$

which is iclearly a linear transformation. The image of T is $\mathbb{C}$, because $T(\vec{v} \otimes \vec{w}) = 1$. So T is surjective. Moreover, $R \subset \ker(T)$, so by the universal property of quotients, $T : V \otimes W \to \mathbb{C}$ is well-defined.

Theorem 2.1.6. [Universal property of quotients]
Let U be a vector space, $K \subset U$ any subspace, $q : U \to U/K$ the "reduce mod K" linear transformation. Let $T : U \to V$ be a linear transformation. Then there is a linear transformation $\tilde{T} : U/K \to V$ satisfying $T = \tilde{T} \circ q$ iff $K \subset \ker(T)$. That is, making the diagram below commute:



Also, $\text{Im}(T) = \text{Im}(\tilde{T})$ and $\ker(\tilde{T}) = q(\ker(T))$.

Theorem 2.1.7. Let V, W be finite-dimensional vector spaces. Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ and $\{\vec{w}_1, \ldots, \vec{w}_m\}$ be bases for V and W, respectively. Then $\{\vec{v}_i \otimes \vec{w}_j\}$ is a basis for $V \otimes W$. In particular, $\dim(V \otimes W) = \dim(V)\dim(W)$.

Proof: Let $\sum_k x_k \otimes y_k$ be an arbitrary element of $V \otimes W$. Then

$$\sum_k x_k \otimes y_k = \sum_k \left(\sum_i a_{ik} \vec{v}_i\right) \otimes \left(\sum_j b_{jk} \vec{w}_j\right) = \sum_{i,j,k} a_{ik} b_{jk} (\vec{v}_i \otimes \vec{w}_j) \in \text{span}\{\vec{v}_i \otimes \vec{w}_j\}$$

To get linear independence, define a linear transformation

$$T : H \to \mathbb{C}^{nm} \qquad \text{by} \qquad T\left(\sum_k c_k \left(\sum_i a_{ik} \vec{v}_i\right) \otimes \left(\sum_j b_{jk} \vec{w}_j\right)\right) = \sum_{i,j,k} c_k a_{ik} b_{jk} \vec{e}_{ij}$$

where $\{\vec{e}_{ij}\}$ are the standard unit basis vectors in $\mathbb{C}^{nm}$, $\mathbb{C}^{nm}$ being viewed as $n \times m$ matrices and $\vec{e}_{ij}$ the matrix with all zeros except for a 1 in the (i,j)-entry. Since $T(\vec{v}_i \otimes \vec{w}_j) = \vec{e}_{ij}$, T is injective. It is easy to check that $R \subset \ker(T)$. For example,

$$\begin{aligned} T(\vec{v}_1 \otimes \vec{w} + \vec{v}_2 \otimes \vec{w} - (\vec{v}_1 + \vec{v}_2) \otimes \vec{w}) &= T\left(\left(\sum_i a_i \vec{v}_i\right) \otimes \sum_j b_j \vec{w}_j + \sum_i a'_i \vec{v}_i \otimes b_j \vec{w}_j - \sum_i (a_i - a'_i)\vec{v}_i \otimes \sum_j \vec{w}_j\right) \\ &= \sum_{i,j} a_i b_j \vec{e}_{ij} + \sum_{i,j} a'_i b_j \vec{e}_{ij} - \sum_{i,j} (a_i - a'_i) b_j \vec{e}_{ij} \\ &= 0 \end{aligned}$$

By the UPQ, $T : H/R \to \mathbb{C}^{nm}$ is wel-defined, so $T : V \otimes W \to \mathbb{C}^{nm}$ is well-defined and onto. Therefore, since $\{\vec{e}_{ij}\}$ is linearly independent in $\mathbb{C}^{nm}$, their preimages $\{\vec{v}_i \otimes \vec{w}_j\}$ are also linearly independent in $V \otimes W$. ■

Example 2.1.8. Suppose that $T : U \to V$ and $S : W \to X$ are linear transformations. We can define $T \otimes S : U \otimes W \to V \otimes X$ by $(T \otimes S)(\sum \vec{v}_i \otimes \vec{w}_i) = \sum T(\vec{v}_i) \otimes S(\vec{w}_i)$. Let $M_1 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $M_2 = \left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right)$ in $M_2(\mathbb{C})$. Then the matrix of $M_1 \otimes M_2$ with respect to $\{\vec{e}_i \otimes \vec{e}_j\}$ of $\mathbb{C}^2 \otimes \mathbb{C}^2$ is

$$\begin{pmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{pmatrix}$$

Definition 2.1.9. Let $\rho : G \to GL(V)$ and $\tau : G \to GL(W)$ bu representations. Then $\rho \otimes \tau : G \to GL(V \otimes W)$ is given by

$$[(\rho \otimes \tau)(g)]\left(\sum_i \vec{v}_i \otimes \vec{w}_i\right) = \sum_i [\rho(g)](\vec{v}_i) \otimes [\tau(g)](\vec{w}_i).$$

Example 2.1.10. Let $G = S_3$ and $\rho : G \to GL(\mathbb{C})$ be the trivial representation and $\tau : G \to CL(\mathbb{C})$ the sign representation (gives $+1$ to even permutations, and -1 to odd permutations). Then $\rho \otimes \tau : G \to GL(\mathbb{C} \otimes \mathbb{C}) \cong GL(\mathbb{C})$ is given by $(\rho \otimes \tau)(g) = \rho(g)\tau(g) = \tau(g)$.

Example 2.1.11. If $\rho : G \to GL(\mathbb{C})$ is one-dimensional and $\tau : G \to GL(V)$ is any representation, then $\rho \otimes \tau \cong \tau_1$, where $\tau_1(g) = \rho(g)\tau(g)$.

Proposition 2.1.12. If ρ, τ are irreducible representations, then $\rho \otimes \tau$ is not irreducible.

Remark 2.1.13. What if $\rho : G \to GL(V)$ is irreducible, is $\rho \otimes \rho$ irreducible? No, if $\dim(V) < 2$.

Theorem 2.1.14. Let $\rho : G \to GL(V)$ be a representation with $\dim(V) \geqslant 2$. Then $\rho \otimes \rho$ is irreducible.

Proof: Define $\theta : H \to V \otimes V$ (for H the free vector space on $V \times V$) by

$$\theta\left(\sum_i a_i \vec{v}_i \otimes \vec{w}_i\right) = \sum_i a_i \vec{w}_i \otimes \vec{v}_i$$

Then $R \subset \ker(\theta)$ (this is easy to check), so θ is well-defined. Let $\mathrm{Sym}^2(V)$ be the 1-eigenspace of θ, and $\mathrm{Alt}^2(V)$ be the (-1)-eigenspace of θ. We claim that $\mathrm{Sym}^2(V)$ and $\mathrm{Alt}^2(V)$ are G-invariant subspaces of

$V \otimes V$. To see this, note that $\mathrm{Sym}^2(V)$ and $\mathrm{Alt}^2(V)$ have zero intersection, and $V \otimes V = \mathrm{Sym}^2(V) \cup \mathrm{Alt}^2(V)$, so $V \otimes V \cong \mathrm{Sym}^2(V) \oplus \mathrm{Alt}^2(V)$ as vector spaces. So suppose that $\sum_i a_i(\vec{v}_i \otimes \vec{w}_i) \in \mathrm{Sym}^2(V)$. Then

$$\begin{aligned}
\theta\left([(\rho \otimes \rho)(g)]\left(\sum_i a_i(\vec{v}_i \otimes \vec{w}_i)\right)\right) &= \theta\left(\sum_i a_i([\rho(g)](\vec{v}_i) \otimes [\rho(g)](\vec{w}_i))\right) \\
&= \sum_i \theta\left(([\rho(g)](\vec{v}_i)) \otimes ([\rho(g)](\vec{w}_i))\right) \\
&= \sum_i a_i [\rho(g)](\vec{v}_i) \otimes [\rho(g)](\vec{v}_i) \\
&= [(\rho \otimes \rho)(g)]\left(\sum_i \vec{w}_i \otimes \vec{v}_i\right) \\
&= [(\rho \otimes \rho)(g)\theta\left(\sum_i a_i \vec{v}_i \otimes \vec{w}_i\right) \\
&= [(\rho \otimes \rho)(g)]\left(\sum_i \vec{v}_i \otimes \vec{w}_i\right)
\end{aligned}$$

So $[(\rho \otimes \rho)(g)](\sum_i a_i(\vec{v}_i \otimes \vec{w}_i)) \in \mathrm{Sym}^2(V)$. Thus $\mathrm{Sym}^2(V)$ is G-invariant and we're done. The case for $\mathrm{Alt}^2(V)$ is similar. ∎

Remark 2.1.15. Note that if $\dim(V) = n$, then $\dim(V \otimes V) = n^2$, and $\mathrm{Sym}^2(V) = n + \binom{n}{2} = \frac{n(n+1)}{2}$.

2.2 Characters

Definition 2.2.1. Let $\rho : G \to GL(V)$ be a representation. The *character* of ρ is the function $\chi_\rho : G \to \mathbb{C}$ given by $\chi_\rho(g) = \mathrm{trace}(\rho(g))$. If ρ is irreducible, then χ_ρ is termed an *irreducible character*.

Remark 2.2.2. The following are some elementary properties of characters:

- If $\dim(\rho) = 1$, then $\chi_\rho = \rho$
- If $\rho \cong \tau$, then $\chi_\rho = \chi_\tau$, as trace is invariant under linear transformations
- $\chi_\rho(1) = \dim(V)$ for any ρ
- $\chi_\rho(g^{-1}hg) = \chi_\rho(h)$

If we add the stipulation that g is finite and $\dim(V)$ is finite, then we also have:

- $\chi_{\rho\oplus\tau} = \chi_\rho + \chi_\tau$
- $\chi_{\rho\otimes\tau} = \chi_\rho \cdot \chi_\tau$
- $\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$

Theorem 2.2.3. Let ρ, τ be representations of G in V, W, respectively. Let $T : V \to W$ be any linear transformation. Then $T' = \sum_g \tau(g^{-1}) \circ T \circ \rho(g)$ is a morphism $\rho \to \tau$. In particular, if ρ and τ are irreducible, and $\rho \cong \tau$, then $T' = 0$.

Definition 2.2.4. Define $\mathbb{C}[G] = \{f : G \to \mathbb{C}\}$ to be the *complex group ring* (or *algebra*). Note that $\mathbb{C}[G]$ has a natural inner product

$$\langle \varphi, \psi \rangle = \frac{1}{\#G} \sum_{g \in G} \varphi(g)\overline{\psi(g)}$$

This corresponds to the usual inner product in $\mathbb{C}^n$, divided by $\#G$, via $f \leftrightarrow (f(g_1), \ldots, f(g_n))$.

Theorem 2.2.5. Irreducible characters are an orthonormal set in $\mathbb{C}[G]$.

Proof: We show that if $\rho,\ \tau$ are irreducible representations, then $\langle \chi_\rho, \chi_\tau \rangle = \begin{cases} 1 \text{ if } \rho \cong \tau \\ 0 \text{ if } \rho \not\cong \tau \end{cases}$
Let $\rho : G \to GL(V)$ with a fixed basis of V, and $\tau : G \to GL(W)$ wih a fixed basis of W. For each $g \in G$, write $\rho(g) = r_{ij}(g)$ and $\tau(g) = t_{ij}(g)$ as matrices. Then

$$\langle \chi_\rho, \chi_\tau \rangle = \frac{1}{\#G} \sum_{g,i,j} r_{ii}(g)\overline{t_{jj}(g)} = \frac{1}{\#G} \sum_{g,i,j} r_{ii}(g) t_{jj}(g^{-1})$$

Let $T : V \to W$ be any linear transformation. Then $T' = \sum_g \tau(g^{-1}) \circ T \circ \rho(g)$ is a morphism $\rho \to \tau$. If $\rho \not\cong \tau$, then $T' = 0$, so if $T = [T_{ij}]$, we have

$$\sum_{g,i,j} t_{ki}(g^{-1}) T_{ij} r_{j\ell} \ \ \forall\, k, \ell$$

So we set $T_{ij} = 0$ for all i, j except k, ℓ, for which $T_{k\ell} = 1$. Then $\sum_g t_{rr} r_{\ell\ell}(g^{-1}) = 0$, so summing over k, ℓ gives $\langle \chi_\rho, \chi_\tau \rangle = 0$.

If $\rho \cong \tau$, then $\chi_\rho \cong \chi_\tau$, so we assume $\rho = \tau$. If $\rho = \tau$, then $T' = \lambda I$ by our theorem, and $\lambda = \#G \cdot \text{trace}(T) \cdot (\dim(V))^{-1}$. Exactly the same argument as before gives $\langle \chi_\rho, \chi_\tau \rangle = 1$. ■

This means that there are infinitely many irreducible representations of G up to isomorphism, because the corresponding characters are an orthogonal set in a finite-dimensional vector space.

