Characterizations of the Chern characteristic classes

by

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Author's declaration

I hereby declare that I am the sole author of this research project. This is a true copy of the research project, including any required final revisions, as accepted by my examiners.

I understand that my research project may be made electronically available to the public.

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Abstract

Chern's characteristic class may be defined by several different ways, including algebraic topology, differential geometry, and sheaf theory. All these approaches are presented, with the main goal to show that even though the definitions lie in different spaces, they all satisfy the Chern class axioms and are isomorphic by various theorems. To reach this goal, strong background machinery is constructed, including the complex Grassmannian as a CW-complex, a detailed setup for the splitting principle, and a thorough proof of the Chern–Weil theorem.

Kopsavilkums

Černa harakteristiskās klases var definēt vairākos veidos, tostarp ar rīkiem no algebriskās topoloģijas, diferenciālās ģeometrijas un kūlu (jeb *sheaf*) teorijas. Galvenais projekta mērķis ir pasniegt šīs atšķirīgās definīcijas un pierādīt, ka apmierinot īpašās Černa klašu aksiomas, definīcijas dod vienādus objektus. Lai nonāktu līdz šim mērķim, dažādas struktūras un mehānismi tiek konstruēti, tostarp kompleksais Grasmaniāns, pamati šķelšanās principam un detalizēts pierādījums Černa–Veja teorēmai.

Acknowledgments

This project was made readable largely thanks to my supervisors Benoit Charbonneau and Spiro Karigiannis. Their readiness to help with all of my problems made this undertaking feasible and I have had lots of fun learning math with them. Lots of fun except for the last month, which was at times unbearable and overwhelming and made me want to quit. Thanks also to my fellow students whose knowledge and insight kept me going whenever I was stuck and whose good company kept me balanced whenever I was overworked.

Pateicības

Pateicos tēvam, mātei, ģimenei, Latvijai par gudrību, mīlestību un sapratni. Tie visi man deva prieku un atsvabinājumu no šīs reizēm bezjēdzīgās urbšanās. Dziļa pateicība arī Ainai Galējai-Dravniecei, ar kuras palīdzību varēju izglītoties. Visbeidzot, gudrības tikums:

Ai dzīvīte, ai dzīvīte,
Pie dzīvītes vajadzēja
Vieglu roku, vieglu kāju,
Laba, gudra padomiņa.

Priecājos un esmu pateicīgs par apstākļiem, kas man ļāva mācīties matemātiku. Zinu, ka matemātika nav vienīgais dzīvē, bet tā ir tik jautra un pacilājoša un saistoša un nekādīgi nevarētu bez tās iztikt.

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Contents

1 Background, motivation, and layout

Characteristic classes provide information to aid in distinguishing vector bundles on manifolds. The information encoded in the classes allows them to discern manifolds by their tangent bundles.

Characteristic classes were first introduced in the mid-1930s, independently by Eduard Stiefel and Hassler Whitney, although without most of the homological language, which was being developed concurrently (see Chapter IV in Part 3 of [Die89] for more history). The language of homology and Grassmannians was introduced to these ideas in 1942 by Lev Pontrjagin, even though he only considered tangent bundles. By 1945, Shiing-Shen Chern had applied Pontrjagin's ideas to complex vector bundles, using instead cohomology, but it was only in 1952 when this structure was applied to arbitrary vector bundles, by Wu Wen Tsün.

However, the language used within to describe characteristic classes had already been developed by this time. The cell structure of the Grassmannian described in §4.2 was first introduced by Charles Ehresmann in the early 1930s, using the structure of Schubert symbols and cells, defined by Hermann Schubert in 1879.