Remark 2.2.6. Note that if $\rho = (m_1\rho_1) \oplus \cdots \oplus (m_r\rho_r)$ for $m_i \in \mathbb{Z}$ and ρ_i pairwise non-isomorphic representations, then

$$m_i = \langle \chi_\rho, \chi_{\rho_i} \rangle$$

Note even further that irreducible decompositions of representations are unique up to isomorphism. Moreover, non-isomorphic representations have different characters, because the character determines an irreducible decomposition. Finally, with ρ as above, we have that

$$\langle \chi_\rho, \chi_\rho \rangle = m_1^2 + \cdots + m_n^2 \qquad \text{and} \qquad \langle \chi_\rho, \chi_\rho \rangle = 1 \iff \rho \text{ is irreducible.}$$

Example 2.2.7. Recall that we had a representation of S_3 in $\mathbb{C}^2$ on the assignment.

$\rho(g)$	$\rho(g) \in \mathbb{C}^2$	character of $\rho(g)$
$\rho(1)$	$\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$	2
$\rho(12)$	$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$	0
$\rho(13)$	$\left(\begin{smallmatrix} -1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$	0
$\rho(23)$	$\left(\begin{smallmatrix} 1 & -1 \\ 0 & -1 \end{smallmatrix}\right)$	0
$\rho(123)$	$\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$	-1
$\rho(132)$	$\left(\begin{smallmatrix} -1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$	-1

Hence $\langle \chi_\rho, \chi_\rho \rangle = \frac{1}{6}(2^2 + 0^2 + 0^2 + 0^2 + (-1)^2 + (-1)^2) = 1$, and so ρ is irreducible.

Example 2.2.8. Note that there are only 3 irreducible representations of S_3 - the trivial, the sign, and the one given in the previous example. All others that are irreducible are isomorphic to one of them.

Now, let $\rho : S_3 \to GL(\mathbb{C}^3)$ be the permutation representation. How can we write ρ as a sum of irreducible representations? We note the following facts:

$$\chi_\rho(1) = 3 \quad \chi_\rho(\text{2-cycles}) = 1 \quad \chi_\rho(\text{3-cycles}) = 0$$

$$\langle \chi_\rho, \chi_\rho \rangle = \frac{1}{6}(3^2 + 1^2 + 1^2 + 1^2 + 0^2 + 0^2) = 2$$

Let Δ be the representation that was ρ in the previous example. Then

$$\langle \chi_\rho, \chi_\Delta \rangle = \frac{1}{6}(3 \cdot 2 + 1 \cdot 0 + 1 \cdot 0 + 1 \cdot 0 + 0 \cdot (-1) + 0 \cdot (-1)) = 1$$

Therefore $\rho = \Delta \oplus \rho'$ for some representation ρ' whose irreducible decomposition contains no Δ. We note further that

$$\langle \chi_\rho, \chi_{\text{triv}} \rangle = \frac{1}{6}(3 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 0 \cdot 1 + 0 \cdot 1) = 1$$

And so $\rho = \Delta \oplus (\text{triv})$.

Example 2.2.9. Let G be any finite group, and $V = \bigoplus_{g \in G} \mathbb{C}g$. Let $\rho : G \to GL(V)$ be given by $[\rho(g)] \left(\sum a_i g_i\right) = \sum a_i g g_i$. This ρ is termed the *left-regular* representation of G.

Then $\chi_\rho(g) = \#G$ if $g = 1$, and $\chi_\rho(g) = 0$ if $g \neq 1$. Therefore $\langle \chi_\rho, \chi_\rho \rangle = \#G$, so unless G is the trivial group, it is never irreducible.

And if τ is any irreducible representation of G, then $\langle \chi_\rho, \chi_\tau \rangle = \dim(\tau) = 1$, so $\rho \cong (d_1\rho_1) \oplus \cdots \oplus (d_r\rho_r)$ where $\rho_1, \ldots, \rho_r$ are all irreducible representations of G up to isomorpism, and $d_i = \dim(i)$. This means that $\langle \chi_\rho, \chi_\rho \rangle = d_1^2 + \cdots + d_r^2$, and as $\langle \chi_\rho, \chi_\rho \rangle = \#G$, we have that $\#G = d_1^2 + \cdots + d_r^2$.

Remark 2.2.10. Irreducible characters in general do not span $\mathbb{C}[G]$. Note that for all $g, h \in G$,

$$\chi(g^{-1}hg) = \chi(h)$$

so χ remains constant on conjugacy classes.

Definition 2.2.11. A *conjugacy class* in a group G is a set of elements C such that for all $c, d \in C$, $c^{-1}dc \in C$. A *class function* on a group G is a function $f : G \to \mathbb{C}$ that is constant on conjugacy classes. Class functions are a subspace of $\mathbb{C}[G]$.

Theorem 2.2.12. Let G be a finite group. The irreducible characters of G form the orthonormal basis of the space V of class functions on G.

Hence if G has k conjugacy classes, then G has k irreducible characters.

Proof: We will show that if $W = \text{span}\{\text{irreducible characters}\}$, then $W^\perp \cap V = \{0\}$. This wil imply $W = V$, as $W \subset V$. Note that $W = \overline{W}$, and thus $W^\perp = \overline{W}$, so $f \in W$ iff $\overline{f} \in \overline{W^\perp}$. So first suppose that $f \in W^\perp$. Then $\overline{f} \in W^\perp$, so $\langle \chi, \overline{f} \rangle = 0$ for any irreducible character χ. Define

$$T_f^\rho : V \to V \qquad \text{by} \qquad T_f^\rho(\vec{v}) = \sum_{g \in G} f(g)[\rho(g)](\vec{v})$$

for any representation ρ of G. It is straightforward to check that T_f^ρ is a morphism $\rho \to \rho$. If ρ is irreducible, then $T_f^\rho = \lambda \text{id}$, for some $\lambda \in \mathbb{C}$. However, as

$$0 = \langle \chi, f \rangle = \sum_{g \in G} \chi_\rho(g) f(g) = \sum_{g \in G} f(g)\text{trace}(\rho(g)) = \text{trace}(T_f^\rho)$$

we have that $\lambda = 0$. Hence $T_f^\rho = 0$ for any irreducible representation ρ. Then by linearity, $T_f^\rho = 0$ for any representation ρ.

Let ρ be the left-regular representation with $\vec{v} = g_i$. Then

$$0 = T_f^\rho(\vec{v}) = \sum_{g \in G} f(g)[\rho(g)](\vec{v}) = \sum_{g \in G} f(g) g g_i$$

so since $\{gg_i\} = \{g_i\}$ is linearly independent, it follows that $f(g) = 0$ for all $g \in G$. This means that $W^\perp \cap V = \{0\}$ as desired, so $W = V$ and the irreducible characters span the space of class functions. ■

2.3 Character tables

Example 2.3.1. Let $G = D_4$, the dihedral group of order 4. Let's write down a list of all the irreducible characters of G. This will be called a *character table* for G. So let x represent counter-clockwise rotation by $90°$, and y represent reflection along the axis of symmetry. The facts we know are:

$$D_4 = \{x, y \ : \ yx = x^{-1}y, \ x^4 = y^2 = 1\}$$
$$\text{conjugacy classes: } \{1\}, \ \{x^2\}, \ \{x, x^3\}, \ \{xy, x^3y\}, \ \{y, x^2y\}$$

So there are 5 irreducible characters. They may be classified as follows:

· The trivial character, χ_t, is always irreducible.

· Using the homomorphism $\varphi : D_4 \to \mathbb{Z}_2$, with $\varphi(x^a y^b) = b \pmod 2$, we get a "sign" representation on D_4, with $\rho(g) = (-1)^{\varphi(g)}$.

· Using the homomorphism $\psi : D_4 \to (\mathbb{Z}_2)^2$, with $\psi(x^a y^b) = (a \pmod 2, b \pmod 2)$, we get two more irreducible characters, corresponding to the representations

$$\rho_1(x^a y^b) = (-1)^{a \pmod 2} \qquad \text{and} \qquad \rho_2(x^a y^b) = (-1)^{a+b \pmod 2}$$

· The final character may then be calculated from the orthonormality condition.

This gives us the following table:

	$\{1\}$	$\{x^2\}$	$\{x, x^3\}$	$\{xy, x^3y\}$	$\{y, x^2y\}$
χ_t	1	1	1	1	1
χ_{sgn}	1	1	1	-1	-1
χ_{ρ_1}	1	1	-1	-1	1
χ_{ρ_2}	1	1	-1	1	-1
$\chi_\bullet$	2	-2	0	0	0

Note that there are lots of ways to find the last character of D_4. The easiest is to use the fact that, with the other 4 characters, it is an orthonormal basis of the space of class functions. Or you could just guess it - it is the realization of D_4 as the symmetries of a square.

Note that our approach above was to find homomorphisms from D_4 to simpler groups, and then find representations of those groups.

Remark 2.3.2. For L_a the left-regular representation, we have that

$$\bigoplus_{\rho_i \text{ irreducible}} \chi_{\rho_i}(1) \cdot \chi_{\rho_i} = \chi_{L_a} = (\dim(G) \ 0 \ 0 \ \cdots \ 0).$$

Example 2.3.3. Compute the character table for A_5. First note that the only normal subgroups of A_5 are $\{0\}$ and A_5. For $\{0\}$, homomorphisms to simpler groups are not simpler. For A_5, the only homomorphism to a simpler group gives the trivial representation. First we note that conjugacy class representatives of A_5, and the class sizes, are given by:

$$\Big\{ \underbrace{(1) \quad (1\ 2\ 3) \quad (1\ 2)(3\ 4)}_{\text{conjugacy classes in } S_5} \quad (1\ 2\ 3\ 4\ 5) \quad (1\ 3\ 4\ 5\ 2) \Big\}$$

1 cycle $\quad \binom{5}{3} \cdot 2 = 20$ cycles $\quad 5 \cdot 3 = 15$ cycles $\quad 24/2 = 12$ cycles $\quad 24/2 = 12$ cycles

So we need 5 representations. We always have the trivial representation with character χ_t. Next consider the permutation representation, whose trace χ_{perm} is the number of fixed vectors. This representation, for which $\chi_{perm} = (5, 2, 1, 0, 0)$, is not irreducble, as

$$\langle \chi_{perm}, \chi_{perm} \rangle = \frac{1}{60}(5^2 \cdot 1 + 2^2 \cdot 20 + 1^2 \cdot 15 + 0^2 \cdot 12 + 0^2 \cdot 12) = \frac{1}{60}(25 + 80 + 15) = 2$$

We hope that $\chi_{perm} = \chi_t \oplus \chi$, because then we will know χ_{perm}. So we check if the trivial representation appears in it:

$$\langle \chi_{perm}, \chi_t \rangle = \frac{1}{60}((5 \cdot 1) \cdot 1 + (2 \cdot 1) \cdot 20 + (1 \cdot 1) \cdot 15) = \frac{1}{60}(5 + 40 + 15) = 1$$

So we define $\chi_{pt} = \chi_{perm} - \chi_t$, which is irreducible. However, we need 3 more representations, so we try $\mathrm{Sym}^2(pt)$ and $\mathrm{Alt}^2(pt)$. Before we do that, we need to formaulate their characters. So we include the final table below, with the work done for finding the entries of the below further down.

	(1)	(1 2 3)	(1 2)(3 4)	(1 2 3 4 5)	(1 3 4 5 2)
χ_t	1	1	1	1	1
χ_{pt}	4	1	0	-1	-1
χ_n	5	-1	1	0	0
χ_1	3	0	-1	φ	$-1/\varphi$
χ_2	3	0	-1	$-1/\varphi$	φ

The symbol φ is the golden ratio, $(1 + \sqrt{5})/2$.

Theorem 2.3.4. Let $\rho : G \to GL(V)$ be a representation of a finite group. Let ρ_s and ρ_a be the $\mathrm{Sym}^2(\rho)$ and $\mathrm{Alt}^2(\rho)$ representations. Let χ, χ_s, χ_a be the characters of ρ, ρ_s, ρ_a, respectively. Then

$$\chi_s(g) = \frac{1}{2}\left(\chi(g)^2 + \chi(g^2)\right) \qquad \text{and} \qquad \chi_a(g) = \frac{1}{2}\left(\chi(g)^2 - \chi(g^2)\right)$$

Proof: Since G is finite (more generally, since ρ is unitary), $\rho(g)$ is diagonalizable. Let $\{e_1, \dots, e_n\}$ be an eigenbasis of V with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$. Then $\chi(g) = \sum \lambda_i$ and $\chi * g^2) = \sum(\lambda_i)^2$. Now,

$$0\rho_s(g)](\vec{e}_i \otimes \vec{e}_i) = [\rho(g)](\vec{e}_i) \otimes [\rho(g)](\vec{e}_i) = \lambda_i^2(\vec{e}_i \otimes \vec{e}_i)$$
$$[\rho_s(g)](\vec{e}_i \otimes \vec{e}_j + \vec{e}_j \otimes \vec{e}_i) = \lambda_i\lambda_j(\vec{e}_i \otimes \vec{e}_j + \vec{e}_j \otimes \vec{e}_i)$$

So $\chi_s(G) = \sum_(\lambda_i)^2 + \sum_{i<j} \lambda_i\lambda_j = \frac{1}{2}(\chi(g)^2 - \chi(g^2))$. A similar calculation works for χ_a. ■

What are the characters of $\mathrm{Sym}^2(\sigma)$ and $\mathrm{Alt}^2(\sigma)$? First, check the irreducibility of χ_s by

$$\chi_s(g) = \frac{1}{2}(\chi(g)^2 + \chi(g^2)) = (10, 1, 2, 0, 0).$$

The irreducibility is then checked with the other characters.

$$\langle \chi_s, \chi_s \rangle = \frac{1}{60}(10^2 + 1^2 \cdot 20 + 2^2 \cdot 15) = 3$$
$$\langle \chi_s, \chi_t \rangle = \frac{1}{60}(10 \cdot 1 + 1 \cdot 1 \cdot 20 + 2 \cdot 1 \cdot 15) = 1$$
$$\langle \chi_s, \chi_{pt} \rangle = \frac{1}{60}(10 \cdot 4 + 1 \cdot 1 \cdot 20) = 1$$

This implies that χ_s is the sum of three irreducible representations, with χ_t being one and χ_s being another. That is, $\chi_s = \chi_t \oplus \chi_{pt} \oplus \chi_n$ for some irreducible representation ρ_n with $\chi_n = (5, -1, 1, 0, 0)$. This gives us one more irreducible representation ρ_n, but we still need two more. Next, check the irreducibility of χ_a. We note that

$$\chi_a(g) = \frac{1}{2}(\chi(g)^2 - \chi(g^2)) = (6, 0, -2, 1, 1) \qquad \text{with} \qquad \langle \chi, a\chi_a \rangle = 2.$$

So this representation is the sum of two irreducible representations. However, we have

$$\langle \chi_a, \chi_t \rangle = 0 \qquad \langle \chi_a, \chi_{pt} \rangle = 0 \qquad \langle \chi_a, \chi_n \rangle = 0$$

So $\chi_a = \chi_1 \oplus \chi_2$, for the two represntations that are not yet in the table. Note that both must be of dimension 3. By trial and error, we find that

$$\chi_1 = (3, a, b, c, d) \qquad \text{and} \qquad \chi_2 = (3, -a, -2 - b, 1 - c, 1 - d).$$

We know that $\langle \chi_1, \chi_t \rangle = \langle \chi_1, \chi_{pt} \rangle = \langle \chi_1 \chi_n \rangle = 0$, which gives 3 linear equations in 4 variables. We also have that $\langle \chi_1, \chi_1 \rangle = 1$, which is a quadratic equation in a, b, c, d. Ultimately, we find that

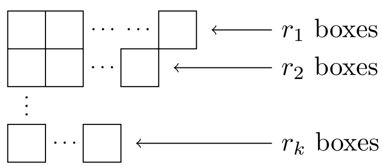
$$\chi_1 = (3, 0, -1, \varphi, -1/\varphi) \qquad \text{and} \qquad \chi_2 = (3, 0, -1, -1/\varphi, \varphi),$$

where φ is the golden ratio. This completes the table.

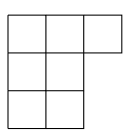
2.4 The symmetric group S_n

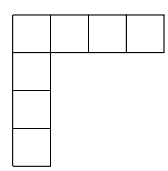
A good question to ask is what are the irrodocible representations of S_n. We can start by saying that the conjugacy classes of S_n will be the sets of permutations with the same cycle structure.

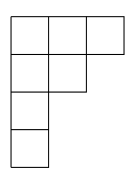
Definition 2.4.1. Conjugacy classes of S_n are in a 1-1 correspondence with *Young tableaux.* The young tableau associated to a conjugacy class $(r_1, r_2, \ldots, r_k)$, with $\sum r_i = n$ and $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_k$ is



Example 2.4.2. Consider the following permutations and their associated Young tableaux.

$(1\ 2\ 3)(4\ 5)(6\ 7) \in S_7 \quad \leftrightarrow$ 

$(1\ 2\ 3\ 4) \in S_7 \quad \leftrightarrow$ 

$(1\ 2)(3\ 4\ 5) \in S_7 \quad \leftrightarrow$ 

Definition 2.4.3. A *numbering* of a Young tableau is an injective function from $\{1, \ldots, n\}$ to the boxes in the tableau.

Note that S_n acts on the set of numberings of a fixed tableau.

Definition 2.4.4. A *tabloid* is an equivalence class of numberings of some fixed tableau, where two numberings are equivalent iff for each row both numberings contain the same set of numbers, albeit possibly in a different order.

So S_n also acts on the set of tabloids of a given fixed shape.

Definition 2.4.5. Let S be a fixed shape of a Young tableau. Let $M^S = \bigoplus_T \mathbb{C}T$, where T ranges over all tabloids of shape S. This is a permutation representation of S_n in M^S, coming from the S_n action on the tabloids.

Let T be a numbering of a tableau, and let $C(T)$ be a subgroup of S_n of elements $\sigma \in S_n$ that preserve the columns of T. That is, m and $\sigma(m)$ are in the same column for all m. Define $R(T)$ to be an anologous object, except for rows.

Let $[T]$ be the tabloid associated to T. Define $\vec{v}_T = \sum_{\sigma \in C(T)} \operatorname{sgn}(\sigma)[\sigma(T)]$, where $\operatorname{sgn}(\sigma) = \left\{ \begin{smallmatrix} -1 & \text{if } \sigma \text{ is odd} \\ 1 & \text{if } \sigma \text{ is even} \end{smallmatrix} \right.$

Remark 2.4.6. It may be checked that $\vec{v}_T$ does not depend on T, only on $[T]$. Let $S^s = \operatorname{span}(\vec{v}_T)$. We will show that S^s is an S_n-invariant subspace of M^S. For any $\tau \in S_n$, we have

$$\begin{aligned}
\tau(\vec{v}_T) &= \tau\left(\sum_\sigma \operatorname{sgn}(\sigma)[\sigma(T)]\right) \\
&= \sum_\sigma \operatorname{sgn}(\sigma)[\tau\sigma(T)] \\
&= \sum_{\sigma \in C(\tau(T))} \operatorname{sgn}(\sigma)[\tau^{-1}\tau\sigma\tau] && (\text{as } C(\tau(T)) = \tau C(T)\tau^{-1}) \\
&= \sum_{\sigma \in C(\tau(T))} \operatorname{sgn}(\sigma)[\sigma\tau(T)] \\
&= \vec{v}_{\tau(T)}
\end{aligned}$$

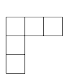
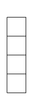
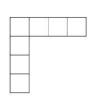
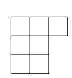
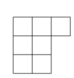
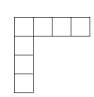
So S_n permutes the $\vec{v}_T$, meaning that S^s is S_n invariant. Note that S^s is called the *Specht module* associated to the shape S.

Our plan now is to show that S^s is irreducible, and that $S^s \ncong S^t$ if $s \neq t$. This will imply that $\{S^s\}$ is a complete list of irreducible representations of S_n up to isomorphism.

Definition 2.4.7. Let s, t be Young tableaux of the same size (same number of boxes in each row). Write $s = (s_1, \ldots, s_k)$ and $t = (t_1, \ldots, t_k)$ (set s_i or t_i to 0 if necessary, to give t and s the same number of rows). We say that s is *dominates* t iff for all m, $s_1 + \cdots + s_m \geqslant t_1 + \cdots + t_m$.

Further, s *strictly dominates* t iff s dominates t and $s \neq t$.

Example 2.4.8. To show how dominating works, note that

( dominates ) but ( does not dominate ) and ( does not dominate ).

Theorem 2.4.9. Let T, T' be numberings of shpaes s, s' of the same size with s not strictly dominating s'. Then either

1. there are 2 different numbers in the same row of T and the same column of T', or
2. $s = s'$ and there is some $p' \in R(T')$, $q \in C(T)$ such that $p'(T) = q(T)$.

Proof: Assume that **1.** is not true. We will prove that **2.** holds. We want to find $p' \in R(T')$ and $q \in C(T)$ such that $p1(T') = q(T)$. Choose $q_1 \in C(T)$ so that all the numbers in the first row of T' are in the first row of $q_1(T)$, which is possible, because all the numbers are in different columns of T by the negation af **1.**.

Choose $q_2 \in C(T)$ to be the first row of numbers from T', and permute the second row of numbers from T'

to the second row (or higher) of $q_2q_1(T)$. Keep going until you get a $q = q_k \cdots q_1 \in C(T)$ such that for every i, all the numbers in the ith row of T' are in the ith row of $q(T)$, or higher.

This means that s dominates s'. By assumption, $s = s'$, and each row of T' has the same set of numbers at the corresponding row of $q(T)$. So there is some $p' \in R(T')$ such that $p'(T') = q(T)$. ∎

Example 2.4.10. Consider the following situation:

$$T = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \text{and} \qquad T' = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

There is no way to make $T = T'$ by permuting each row of T' and each column of T.

Definition 2.4.11. Define the value

$$b_T = \sum_{\sigma \in C(T)} \operatorname{sgn}(\sigma)\sigma \in \bigoplus_g \mathbb{C}g.$$

Then $\vec{v}_T = b_T(T)$. Moreover, for any $\sigma \in C(T)$,

$$b_T\sigma = \operatorname{sgn}(\sigma)b_T \qquad \text{and} \qquad b_Tb_T = \sum\sum \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\sigma\tau = \#C(T)$$

Theorem 2.4.12. Let T, T' and s, s' be as above. If **1.** holds, then $b_T(T') = 0$. Otherwise, $b_T(T') = \pm\vec{v}_T$.

<u>*Proof:*</u> If **1.** holds, let τ be the transposition switching the two numbers. Then $\tau \in R(T')$ and $\tau \in C(T)$, so $b_T\tau = -b_T$, and $b_T(T') = 0$ because

$$b_T(T') = b_T(\tau(T')) = (b_T\tau)(T') = -b_T(T')$$

If **2.** holds, then choose $p \in R(T')$ and $q \in C(T)$, such that $p'(T') = q(T)$. Then

$$b_T(T') = b_T(p'(T')) = b_T(q(T)) = \operatorname{sgn}(q)b_T(T) = \pm v_T$$

∎

Proposition 2.4.13. The space S^s is irreducible, and $S^s \ncong S^t$ if $s \neq t$.

<u>*Proof:*</u> Assume that $S^s = V \oplus W$ for some subspaces V, W that are S_n-invariant. Let T be any numbering of s. Then

$$b_T(S^s) = b_T\left(\operatorname{span}_{T'}\{\vec{v}_{T'}\}\right) = \operatorname{span}\{\vec{v}_T\},$$

which is 1-dimensional. So $(b_T(v)) \oplus (b_T(w))$ is 1-dimensional, meaning that (WLOG) $b_T(v) = \operatorname{span}\{\vec{v}_T\}$. But V is S_n-invariant, so $b_T(V) \subset V$, giving $\vec{v}_T \in V$. But $S^s = \operatorname{span}_\sigma\{\sigma(\vec{v}_T)\}$, and thus $V = S^s$, as S^s is the smallest S_n-invariant subspace containing $\vec{v}_T$.

Next, if $s \neq t$, then (WLOG) s does not strictly dominate t. Then if T is any numbering of s, we get that $b_T(S^t) = 0$, but $b_T(S^t) \neq 0$. This means that $S^s \ncong S^t$ if $s \neq t$, as desired. So we have found all the irreducible representations of S_n. ∎

Example 2.4.14. Compute a character table for S_4. Recalling that the trace of a permutation matrix is the number of fixed points, we may calculate the characters of the 5 permutation representations of S_4.