This paper will not proceed historically, but by necessity - when new structures need to be used, they will be defined, all subordinate to the goal of defining Chern classes from several different perspectives. Due to the wide approach taken, some sections are rooted firmly in a specific mathematical area, while others are abstract and wide-reaching. The three main subjects from which structures are taken are given in the diagram below. Due to the varied approach, a broader, theory independent (sometimes category theoretic), view is often taken to describe objects, which may be specialized to any one of the three subjects.

$$\textbf{topological} \xleftrightarrow{\text{Section 5.4}} \textbf{differential geometric} \xleftrightarrow{\text{Section 5.4}} \textbf{sheaf theoretic}$$

topological	**differential geometric**	**sheaf theoretic**	**theory independent**
Section 2.4	Section 2	Section 4.4	Section 2.4
Section 3.1	Section 4.3		Section 3.2
Section 4.1	Section 6.1		Section 5.2
Section 4.2			Section 5.3

A general knowledge of differential geometric and topological concepts is assumed, though the more complicated relevant ones will be reviewed in the succeeding section. Most of the geometric definitions come from [MS74], but notation is as in the (somewhat) more current [BT82] and [Huy05]. The proofs of several theorems are given in sketch forms, though sources for complete justifications are always given.

Note: Every manifold is assumed to be connected and smooth, unless otherwise noted.

2 Auxiliary objects

2.1 Vector bundles

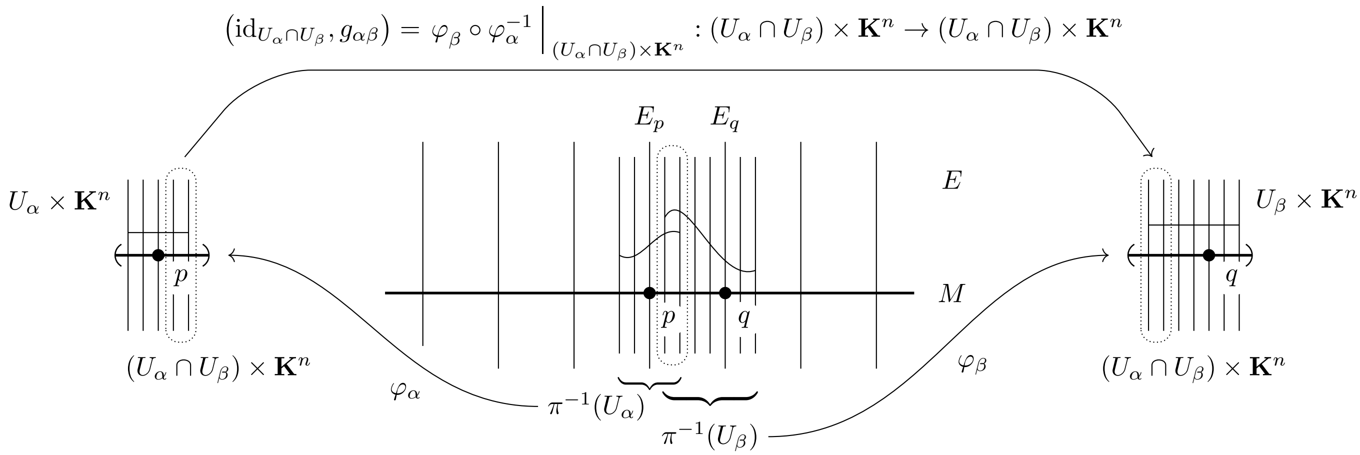
Definition 2.1.1. Let M be a topological manifold and $\mathbf{K}$ a field. A $\mathbf{K}$-*vector bundle over* M is a triple (E, M, π), usually denoted by just E, where

1. E is a topological space called the *total space*,
2. $\pi : E \to M$ is a continuous map called the *projection map*,
3. for each $p \in M$, the *fiber* $E_p := \pi^{-1}(p)$ has a vector space structure, and
4. for each $p \in M$, there exists a neighborhood $U \ni p$, a homeomorphism $\varphi : \pi^{-1}(U) \to U \times \mathbf{K}^n$ so that $v \in \pi^{-1}(\{p\}) \mapsto \varphi(p, v) \in \mathbf{K}^n$ is a linear isomorphism (this is called the *local triviality* condition).