	1 (1)	**6** (1 2)	**8** (1 2 3)	**6** (1 2 3 4)	**3** (1 2)(3 4)
$M(\ydiagram{4})$	1	1	1	1	1
$M\left(\ydiagram{3,1}\right)$	4	2	1	0	0
$M\left(\ydiagram{2,2}\right)$	6	2	0	0	2
$M\left(\ydiagram{2,1,1}\right)$	12	2	0	0	0
$M\left(\ydiagram{1,1,1,1}\right)$	24	0	0	0	0

To find the related irreducible representations, we first note that $S(\ydiagram{4}) = M(\ydiagram{4})$, as it the trivial representation. Next, note that

$$\left\langle M\left(\ydiagram{3,1}\right), S(\ydiagram{4})\right\rangle = \frac{1}{24}(4 + 2\cdot 6 + 1\cdot 8) = 1 \qquad \text{so} \qquad M\left(\ydiagram{3,1}\right) = S(\ydiagram{4}) \oplus R,$$

where $\chi_R = (3, 1, 0, -1, -1)$, which is irreducible. So since $S\left(\ydiagram{3,1}\right)$ is isomorphic to a subrepresentation of $M\left(\ydiagram{3,1}\right)$, it follows that $S\left(\ydiagram{3,1}\right) = R$. Next, observe that

$$M\left(\ydiagram{2,2}\right) \cong S(\ydiagram{4}) \oplus S\left(\ydiagram{3,1}\right) \oplus R \qquad \text{so} \qquad R \cong S\left(\ydiagram{2,2}\right),$$

with $\chi_R = (6, 2, 0, 0, 2) - (1, 1, 1, 1, 1) - (3, 1, 0, -1, -1) = (2, 0, -1, 0, 2)$. We do not proceed further, but all the associated irreducible representations may be found in this manner.

2.5 Commutators and dimension bounds

Remark 2.5.1. A finite group G is abelian iff all of its irreducible representations are 1-dimensional.

Remark 2.5.2. Let G be a group with $H \lhd G$ and G/H abelian. This means that there is a surjective homomorphism $q : G \to A$ such that A is abelian and $\ker(q) = H$. Then

$$q(xy) = q(x)q(y) = q(yx)\ \forall\ x, y \in G \qquad \text{so} \qquad q(xyx^{-1}y^{-1}) = 1.$$

This means that $xyx^{-1}y^{-1} \in H$ for all $x, y \in G$. The element $xyx^{-1}y^{-1}$ is called the *commutator* of x and y, and is sometimes written $[x, y]$.

Definition 2.5.3. Let G be a group. Define the *commutator subgroup* N of G to be the subgroup generated by all commutators of G. It is a normal subgroup of G, and is denoted by $[G, G]$.

From the above statements we may conclude that, for $H \leqslant G$, the group G/H is abelian iff H contains tho commutator subgroup of G.

Remark 2.5.4. Let N be the commutator subgroup of a finite group G, and $A = G/N$. Then every 1-dimensional representation $\rho : A \to GL(\mathbb{C})$ gives rise to a 1-dimensional representation $\rho \circ q : G \to GL(\mathbb{C})$, for $q : G \to A$ the quotient homomorphism.

Conversely, if $\tau : G \to GL(\mathbb{C})$ is a 1-dimensional representation, then since $GL(\mathbb{C})$ is abelian, $N \subset \ker(\tau)$. By the universal property of quotients, τ induces a homomorphism $\tilde{\tau} : G/N \to GL(\mathbb{C})$, which is a representation.

The above remark proves the following:

Theorem 2.5.5. The 1-dimensional representations of a finite group G are in a 1-1 correspondence with the 1-dimensional representations of G/N, where N is the commutator subgroup of G.

Here, G/N is called the *abelianization* of G.

Theorem 2.5.6. Let G be a finite group and $A \leqslant G$ abelian. Then any 1-dimensional representation of G has dimension less than or equal to $\#G/\#A$.

Proof: Let $\rho : G \to GL(V)$ be an irreducible representation of G. Let $\tau = \rho|_A$. Let $W \subset V$ be an A-invariant subspace of V. Then $\dim(W) = 1$. Let $W = \text{span}\{\vec{w}\}$, and let $W' = \text{span}\{[\rho(g)](\vec{w})\}$, where the span is over all $g \in G$. Then W' is G-invariant. But if g_1, g_2 are in the same coset of xA of A in G, then $g_1 = g_2 a$ for some $a \in A$, so $g_1(\vec{w}) = (g_2 a)(\vec{w}) = g_2(\lambda \vec{w})$ for some $\lambda \in \mathbb{C}$. So the dimension of W' can be no greater than the number of left cosets of A in G, which is $\#G/\#A$. Since V is irreducible (and if we choose $W' \neq 0$), then $W' = V$, and we are done. ■

Remark 2.5.7. Since the dihedral group D_n has an abelian (cyclic) subgroup of order n, it follows that every irreducible representation of D_n has dimension 1 or 2.

2.6 Induced representations

Let G_1, G_2 be finite groups. What are the irreducible representations of $G_1 \times G_2$? Let us first consider $\rho_i : G_i \to GL(V_i)$ as representations for $i = 1, 2$. Define $\rho : G_1 \times G_2 \to GL(V_1 \otimes V_2)$, for any $g_i \in G_i$ and $\vec{v} \in V_1$, $\vec{w} \in V_2$ by

$$[\rho(g_1, g_2)] \left(\sum \vec{v}_j \otimes \vec{w}_j \right) = \sum [\rho_1(g_1)](\vec{v}_j) \otimes [\rho_2(g_2)](\vec{w}_j)$$

This ρ is usually expressed as $\rho = \rho_1 \otimes \rho_2$. It is easy to see that ρ is a representation. It is also true that if ρ_1, ρ_2 are irreducible, then ρ is also irreducible. To see this, note that the character of ρ satisfies $\chi_\rho(g_1, g_2) = \chi_{\rho_1}(g_1)\chi_{\rho_2}(g_2)$, so

$$\begin{aligned}
\langle \chi_\rho, \chi_\rho \rangle &= \frac{1}{\#G_1 \cdot \#G_2} \left(\sum_{g_1, g_2} \chi_\rho(g_1, g_2) \overline{\chi_\rho(g_1, g_2)} \right) \\
&= \frac{1}{\#G_1} \cdot \frac{1}{\#G_2} \left(\sum_{g_1, g_2} \chi_{\rho_1}(g_1)\chi_{\rho_2}(g_2) \overline{\chi_{\rho_1}(g_1)\chi_{\rho_2}(g_2)} \right) \\
&= \frac{1}{\#G_1} \left(\sum_{g_1, g_2} \chi_{\rho_1}(g_1) \overline{\chi_{\rho_1}(g_1)} \right) \frac{1}{\#G_2} \left(\sum_{g_2} \chi_{\rho_2}(g_2) \overline{\chi_{\rho_2}(g_2)} \right) \\
&= \langle \chi_{\rho_1}, \chi_{\rho_1} \rangle \langle \chi_{\rho_2}, \chi_{\rho_2} \rangle \\
&= 1
\end{aligned}$$

We know that if G_i has a_i conjugacy classes, then $G_1 \times G_2$ has $a_1 a_2$ conjugacy classes. So $G_1 \times G_2$ has $a_1 a_2$ irreducible representations (up to isomorphism), and we have constructed $a_1 a_2$ of them. To see that $\rho_1 \otimes \rho_2 \not\cong \rho_1' \otimes \rho_2'$ if $(\rho_1, \rho_2) \not\cong (\rho_1', \rho_2')$, note that by a calculation similar to above,

$$\langle \chi_\rho, \chi_{\rho'} \rangle = \langle \chi_{\rho_1}, \chi_{\rho_1} \rangle \langle \chi_{\rho_2}, \chi_{\rho_2} \rangle$$

So if $\rho_i \not\cong \rho_i'$ for some d, then $\langle \chi_\rho, \chi_{\rho_i} \rangle = 0$, giving $\langle \chi_\rho, \chi_{\rho'} \rangle = 0$.

Definition 2.6.1. Let G be a finite group with $H \leqslant G$ and $\rho : H \to GL(V)$ a representation. Choose $g_1, \ldots, g_n \in G$ so that $g_1 H, \ldots, g_n H$ is a complete and distinct set of left cosets of H in G. Define the vector space

$$W = g_1 V_1 \oplus \cdots \oplus g_n V_n \qquad \text{where} \qquad g_i V_i = V$$

Define a homomorphism

$$\begin{array}{rcl} \mathrm{Ind}_H^G \rho = \tau : \quad G & \to & GL(W) \\ [\tau(g)](g_1\vec{v}_1 + \cdots + g_n\vec{v}_n) & = & g_{j1}[\rho(h_1)](\vec{v}_1) + \cdots + g_{jn}[\rho(h_n)](\vec{v}_n) \end{array}$$

where for each r, $gg_r = g_{jr}h_r$. A simple check shows that this is a representaton of G.

Example 2.6.2. Let G be any group and $H = (1)$. So $\rho : H \to GL(\mathbb{C})$ must be the trivial representation. Then $\mathrm{Ind}_H^G \rho$ is the left-regular representation. This is because we have $gg_r = g_{jr}$ for all r.

Example 2.6.3. Let G be any group and $H = G$. Then $\mathrm{Ind}_H^G \rho = \rho$.

Example 2.6.4. Let $G = \mathbb{Z}_4$ and $H = \langle 2 \rangle$. If $\rho : H \to GL_1(\mathbb{C})$ is trival, then $\tau = \mathrm{Ind}_H^G \rho$ is the permutation representation of G acting on left H-cosets by left-multiplication.

Example 2.6.5. Let $G = \mathbb{Z}_4$ and $H = \langle 2 \rangle$. Let $\rho : H \to GL_1(\mathbb{C})$ be the sign representation, i.e. $\rho(0) = 1$ and $\rho(2) = -1$. Let $\tau = \mathrm{Ind}_H^G \rho$. Note in general that $\dim(\mathrm{Ind}_H^G \rho) = [G : H] \dim(\rho)$, where $[G : H]$ is the index of H in G, or equivalently, the number of left cosets of H in G.

Here we have that $\dim(\tau) = 2$. A basis for $W = 0\mathbb{C} \oplus 1\mathbb{C}$ is $\{0(1), 1(1)\}$. This turns $\tau(n)$ into a matrix:

$$\begin{array}{l} \tau(0) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \\ [\tau(1)](0(1)) = (1+0)(1) = 1(1) \implies \tau(1) = \left(\begin{smallmatrix} 0 & - \\ 1 & - \end{smallmatrix}\right) \\ [\tau(1)](1(1)) = (1+1)(1) = (0)[\rho(2)](1) = 0(-1) = -0(1) \implies \tau(1) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \\ [\tau(2)](0(1)) = (2+0)(1) = (0)[\rho(2)](1) = 0(-1) = -0(1) \implies \tau(2) = \left(\begin{smallmatrix} -1 & - \\ 0 & - \end{smallmatrix}\right) \\ [\tau(2)](1(1)) = (2+1)(1) = (1)[\rho(2)](1) = 1(-1) = -1(1) \implies \tau(2) = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \end{array}$$

Similarly, $\tau(-1) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Notice that $\tau = \rho_1 \oplus \rho_2$, where ρ_1, ρ_2 are the 2 extensions of ρ to G.

Theorem 2.6.6. If $\tau = \mathrm{Ind}_H^G \rho$, then

$$\chi_\tau(g) = \sum_{g_i^{-1} g g_i \in H} \chi_\rho(g_i^{-1} g g_i) = \frac{1}{\#H} \sum_{x^{-1} g x \in H} \rho(x^{-1} g x)$$

Proof: Choose a basis $\{g_i\vec{v}_i\}$ for $\bigoplus g_i V_i$ and pick $g \in G$. For each i, $gg_i = g_i h_i$ for some $h_i \in H$ and index i. The trace of $\tau(g)$ is the sum of all the coefficients of $g_i\vec{v}_j$ in $[\tau(g)](g_i\vec{v}_j)$. If $g_i \neq g_j$, then the coefficient of $g_i\vec{v}_j$ in $[\tau(g)](g_i\vec{v}_j)$ is 0. But $g_j = g_i$ iff $h_i = g_i^{-1} g g_i$, so $[\tau(g)](g_i\vec{v}_j) = g_i[\rho(g_i^{-1} g g_i)](\vec{v}_j)$, thereby increasing the trace of $\tau(g)$ by the trace $\chi_\rho(g_i^{-1} g g_i)$ of $\rho(g_i^{-1} g g_i)$.

This establishes the first formula. The second follows immediately by group theory. ■

Example 2.6.7. Let $G = D_4$ and $H = \langle y \rangle$, where y has order 4. Let $\rho : H \to GL(\mathbb{C})$ be given by $\rho(y^a) = i^a$.How do we extend ρ to G, or what is $\mathrm{Ind}_H^G \rho = \tau$?

First define the space $W = \mathbb{C} \otimes x\mathbb{C}$, where $x \in G$ is a reflection, and we must have that $y(x\vec{v}) = (yx)\vec{v}$. But we also have that $(yx\vec{v}) = (xy^{-1})(\vec{v})$ by the properties of D_4. And

$$(xy^{-1})(\vec{v}) = x(y^{-1}\vec{v}) = x(-i\vec{v}) = -i(x\vec{v})$$

As an aside, we note that if the order of x was 3, then $w = \mathbb{C} \oplus x\mathbb{C} \oplus x^2\mathbb{C}$. Further, if x was in a different coset than y, then we would have to make a new definition. Back to the example, we now have that everything is forced, so

$$\tau(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \tau(y) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad \tau(y^2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \tau(y^3) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$\tau(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau(xy) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad \tau(xy^2) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \qquad \tau(xy^3) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

Therefore the character of τ is

$$\chi_\tau(1, y, y^2, y^3, x, xy, xy^2, xy^3) = (2, 0, -2, 0, 0, 0, 0, 0)$$

This is the same answer that the theorem above gives, which is good.

3 Group rings

3.1 Modules and rings

Definition 3.1.1. Let G be a group, not necessarily finite. The complex group ring of G is $\mathbb{C}[G] = \bigoplus_{g \in G} \mathbb{C}g$, where multiplication is defined by

$$\left(\sum_i a_i g_i\right)\left(\sum_j b_j g_j\right) = \sum_{i,j} a_i b_j g_i g_j$$

Then $\mathbb{C}[G]$ is a ring, commutative if and only if G is abelian.

Proposition 3.1.2. $\mathbb{C}[G]$, the vector space spanned by elements in G with multiplication defined in the usual associative way, is a commutative ring iff G is abelian.

Definition 3.1.3. Let R be a ring with unity. A *left R-module* is an abelian group M with operation $\cdot : R \times M \to M$ such that:

(i). $r \cdot (b + c) = r \cdot b + r \cdot c$
(ii). $r_1 \cdot (r_2 \cdot a) = (r_1 \cdot r_2) \cdot a$
(iii). $1 \cdot a = a$
(iv). $(r_1 + r_2) \cdot a = r_1 \cdot a + r_2 \cdot a$

for all $r, r_1, r_2 \in R$ and $a, b, c \in M$

Example 3.1.4. If R is a commutative field, then any R-module is a vector space over $\mathbb{R}$ with finite dimension. That is, they are *free modules.*

Example 3.1.5. $\mathbb{Z}$-modules are abelian groups with $n \cdot m = \underbrace{m \cdot m \cdots m}_{n \text{ times}}$. For example:

· $\mathbb{Z}^n$ is a $\mathbb{Z}$-module
· $\mathbb{Z}/n\mathbb{Z}$ is a $\mathbb{Z}$-module given by $mx = mx \pmod{n}$
· Any abelian group is a $\mathbb{Z}$-module
· Let R be any ring. If I is a left ideal of R, then I is an R-module

Remark 3.1.6. A left ideal is an R-submodule, so R-submodules may be factored using factorizations of ideals (into prime ideals). Begin by letting $\rho : G \to GL(V)$ be a representation of G. Then V is a $\mathbb{C}[G]$-module given by

$$\left(\sum_i a_i g_i\right)\vec{v} = \sum_i a_i \rho(g_i)[\vec{v}]$$

Given any left $\mathbb{C}[G]$-module, we may make a representation of G by $i : G \to \mathbb{C}[G]$ the inclusion map.

Example 3.1.7. $\rho(g)[\vec{v}] = g\vec{v}$, which is clearly in $GL(V)$, and clearly a homomorphism $\rho : G \to GL(V)$. Hence as a representation of G, V is naturally a $\mathbb{C}[G]$-module.

Example 3.1.8. Let $G = \mathbb{Z}_n$, and $\rho : G \to GL(\mathbb{C})$, with $\rho(1) = -1$. What is $\mathbb{C}[G]$?

Well, we must have that $\mathbb{C}[G] = \mathbb{C} \cdot 0 \oplus \mathbb{C} \cdot 1$. This means that $\mathbb{C}$ is a $\mathbb{C}[G]$-module, multiplication being

$$(c_1 \cdot 0 + c_2 \cdot 1) \cdot z = c_1 \cdot \rho(0) \cdot z + c_2 \cdot \rho(1) \cdot z = c_1 \cdot z - c_2 \cdot z$$

Example 3.1.9. Polynomial rings are $\mathbb{C}[t]$-modules. Evaluate $p(t) \in \mathbb{C}[t]$ at $z_0 \in \mathbb{C}$. For instance, we might have $e_{z_0}(p) = p(z_0)$. More generally, $\mathbb{C}[t] \to \mathbb{C}[G]$ by $\varphi : t \mapsto g \in G$. Clearly this map is onto, so $\varphi(p) = p(1)$, so

$$\mathbb{C}[G] \cong \mathbb{C}[t]/\ker(\varphi)$$

However, $\ker(\varphi) = \langle t^2 - 1 \rangle$. Hence this map is a linear transformation that is onto, so it is also injective, as $\mathbb{C}[t]/\ker(\varphi)$ is 2-dimensional, as is $\mathbb{C}[G]$, so the universal property of quotients says that φ gives a homomorphism

$$\mathbb{C}[t]/\langle t^2 - 1 \rangle \to \mathbb{C}[G]$$

So φ' is an isomorphism and $\mathbb{C}[G] \cong \mathbb{C}[t]/\langle t^2 - 1 \rangle$. By the Chinese remainder theorem, $\mathbb{C}[T]/\langle t^2 - 1 \rangle \cong \mathbb{C} \oplus \mathbb{C}$, so $\mathbb{C}[G] \cong \mathbb{C} \oplus \mathbb{C}$ as rings.

Definition 3.1.10. Let R be a ring with unity (not necessarily commutative). Let M, N be left R-modules. an *R-module homomorphism* from M to N is a function $f : M \to N$ such that $f(m_1 + m_2) = f(m_1)_f(m_2)$, and $f(\lambda m) = \lambda f(m)$ for all $m, m_1, m_2 \in M$ and $\lambda \in R$.

Note that if $R = \mathbb{C}[G]$, then an R-module homomorphism is exactly a morphism of representations. An *isohomprihsm* of R-modules is a homomorphism of R-modules that has an inverse homomorphism.

Definition 3.1.11. Let R be a ring and M an R-module. Define

$$\mathrm{End}_R(M) = (\text{the endomorphism ring of } M \text{ over } R) = \{R\text{-module homomorphisms } f : M \to M\}$$

The operations are pointwise function addition and composition. Note that an *endomorphism* of M is a homomorphism $M \to M$.

Note that Schur's lemma says that $\mathrm{End}_{\mathbb{C}[G]}(V) = \mathbb{C}$ if V is an irreducible representation of G.

Definition 3.1.12. Let M, N be R-modules. Define

$$\mathrm{Hom}_R(M, N) = \{f : M \to N \ : \ f \text{ is an } R\text{-module homomorphism}\}$$

Note that $\mathrm{Hom}_R(M, N)$ is an R-module via $(f_1 + f_2)(m) = f_1(m) + f_2(m)$ and $(cf_1)(m) = c \cdot f_1(m)$ for all $m \in R$.

Recall also that $V_1 \ncong V_2$ with V_1, V_2 irreducible, then $\mathrm{Hom}_{\mathbb{C}[G]}(V_1, V_2)$.

Note that if M, N are R-modules, then $M \oplus N$ is also an R-module with $(m_1, n_1) + (m_2, n_2) = (m_1 + m_2, n_1 + n_2)$ and $r(m, n) = (rm, rn)$. With this definition, the direct sum of $\mathbb{C}[G]$-modules corresponds to the direct sum of representations.

Definition 3.1.13. An R-module M is *simple* iff its only R-submodules are 0 and M. Note that simple $\mathbb{C}[G]$-modules correspond precisely to irreducible representations of G. Thus every $\mathbb{C}[G]$-module that is a finite-dimensional vector space is isomorphic to a direct sum of simple $\mathbb{C}[G]$-modules.

Definition 3.1.14. An R-module is called *semi-simple* iff for every submodule N of M there is a submodule $N^\perp$ of M such that $M = N \oplus N^\perp$.

Theorem 3.1.15. [Maschke]
Every $\mathbb{C}[G]$-module is semi-simple if G is finite.

Proof: Let V be a $\mathbb{C}[G]$-module, $W \subset V$ any submodule. We want to find a submodule $W' \subset V$ such that $V \cong W \oplus W'$ as $\mathbb{C}[G]$-modules.

Let $f : V \to W$ be some $\mathbb{C}$-linear projection, i.e. $f(\vec{w}) = \vec{w}$ for all $\vec{w} \in W$. If only f were a $\mathbb{C}[G]$-module homomorphism, then $W' = \ker(f)$ would give us what we want. However, f may not be a $\mathbb{C}[G]$-module homomorphism, so we have to make it work. Define $h : V \to W$ by

$$h(\vec{v}) = \frac{1}{\#G} \sum_g g^{-1} f(g\vec{v})$$

Then $h(\vec{v}) \in W$, beause $f(g\vec{v}) \in W$, and W is G-invariant. Notice also that $g(\vec{w}) = \vec{w}$ for all $\vec{w} \in W$, so h is a projection onto W. Now we can show that h is a $\mathbb{C}[G]$-module homomorphism, with

$$h\left(\sum_i a_i g_i\right) = \sum_i a_i h(g_i \vec{v}) = \frac{1}{\#G}\sum_{g,i} a_i g_i^{-1} f(g g_i \vec{v}) = \frac{1}{\#G}\sum_{i,g} a_i g_i (g_i')^{-1} f(g_i' \vec{v}) = \left(\sum_i a_i g_i\right) h(\vec{v})$$

where $g_i' = g g_i$. This is as desired. So $V \cong W \oplus \ker(h)$ as vector spaces, and W and $\ker(h)$ are both $\mathbb{C}[G]$-modules. So $V \cong W \oplus \ker(h)$ as $\mathbb{C}[G]$-modules. ■

Note we have already proved the above theorem for the $\mathbb{C}[G]$-module is a finite-dimensional $\mathbb{C}$-vector space.

Remark 3.1.16. Our next goal is to understand the ring $\mathbb{C}[G]$ better. As a left $\mathbb{C}[G]$-module, $\mathbb{C}[G]$ corresponds to the left-regular representation. So as a $\mathbb{C}[G]$-module, we have

$$\mathbb{C}[G] \cong n_1 V_1 \oplus \cdots \oplus n_r V_r$$

where $V_1, \ldots, V_r$ are the simple $\mathbb{C}[G]$-modules (corresponding to irreducible representations) with $n_i = \dim(V_i)$, where $\dim(V_i)$ means dimension as a $\mathbb{C}$-vector space.

Proposition 3.1.17. $\mathbb{C}[G]$ is isomorphic to $\mathrm{End}(\mathbb{C}[G])$ as a ring, if we make $\mathbb{C}[G]$ into a $\mathbb{C}[G]$-module by left-multiplication.

<u>*Proof:*</u> Let $\varphi : \mathbb{C}[G] \to \mathrm{End}(\mathbb{C}[G])$ be given by

$$\varphi\left(\sum_i a_i g_i\right) = \left[\sum_i b_i g_i \mapsto \left(\sum_i b_i g_i\right)\left(\sum_i a_i g_i^{-1}\right)\right]$$

It is easy to see that φ is well-defined and $\mathbb{C}[G]$-linear. It is also clearly injective, so we just need to check that φ is onto. So let $f \in \mathbb{C}[G]$ be an arbitrary endomorphism. We want to show that $f = \varphi(\sum a_i g_i)$ for some $\sum a_i g_i \in \mathbb{C}[G]$. Let $a = f(1) \in \mathbb{C}[G]$. Then for all $b \in \mathbb{C}[G]$, we have $f(b) = bf(1) = ba$, so we conclude that $f \in \mathrm{Im}(\varphi)$ as desired. ■

Our next question is: what kind of a ring is $\mathbb{C}[G]$? As a left $\mathbb{C}[G]$-module, $\mathbb{C}[G] \cong d_1 V_1 \oplus \cdots \oplus d_r V_r$, where $V_1, \ldots, V_r$ are the irreducible representations of G, up to isomorphism, and $d_i = \dim(V_i)$. Note the representations are actually $\rho_i : G \to GL(V_i)$.

Theorem 3.1.18. Let V be a simple $\mathbb{C}[G]$-module (that is, a $\mathbb{C}[G]$-module corresponding to an irreducible representation), and $n \in \mathbb{Z}_+$. Then $\mathrm{End}(nV) = M_n(\mathbb{C})$ for $nV = V \oplus \cdots \oplus V$, the sum of n copies of V.

<u>*Proof:*</u> Define $\varphi : M_n(\mathbb{C}) \to \mathrm{End}(nV)$ by

$$\varphi(M) = \left((\vec{v}_1, \ldots, \vec{v}_n) \mapsto M\begin{pmatrix}\vec{v}_1\\ \vdots \\ \vec{v}_n\end{pmatrix} \in nV\right)$$

It is easy to check that φ is a well-defined ring homomorphism, and that it is injective (as $\ker(\varphi) = 0$). For surjectivity, let $f \in \mathrm{End}(V)$. Write $f = (f_1, \ldots, f_n)$, where $f_i : nV \to V$ is a $\mathbb{C}[G]$-module homomorphism. By restricting f_i to the jth coordinate, we get a $\mathbb{C}[G]$-module homomorphism $f_{ij} : V \to V$. By Schur's lemma, $f_{ij} = a_{ij}\mathrm{id}_V$ for some $a_{ij} \in \mathbb{C}$, so

$$\begin{aligned}
f(\vec{v}_1, \ldots, \vec{v}_n) &= (f_1(\vec{v}_1, \ldots, \vec{v}_n), \ldots, f_n(\vec{v}_1, \ldots, \vec{v}_n)) \\
&= (f_{11}(\vec{v}_1) + \cdots + f_{in}(\vec{v}_n), \ldots, f_{n1}(\vec{v}_1) + \cdots + f_{nn}(\vec{v}_n)) \\
&= (a_{11}\vec{v}_1 + \cdots + a_{in}\vec{v}_n, \ldots, a_{n1}\vec{v}_1 + \cdots + a_{nn}\vec{v}_n) \\
&= M\begin{pmatrix}\vec{v}_1\\ \vdots \\ \vec{v}_n\end{pmatrix}
\end{aligned}$$

for $M_{ij} = a_{ij}$. So φ is surjective, and so is a bijection, and so is a ring homomorphism. ■

Remark 3.1.19. The above showed us that

$$\begin{aligned}\mathbb{C}[G] &\cong \mathrm{End}(\mathbb{C}[G]) \\ &\cong \mathrm{End}(d_1V_1 \oplus \cdots \oplus d_rV_r) \\ &\cong \mathrm{End}(d_1V_1) \oplus \cdots \oplus \mathrm{End}(d_rV_r) \\ &\cong M_{d_1}(\mathbb{C}) \oplus \cdots M_{d_r}(\mathbb{C})\end{aligned}$$

But what is this isomorphism described by? Let $\rho_i : G \to GL(V_i)$ be the irreducible representation corresponding to V_i. Define $\rho : \mathbb{C}[G] \to M_{d_1}(\mathbb{C}) \oplus \cdots \oplus M_{d_r}(C)$ by

$$\rho\left(\sum_i a_i g_i\right) = \left(\sum_i a_i \rho_1(g_i), \ldots, \sum_i a_i \rho_r(g_i)\right) = \sum_i a_i(\rho_1(g_i), \ldots, \rho_r(g_i))$$

It is clear that ρ is a ring homomorphism. It is injective because if $\rho(\sum a_i g_i) = 0$, then the representations ρ_i would be linearly dependent, meaning that their characters are linearly dependent, but they are orthonormal. Hence ρ is injective. Since we already know that $\mathbb{C}[G] \cong M_{d_1}(\mathbb{C}) \oplus \cdots \oplus M_{d_r}(\mathbb{C})$ as $\mathbb{C}$-vector spaces, we conclude that ρ is also surjective.

The above statement also means that the ith component $\rho_i : \mathbb{C}[G] \to M_{d_i}(\mathbb{C})$ of ρ is surjective. That is, every linear transgormation $f : \mathbb{C}^{d_i} \to \mathbb{C}^{d_i}$ can be realized as a linear combination of the matrices $\rho_i(g)$, for $g \in G$.

Remark 3.1.20. For $Z(V)$ the center of V,

$$Z(\mathbb{C}[G]) \cong Z(M_{d_1}(\mathbb{C}) \oplus \cdots \oplus M_{d_r}(\mathbb{C})) = \mathbb{C}I \oplus \cdots \oplus \mathbb{C}I \cong \mathbb{C} \oplus \cdots \oplus \mathbb{C}.$$

As a subring of $\mathbb{C}[G]$, the center is a complex vector space of dimension $r = \#$ of conjugacy classes of G.

Suppose that $(\sum a_i g_i)g = g(\sum a_i g_i)$ for all $g \in G$. This is the same as $\sum a_i g_i g = \sum a_i g g_i$. Thus if $\sum a_i g_i$ is in the center of $\mathbb{C}[G]$, then for every g, i, j such that $gg_i = g_j g$ we must have that $a_i = a_j$. But note that $gg_i = g_j g$ iff$g^{-1}g_j g = g_i$, so in order for $\sum a_i g_i$ to be in the center of $\mathbb{C}[G]$, we need $a_i = a_j$ if g_i and g_j are in the same conjugacy class.

Thus every element of the center of $\mathbb{C}[G]$ is in the span of the elements $\sum_{g \in C} g$, for C a conjugacy class of G. This span has dimension r, so it must equal the center of G.

3.2 Tensor products of modules over an arbitrary ring

Definition 3.2.1. Let R be a commutative ing with unity. Let M, N be left R-modules, and let B be a free R-module on the set $\{m \otimes n \; : \; m \in M, n \in N\}$. That is, $B = \{\sum a_i(m_i \otimes n_i) \; : \; m_i \in M, n_i \in N, a_i \in \mathbb{R}\}$. Define

$$Z = \left\{ \begin{array}{c} \text{R-linear combinations of} \\ \text{the following forms:} \end{array} \quad \begin{array}{c} (m_1 + m_2) \otimes n - m_1 \otimes n - m_2 \otimes n \\ m \otimes (n_1 + n_2) - m \otimes n_1 - m \otimes n_2 \\ (rm) \otimes n - r(m \otimes n) \\ m \otimes (rn) - r(m \otimes n) \end{array} \quad , \quad \begin{array}{c} r \in R \\ m, m_1, m_2 \in M \\ n, n_1, n_2 \in N \end{array} \right\}$$

Define $M \otimes_R N = B/Z$, where the R-module B/Z is the abelian group R/Z with the R-action $r(b+z) = rb+z$.

Example 3.2.2. Consider the folowing examples of tensor products of modules, where R is commutative:

- $M \otimes_R R \cong M$, because every element of $M \otimes_R R$ is equal to $m \otimes 1$ for some $m \in M$
- $R^n \otimes_R R^m \cong R^{nm}$, same as for vector spaces
- For $R = \mathbb{Z}$, $M = \mathbb{Q}$ and $N = \mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \; : \; a, b \in \mathbb{Z}\}$, we have that

$$M \otimes_R N = \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{2}] \cong \mathbb{Q}[\sqrt{2}] \cong \mathbb{Q}(\sqrt{2}).$$

Example 3.2.3. Let's consider the last example in greater detail. Define a map

$$\varphi : \mathbb{Q} \otimes \mathbb{Z}[\sqrt{2}] \to \mathbb{Q}(\sqrt{2}) \qquad \text{by} \qquad \sum_i q_i \otimes (a_i + b_i\sqrt{2}) \mapsto \sum_i q_i(a_i + b_i\sqrt{2}).$$

Then φ is a homomorphism of $\mathbb{Z}$-modules. To see that φ is surjective, note that $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \; : \; a, b \in \mathbb{Q}\}$, so every element in $\mathbb{Q}(\sqrt{2})$ may be written as $(a + b\sqrt{2})/c$, for $a, b, c \in \mathbb{Z}$ and $\varphi(1/c \otimes (a + b\sqrt{2})) = (a + b\sqrt{2})/c$.

To see that φ is injective, it is enough to show that $\mathbb{Q} \otimes \mathbb{Z}[\sqrt{2}]$ and $\mathbb{Q}(\sqrt{2})$ are both 2-dimensional vector spaces, and that φ is a $\mathbb{Q}$-linear transformation. All of these facts are easy to see except the dimension of $\mathbb{Q} \otimes \mathbb{Z}[\sqrt{2}]$, which is proven as follows. First, we note that any element of $\mathbb{Q} \otimes \mathbb{Z}[\sqrt{2}]$ is of the form

$$\sum_i q_i \otimes (a_i + b_i\sqrt{2}) = \sum_i a_i q_i \otimes 1 + \sum_i b_i q_i \otimes \sqrt{2} = \left(\sum_i a_i q_i\right)(1 \otimes 1) + \left(\sum_i b_i q_i\right)(1 \otimes \sqrt{2}).$$

So $\mathbb{Q} \otimes \mathbb{Z}[\sqrt{2}]$ is spanned by $1 \otimes 1$ and $1 \otimes \sqrt{2}$ as a $\mathbb{Q}$-vector space. Since φ is surjective and $\mathbb{Q}(\sqrt{2})$ has dimension 2, we see that $\dim(\mathbb{Q} \otimes \mathbb{Z}[\sqrt{2}]) = 2$, and φ is an isomorphism, as desired.

Definition 3.2.4. Let R be a ring with unity, but not necessarily commutative. Let T be a ring containing R as a subring, and let M be an R-module. Define B to be the free abelian group on $\{t \otimes m \; : \; t \in T, m \in M\}$, so $B = \{\sum t_i \otimes m_i \; : \; t_i \in T, m_i \in M\}$. Define

$$Z = \left\{ \begin{array}{ccc} \mathbb{Z}\text{-linear combinations of} & (t_1 + t_2) \otimes m - t_1 \otimes m - t_2 \otimes m & r \in R \\ \text{the following forms:} & t \otimes (m_1 + m_2) - t \otimes m_1 - t \otimes m_2 \quad , & t, t_1, t_2 \in T \\ & (tr) \otimes m - t \otimes (rm) & m, m_1, m_2 \in M \end{array} \right\}$$

Define $T \otimes_R M = B/Z$ as abelian groups with left T-module structure by $t(\sum t_i \otimes m_i) = \sum (tt_i) \otimes m_i$.

Example 3.2.5. Let $G = \mathbb{Z}_2$, $H = \{0\}$, $V = \mathbb{C}$, with the trivial action. in other words, V is the trivial one-dimensional representation of H, so V is a $\mathbb{C}[H]$-module. Consider $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$. It is a $\mathbb{C}$-vector space. An arbitrary element of $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ is of the form

$$\sum (a_i \cdot 0 + b_i \cdot 0) \otimes z_i = \sum (a_i z_i \cdot 0 + b_i z_i \cdot 1) \otimes 1 = (A \cdot 0 + B \cdot 1) \otimes 1 \in \operatorname{span}_{\mathbb{C}}\{0 \otimes 1, 1 \otimes 1\}.$$

Therefore $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V = \operatorname{span}\{0 \otimes 1, 1 \otimes 1\}$.

Consider the above example in more detail. Let $W = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V = \operatorname{span}\{0 \otimes 1, 1 \otimes 1\}$. Then W is a $\mathbb{C}[G]$-module, with

$$\begin{array}{ll} 0(0 \otimes 1) = (0 + 0) \otimes 1 = 0 \otimes 1 & \qquad 1(0 \otimes 1) = (1 + 0) \otimes 1 = 1 \otimes 1 \\ 0(1 \otimes 1) = (0 + 1) \otimes 1 = 1 \otimes 1 & \qquad 1(1 \otimes 1) = (1 + 1) \otimes 1 = 0 \otimes 1 \end{array}$$

So W corresponds to the 2-dimensional left-regular representation of G, as expected.

Example 3.2.6. Let $H = \mathbb{Z}_4$, $G = D_4$, $\rho(n) = i^n$, and $V = \mathbb{C}$, the corresponding $\mathbb{C}[H]$-module. Let's compute $W = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ as a $\mathbb{C}[G]$-module. This ought to be the representation of D_4 induced by the given representation of $\mathbb{Z}_4$. So an arbitrary element of $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ is of the form

$$\sum_i \left(\sum_j a_{ij} g_j\right) \otimes \vec{v}_i = \sum_i \left(\sum_j A_{ij} y^j\right) \otimes z_i + \sum_i \left(\sum_j B_{ij} x y^j\right) \otimes z'_i$$

where $B_{ij} = a_{ij}$ for j corresponding to xy^j, and $A_{ij} = a_{ij}$ for j corresponding to y^j.

$$
\begin{aligned}
&= \sum_i \left(\sum_j A_{ij} \otimes \rho(y^j) z_i \right) + \sum_i \left(\sum_j B_{ij} x \otimes \rho(y^j) z_i' \right) \\
&= A(1 \otimes 1) + B(x \otimes 1)
\end{aligned}
$$

for $A, B \in \mathbb{C}$. Next, let $\tau : G \to GL(\mathbb{C})$ be the representation associated to W. Then

$$
\begin{aligned}
[\tau(y^n)](1 \otimes 1) &= (y^n) \otimes 1 = y^n \otimes 1 = 1 \otimes \rho(y^n) = 1 \otimes i^n = i^n (1 \otimes 1) \\
[\tau(y^n)](x \otimes 1) &= y^n x \otimes 1 = xy^{-n} \otimes 1 = x \otimes i^{-n} = i^{-n}(x \otimes 1) \\
[\tau(xy^n)](1 \otimes 1) &= xy^n \otimes 1 = x \otimes i^n = i^n (x \otimes 1) \\
[\tau(xy^n)](x \otimes 1) &= xy^n x \otimes 1 = xxy^{-n} \otimes 1 = y^{-n} \otimes 1 = i^{-n}(1 \otimes 1)
\end{aligned}
$$

Theorem 3.2.7. Let G be a finite group, $H \subset G$ a subgroup, $\rho : H \to GL(V)$ a representation, and $\tau : G \to GL(W)$ the induced representation. Then τ corresponds to the $\mathbb{C}[G]$-module $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$.

Proof: To cnstruct τ, we choose $g_1, \ldots, g_r \in G$ so that $g_1 H, \ldots, g_r H$ is a complete and irredundant list of left H-cosets in G. Let $W = g_1 V \oplus \cdots \oplus g_r V$, with $[\tau(g)]$ defined by

$$[\tau(g)](g_i \vec{v}) = g_i [\rho(h)](\vec{v})$$

where $gg_i = g_j h$ for some $h \in H$. Define $W' = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ with $\mathbb{C}[G]$-module structure induced by multiplication on the left. We want to shown that W, W' are isomorphic, so we construct an isomorphism between them. Define a map

$$
\begin{array}{rcl}
\varphi : \ W & \to & W' \\
\sum g_i \vec{v}_i & \mapsto & \sum g_i \otimes \vec{v}_i
\end{array}
$$

This is an isomorphism of $\mathbb{C}[G]$-modules. It is straightforward to show that φ is a homomorphism of $\mathbb{C}[G]$-modules. It is also surjective because W' is spanned by elements of the form $g \otimes \vec{v}$, and $g_i h \otimes \vec{v} = g_i \otimes h(\vec{v})$. To show that φ is an isomorphism, it is enough to show that $\dim(W) = \dim(W')$, as φ is a $\mathbb{C}$-linear transformation. First note that $\dim(W) = [G : H] \dim(V)$. To compute $\dim(W')$, define a map

$$
\begin{array}{rcl}
T : \ \mathbb{C}[G] \otimes_{\mathbb{C}} V & \to & \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V \\
\alpha \otimes \vec{v} & \mapsto & \alpha \otimes \vec{v}
\end{array}
$$

and extend linearly. It is not hard to see this is a linear transformation and is surjective. The kernel of T is the span of elements of the form $gh \otimes \vec{v} - g \otimes h\vec{v}$. The kernel is thus also spanned by elements of the form $g_i h \otimes \vec{v}_j - g_i \otimes h\vec{v}_j$, where $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for V. In this list we may further neglect terms with $h = 1$. So $\ker(T)$ is spanned by $\{g_i h \otimes \vec{v}_j - g_i \otimes h\vec{v}_j\}$, where h ranges over the non-trivial elements of H. There are $r \cdot (\#H - 1) \cdot \dim(V)$ such elements, so $\ker(T)$ has dimension at most $[G : H](\#H - 1)\dim(V)$, and therefore

$$\dim(W') \geqslant \dim(V)\dim(\mathbb{C}[G]) - \dim(\ker(T)) = \dim(V)\#G - \dim(V)[G : H](\#H - 1) = \dim(V)[G : H]$$

Since φ maps W onto W', we conclude, by counting dimensions, that φ is an isomrphism. ■

3.3 Integrals and integers

A question that often arises is why are the numbers $\chi(g)$ always so "nice"? Mainly because they are sums of roots of unity. But how does this affect their niceness?

Definition 3.3.1. A commutative ring R is *Noetherian* iff every ideal of R is finitely generated.

Definition 3.3.2. Let R be a Noetherian ring contained in some bigger commutative ring T. An element $\alpha \in T$ is said to be *integral* over R iff $R[\alpha]$ is a finitely-generated R-module.

Example 3.3.3. Suppose $\frac{a}{b} \in \mathbb{Q}$ is a rational number. If $\frac{a}{b} \in \mathbb{Z}$, then $\mathbb{Z}[\frac{a}{b}] \cong \mathbb{Z}$ is a finitely-generated $\mathbb{Z}$-module, so $\frac{a}{b}$ is integral over $\mathbb{Z}$. Conversely, assume that $\frac{a}{b} \notin \mathbb{Z}$. Then there is some prime p that divides b but not a. For any finite set $x_1, \ldots, x_n \in \mathbb{Z}[\frac{a}{b}]$, there is some maximal power ℓ of p dividing the denominator of any x_i, so the $\mathbb{Z}$-module generated by $x_1, \ldots, x_n$ cannot contain $\frac{a^m}{b^m}$ for $m > \ell$.

This implies that a rational number is integral over $\mathbb{Z}$ iff it is an integer.

Theorem 3.3.4. Let T be a commutative ring, $R \subset T$ a subring, $\alpha \in T$ any element. Then α is integral over R iff there is some monic polynomial $f(x) \in R[x]$ with $f(\alpha) = 0$.

Proof: Suppose that α is integral over R. Then $R[\alpha]$ is a finitely-generated R-module. Consider the set $\{1, \alpha, \alpha^2, \ldots\} \subset R[\alpha]$. Since $R[\alpha]$ is finitely-generated, there is a finite subset $a_1, \ldots, a_n \in R[\alpha]$ such that every element $\gamma \in R[\alpha]$ can be written as $\gamma = r_1 a_1 + \cdots + r_n a_n$ for some $r_i \in R$. But each a_i is a polynomial in α with coefficients in R, so there is some $N \in \mathbb{Z}$ such that every a_i is an R-linear combination of $1, \alpha, \ldots, \alpha^{N-1}$. Then $\alpha^N = r_1 a_1 + \cdots + r_n a_n$ for some $r_i \in R$, so $\alpha^N = f(\alpha)$ for some polynomial $f(x) \in R[x]$ of degree $< N$. Thus $g(x) = x^n - f(x)$ is a monic polynomial with coefficients in R and $g(\alpha) = 0$.

Suppose that $f(\alpha) = 0$ for some monic polynomial $f(x) \in R[x]$. If $N = \deg(f)$, we get $\alpha^N = r_{N-1}\alpha^{N-1} + \cdots + r_0$. So $R[x]$ is generated by $1, \alpha, \alpha^2, \ldots, \alpha^{N-1}$. ■

Example 3.3.5. Which elements of $\mathbb{Q}[\sqrt{5}]$ are integral over $\mathbb{Z}$? First, we have $\mathbb{Q}[\sqrt{5}] = \{\frac{a}{b} + \frac{c\sqrt{5}}{d} : a, b, c, d \in \mathbb{Z}\}$. If $c = 0$, then we have a rational number, and we know the answer from above. So when $c \neq 0$, the minimal polynomial is

$$\left(x - \frac{a}{b} - \frac{c}{d}\sqrt{5}\right)\left(x - \frac{a}{b} + \frac{c}{d}\sqrt{5}\right) = x^2 - \left(\frac{2a}{b}\right)x + \left(\frac{a^2}{b^2} - \frac{5c^2}{d^2}\right) = x^2 - \left(\frac{2a}{b}\right)x + \frac{a^2d^2 - 5b^2c^2}{b^2d^2}$$

When does this polynomial have integer coefficients? Assume WLOG that $\gcd(a, b) = \gcd(c, d) = 1$. Then $\frac{2a}{b} \in \mathbb{Z}$, which implies that $b \mid 2$, so WLOG $b = 1$ or 2. If $b = 1$, then the polynomial is $x^2 - 2ax + \frac{a^2d^2 - 5c^2}{d^2}$, so $d^2 \mid 5$, meaning that $d = 1$. If $b = 2$, then the polynomial is $x^2 - ax + \frac{a^2d^2 - 20c^2}{4d^2}$. Then $4d^2 \mid (a^2d^2 - 20c^2)$, so $4 \mid a^2d^2$. Then $4 \mid d^2$, since $b = 2$ implies that a is odd, so d is even, meaning that $d = 2d'$ for some integer d'. Then

$$\frac{a^2d^2 - 20c^2}{4d^2} = \frac{4a^2(d')^2 - 20c^2}{16(d')^2} = \frac{a^2(d')^2 - 5c^2}{4(d')^2}.$$

Now, d' must be odd, since otherwise the numerator is odd and $4(d')^2$ is even, so our fraction is not an integer. Furthermore, we must have that $(d')^2 \mid (a^2(d')^2 - 5c^2)$, which implies that $(d')^2 \mid 5c^2$. But then $(d')^2 \mid 5$, so $d' = 1$ and $d = 2$. Thus

$$\frac{a}{b} + \frac{c}{d}\sqrt{5} = \frac{a + c\sqrt{5}}{2}.$$

But this expansion is not always an integer, so there is something more going on here. We also need a and c to be odd, so that $a^2(d')^2 - 5c^2$ can be even. So in general, any integral element of $\mathbb{Q}(\sqrt{5})$ must be of the form $x + y(\frac{1+\sqrt{5}}{2})$ for $x, y \in \mathbb{Z}$. All these elements are integral over $\mathbb{Z}$ because the minimal polynomial of $x + y(\frac{1+\sqrt{5}}{2})$ is

$$\left(t - \left(x + y\left(\frac{1+\sqrt{5}}{2}\right)\right)\right)\left(t - \left(x + y\left(\frac{1-\sqrt{5}}{2}\right)\right)\right) = t^2 - (2x - y)t + (x^2 + 2xy - 2y^2)$$

Theorem 3.3.6. Let T be a commutative ring, $R \subset T$ a subring. Assume that R, T are Noetherian. Then the set of elements of T that are integral over R is a subring of T.

Proof: This amounts to showing that the set of integral elements is non-empty and closed under addition and multiplication. So the set S of R-integral elements of T is non-empty (as it contains 1), therefore it suffices to show that it is closed under $+, -$, and $\cdot$. Thus let $x, y \in S$ be any elements. Then $x+y, x-y, xy$ are all elements of the ring $R[x,y]$, a finitely-generated R-module, generated by $\{x^i, y^j\}$, where i, j range over the same sets as those needed to ensure that $\{x^i\}$ generates $R[x]$ and $\{y^j\}$ generates $R[y]$. To conclude the proof, we need the following lemma:

Lemma 3.3.7. If R is a Noetherian ring and M is a finitely-generated R-module, then every R-submodule of M is also finitely-generated.

Proof: Suppose that $N \subset M$ is an R-submodule. Since M in finitely-generated (by $m_1, \ldots, m_n$), there is a surjective R-module homomorphism $\varphi : R^n \to M$ given by

$$\varphi(r_1, \ldots, r_n) = r_1 m_1 + \cdots + r_n m_n.$$

If we can show that $\varphi^{-1}(N) = \{\vec{v} \in R \ : \ \varphi(\vec{v}) \in N\}$ is a finitely-generated R-module, then N will also be finitely-generated by the images (under φ) of the generators for $\varphi^{-1}(N)$. Thus we may assume that $M = R^n$. If $n = 1$, then N is an ideal of R, so since R is Noetherian, N is finitely-generated. New proceed by induction on n. Define an R-module homomorphism $\pi : R^n \to R$ by $\pi(r_1, \ldots, r_n) = r_1$. Then $\ker(\pi) \cong R^{n-1}$ is a finitely-generated R-module. And $\mathrm{Im}(\pi)$ is also finitely-generated because it is a submodule of R. Furthermore, $\ker(\pi) \cong R^{n-1}$ is finitely-generated because it is a submodule of $\ker(\pi) \cong R^{n-1}$ and $\mathrm{Im}(\pi|_N)$ is finitely-generated because it is a submodule of R.

Thus, N can be finitely generated by the union of a set of generators for $\ker(\pi) \cap N$ and any finite set in N that maps via π to a finite generating set for $\mathrm{Im}(\pi|_N)$. ■

This also proves the theorem. ■

Remark 3.3.8. We know the eigenvalues of all representations are all roots of unity. So if G is finite, $g \in G$ any element, then $\chi(g)$ is always integral over $\mathbb{Z}$.

Theorem 3.3.9. Let G be a finite group and $\rho : G \to GL(V)$ an irreducible representation. Then $\dim(V) \mid \#G$.

Proof: For any conjugacy class C of G, let $e_C = \sum_{g \in C} g \in \mathbb{C}[G]$. Then e_C is in the center of $\mathbb{C}[G]$, and $\bigoplus_C e_C$ is a ring containing e_C that is a finitely-generated $\mathbb{Z}$-module, so e_C is integral over $\mathbb{Z}$. But $\mathbb{C}[G] \cong M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$, and its center is $\mathbb{C}\mathrm{id} \times \cdots \mathbb{C}\mathrm{id}$, and the isomorphism maps

$$\sum_i a_i g_i \to \left(\sum_i a_i \rho_1(g_i), \ldots, \sum_i a_i \rho_r(g_i) \right)$$

where $\rho_i : G \to GL(V_i)$ is the ith irreducible representation of G, up to isomorphism. So $\rho_i(e_C)$ is a scalar multiple of id, meaning that $\mathrm{trace}(\rho_i(e_C))/\dim(V_i)$ is integral over $\mathbb{Z}$. Thus, for any $\alpha = \sum a_i g_i$ in the center of $\mathbb{C}[G]$, we have that $\sum a_i \chi_\rho(g_i)/\dim(V)$ is integral over $\mathbb{Z}$ for any character χ_ρ of any irreducible representation $\rho : G \to GL(V)$. Moreover,

$$\mathrm{span}\{e_C\} = \left\{ \sum f(g_i) g_i \ : \ f \text{ is a class function} \right\}$$

where a class function is a function that is constant on conjugacy classes. So for any class function $f : G \to \mathbb{C}$, the sum $\sum f(g_i)\chi_\rho(g_i)/\dim(V)$ is integral over $\mathbb{Z}$ for any irreducible representation ρ. If we choose $f(g_i) = \chi_\rho(g_i^{-1})$, then

$$\sum_i \chi_\rho(g_i^{-1})\chi_\rho(g_i)/\dim(V) = \sum_i \frac{1}{\dim(V)} = \frac{\#G}{\dim(V)}$$

is integral over $\mathbb{Z}$. Therefore $\dim(V) = \dim(\rho) \mid \#G$. ■

3.4 Induced representations, part II

Definition 3.4.1. Let G be a finite group, $H \leqslant G$, and $f : H \to \mathbb{C}$ a class function of H. Define the *induced class function* on G by

$$[\mathrm{Ind}_H^G(f)](g) = \frac{1}{\#H} \sum_{\substack{t \in G \\ tgt^{-1} \in H}} f(tgt^{-1})$$

This is a class function on G. Also note that $\mathrm{Ind}_H^G(\chi_\rho) = \chi_{\mathrm{Ind}_H^G(\rho)}$.

Theorem 3.4.2. [Frobenius reciprocity]
Let ψ be a class function on H and φ a class function on G. Then

$$\langle \psi, \mathrm{Res}_H(\varphi)\rangle_H = \left\langle \mathrm{Ind}_H^G(\psi), \varphi \right\rangle_G$$

where $\mathrm{Res}_H(\varphi)$ is the restriction of φ to H.

Proof: WLOG we assume that ψ, φ are ireducible characters corresponding to irreducible representations τ, ρ, respectively. Then $\langle \psi, \mathrm{Res}_H(\varphi)\rangle_H$ = the number of copies of τ in the irreducible decomposition of the representation associated to $\mathrm{Res}_H(\varphi)$, and similarly for $\left\langle \mathrm{Ind}_H^G(\psi), \varphi \right\rangle_G$. If $\mu \cong \bigoplus_{i=1}^r a_i\rho_i$ for irreducible representations $\rho_i \not\cong \rho j$ if $i \neq j$, then $\dim(\hom(\rho_i, \mu)) = a_i$ (we actually proved this before, when we computed $\mathrm{End}(\mathbb{C}[G])$. So

$$\langle \psi, \mathrm{Res}_H(\varphi)\rangle_H = \dim(\hom_{\mathbb{C}[G]}(V, W)) \qquad \text{and} \qquad \left\langle \mathrm{Ind}_H^G(\psi), \varphi \right\rangle_G = \dim(\hom_{\mathbb{C}[G]}(\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V, W))$$

where $\tau : H \to GL(V)$ and $\rho : G \to GL(W)$. Define a linear transformation Λ of linear transformations

$$\Lambda : \mathrm{Hom}_{\mathbb{C}[H]}(V, W) \to \mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G] \otimes V, W) \qquad \text{by} \qquad \Lambda(T) = \{g \otimes \vec{v} \mapsto [\rho(g)](t(\vec{v}))\}$$

It is straightforward to check that Λ is well-defined and injective. For surjectivity, let $f : \mathbb{C}[G] \otimes V \to W$ be any homomorphism of $\mathbb{C}[G]$-modules. Then $f = \Lambda(f|_{1\otimes\vec{v}})$. So Λ is an isomorphism. ■

Remark 3.4.3. A natural question to ask now, is when is $\mathrm{Ind}_H^G(\rho)$ irreducible? Or, for $H, K \leqslant G$ where G is finite and $\rho : H \to GL(V)$ a representation of H, what is $\mathrm{Res}_K(\mathrm{Ind}_H^G(\rho))$?

Definition 3.4.4. A *double coset* of (H, K) in G is a coset of the form HgK for some $g \in G$. Note that for any $g_1, g_2 \in G$, either $Kg_1H = Kg_2H$ or $Kg_1H \cap Kg_2H = \emptyset$.

Theorem 3.4.5. Let $\{g_1, \ldots, g_n\}$ be a set of double coset representatives for (H, K) in G, and let $H_i = g_iHg_i^{-1} \cap K$. Define $\rho_i : H_i \to GL(V)$ by $\rho_i(x) = \rho(g_i^{-1}xg_i)$. Then

$$\mathrm{Res}_K(\mathrm{Ind}_H^G(\rho)) \cong \bigoplus_i \mathrm{Ind}_{H_i}^K(\rho_i).$$

Proof: First observe that

$$\begin{aligned} \mathrm{Res}_K(\mathrm{Ind}_H^G(\rho)) &= \mathrm{Ind}_H^G(\rho) && \text{as a } \mathbb{C}[K]\text{-module} \\ &= \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V && \text{as a } \mathbb{C}[K]\text{-module} \end{aligned}$$

We want to show that $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V \cong \bigoplus_i(\mathbb{C}[K] \otimes_{\mathbb{C}[H_i]} V)$ as $\mathbb{C}[K]$-modules. Define

$$\varphi : \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V \to \bigoplus_i(\mathbb{C}[K] \otimes_{\mathbb{C}[H_i]} V) \qquad \text{by} \qquad \varphi(kg_ih \otimes \vec{v}) = (0, \ldots, 0, k \otimes [\rho(h)](\vec{v}), 0, \ldots, 0)$$

where the non-zero entry is the ith coordinate. Extend linearly to complete the definition. To show that φ is well-defined, we need to show that if $k_1 g_i h_1 = k_2 g_i h_2$, then $k_1 \otimes [\rho(h_1)](\vec{v}) = k_2 \otimes [\rho(h_2)]\vec{v}$ for all $\vec{v} \in V$. First note that $k_1 g_i h_1 = k_2 g_i h_2$ implies that $k_1 = k_2(g_i h_2 h_1^{-1} g_i^{-1})$. Next observe that

$$\begin{aligned} k_1 \otimes [\rho(h_1)](\vec{v}) &= k_2(g_i h_2 h_1^{-1} g_i^{-1}) \otimes [\rho(h_1)](\vec{v}) \\ &= k_2 \otimes [\rho_i(g_i h_2 h_1^{-1} g_i^{-1})\rho_i(h_1)](\vec{v}) \\ &= k_2 \otimes [\rho(h_2 h_1^{-1})\rho(h_1)](\vec{v}) \\ &= k_2 \otimes [\rho(h_2)](\vec{v}) \end{aligned}$$

as desired. So φ is a well-defined homomorphism of $\mathbb{C}[K]$-modules. Now we need an inverse. So define

$$\psi : \bigoplus_i (\mathbb{C}[K] \otimes_{\mathbb{C}[H_i]} V) \to \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V \qquad \text{by} \qquad \psi(k_1 \otimes \vec{v}_1, \dots, k_n \otimes \vec{v}_n) = \sum_i k_i g_i \otimes \vec{v}_i.$$

To see that ψ is well-defined, we must show that if $g_i h g_i^{-1} \in H_i$, then the following three equalities hold:

$$\begin{gathered} \psi(k g_i h g_i^{-1} \otimes \vec{v}) = \psi(k \otimes [\rho_i(g_i h g_i^{-1})])(\vec{v}) \\ \psi(0, \dots, 0, k g_i h g_i^{-1} \otimes \vec{v}, 0, \dots,) = k g_i h \otimes \vec{v} = k g_i \otimes [\rho(h)](\vec{v}) \\ \psi(k \otimes [\rho_i(g_i h g_i^{-1})])(\vec{v}) = k g_i [\rho(h)](\vec{v}). \end{gathered}$$

To see that they indeed hold, we note that

$$\psi(\varphi(k g_i h \otimes \vec{v})) = \psi(0, \dots, 0, k \otimes [\rho(h)](\vec{v}), 0, \dots, 0) = k g_i \otimes [\rho(h)](\vec{v}) = k g_i h \otimes \vec{v}$$

and

$$\varphi(\psi(k_1 \otimes \vec{v}, \dots, k_n \otimes \vec{v}_n)) = \varphi\left(\sum_i k_i g_i \otimes \vec{v}\right) = (k_1 \otimes \vec{v}_1, \dots, k_n \otimes \vec{v}_n)$$

So φ, ψ are mutually inverse, which means that they are isomorphisms, as desired. ■

Theorem 3.4.6. [Mackey's irreducibility criterion]
Let G be a finite group, $H \leqslant G$, and $\rho : H \to GL(V)$ a representation. Then $\mathrm{Ind}_H^G(\rho)$ is irreducible iff:

1. ρ is irreducible, and
2. for all $g \in G - H$, $\left\langle \chi_{\rho^g}, \mathrm{Res}_{H_g}(\chi_\rho) \right\rangle_{H_g} = 0$

where $H_g = gHg^{-1} \cap H$, and $\rho^g : H_g \to GL(V)$ is given by $\rho^g(t) = \rho(g^{-1}tg)$.

Proof: Consider the following sequence of equivalent statements:

$$\begin{aligned} \mathrm{Ind}_H^G(\rho) \text{ is irreducible } \quad &\text{iff} \quad \left\langle \mathrm{Ind}_H^G(\chi_\rho), \mathrm{Ind}_H^G(\chi_\rho) \right\rangle_G = 1 \\ &\text{iff} \quad \left\langle \chi_\rho, \mathrm{Res}_H(\mathrm{Ind}_H^G(\chi_\rho) \right\rangle_H = 1 \\ &\text{iff} \quad \left\langle \chi_\rho, \sum_i \mathrm{Ind}_{H_i}^H(\chi_{\rho_i}) \right\rangle_H = 1 \\ &\text{iff} \quad \sum_i \langle \mathrm{Res}_{H_i}(\chi_\rho), \chi_{\rho_i} \rangle_{H_i} = 1 \\ &\text{iff} \quad \langle \mathrm{Res}_H(\chi_\rho), \chi_\rho \rangle_H = 1 \text{ and } \langle \mathrm{Res}_{H_i}(\chi_\rho), \chi_{\rho_i} \rangle_{H_i} = 0 \ \forall\ i \neq 1 \\ &\text{iff} \quad \langle \chi_\rho, \chi_\rho \rangle_H = 1 \text{ and } \langle \mathrm{Res}_{H_i}(\chi_\rho), \chi_{\rho_i} \rangle_{H_i} = 0 \ \forall\ i \neq 1 \\ &\text{iff} \quad \rho \text{ is irreducible and } \left\langle \mathrm{Res}_{H_g}(\chi_\rho), \chi_{\rho^g} \right\rangle_{H_g} = 0 \ \forall\ g \in G - H \end{aligned}$$

Note that for the third line we chose $g_1, \dots, g_n$ double coset representatives, and WLOG let $g_1 = 1$. ■

Corollary 3.4.7. If $H \lhd G$ and $\rho : G \to GL(V)$ is a representation, then $\mathrm{Ind}_H^G(\rho)$ is irreducible iff:

1. ρ is irreducible, and
2. $\langle \chi_{\rho^g}, \chi_\rho \rangle = 0$ for all $g \in G - H$

Proof: Immediate from Mackey. ■

Example 3.4.8. Let $G = D_4 = \langle x, y \ : \ x^2 = y^4 = 1, \ xy = y^{-1}x \rangle$, the dihedral group of order 4, and $\rho : H \to GL(\mathbb{C})$ be given by $\rho(y^a) = i^a$. Is $\mathrm{Ind}_H^G(\rho)$ irreducible?

Since $\dim(\rho) = 1$, ρ is irreducible. Any $g \in G - H$ is $g = xy^a$, so $gHg^{-1} = xHx$, so $Hg = Hx$, and $\rho^g = \rho^x$ for all such g. Further,

$$\rho^x(y^a) = \rho(xy^a x) = \rho(y^{-a}) = i^{-a},$$

so $\rho^x \not\cong \rho$. Thus, since ρ^x is also irreducible, $\langle \chi_{\rho^x}, \chi_\rho \rangle = 0$, so $\mathrm{Ind}_H^G(\rho)$ is irreducible by Mackey.

This completes the mathematical portion of the course.

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