Given a cover $\{U_\alpha\}_{\alpha\in I}$ of M, a pair $(U_\alpha, \varphi_\alpha)$ as in condition **4.** is termed a *local trivialization* for (E, M, π). Since $E_p \cong \mathbf{K}^n$ for all $p \in M$, the bundle E is called a $\mathbf{K}$-*vector bundle of rank* n.

Let φ_α, φ_β be homeomorphisms corresponding to local trivializations. Then the map $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a diffeomorphism of $(U_\alpha \cap U_\beta) \times \mathbf{K}^n$ onto itself such that $(\varphi_\beta \circ \varphi_\alpha^{-1})(p, v) = (p, g_{\alpha\beta}(p)v)$ for $p \in U_\alpha \cap U_\beta$ and

$g_{\alpha\beta}(p)$ a linear isomorphism of $\mathbf{K}^n$ onto itself. That is, we have $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbf{K})$. A general scenario is described in the diagram below.



The two curves above the thickened manifold M are "sections," or smooth functions $s : U \to E$ such that $(\pi \circ s)(u) = u$ for all $u \in U$. They are taken by φ_α and φ_β, respectively, to constant sections (that is, s is constant) on the bundles $U_\alpha \times \mathbf{K}^r$ and $U_\beta \times \mathbf{K}^r$, respectively. Hence the "transition function" $g_{\alpha\beta}$ takes one basis of a vector space to another basis, and so the collection of the $g_{\alpha\beta}$ is enough to uniquely describe the whole vector bundle. Formally, we have the following definition:

Definition 2.1.2. Let E be a vector bundle over a manifold M. In the context of the diagram above, given a cover $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ of M by local trivializations of E, the maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbf{K})$, for all $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$ are called the *transition functions* for this cover by trivializations.

In this text, the field $\mathbf{K}$ is always $\mathbf{R}$ or $\mathbf{C}$. Moreover, the fiber E_p may have more than just the structure of a vector space, for example that of an algebra or a graded ring.

Often the expression "$\pi : E \to M$" or "$E \xrightarrow{\pi} M$" is used for $(E, M\pi)$, or when the context supplies the details, we just write E. There is no universal convention for which symbol to use when referencing a vector bundle, though using just E is widely accepted. Older texts (such as [MS74]) refer to the triple (E, M, π) by another symbol ξ, some write only the fiber E_p, or only the projection map π. Whichever notation is used, some information is hidden.

Next we introduce some special bundles.

Definition 2.1.3.

· *trivial bundle*: Given a topological space M, the trivial bundle $\underline{\mathbf{K}}^n$ over M is a $\mathbf{K}$-vector bundle of rank n with total space $M \times \mathbf{K}^n$ and projection map $\pi(p, v) = p$. The local triviality condition is satisfied by $U = M$ and $\varphi_U = \mathrm{id}_{\pi^{-1}(U)}$ for all $p \in M$.

· *subbundle*: Given a vector bundle $\pi : E \to M$, a subbundle of E is a vector bundle $\tau : F \to M$ such that F is a submanifold of E and the fiber $F_p = F \cap E_p$ has the vector space structure of a subspace of E_p. Equivalently, $\pi|_F : F \to M$ is the subbundle.

· *tangent bundle*: Given an n-dimensional manifold M, the tangent bundle TM is a real rank n vector bundle given by $TM = \bigsqcup_{p \in M} T_pM$, for T_pM the tangent space to M at p. The tangent space is the fiber, so $(TM)_p = T_pM$.

· *dual vector bundle*: Given a $\mathbf{K}$-vector bundle E of rank n over M, the dual vector bundle E^* of E is the vector bundle whose fibers are the dual vector spaces to the fibers of E. That is, $E_p^* = (E_p)^* = \mathrm{Hom}(E_p, \mathbf{K})$.

Definition 2.1.4. Let E be a $\mathbf{K}$-vector bundle of rank n over M and $f : N \to M$ a continuous map. Let $F = \{(q, e) \in N \times E \ : \ f(q) = \pi(e)\}$ be the total space of a bundle $\tau : F \to N$ with projection map $\tau(q, e) = q$. Then F is called the *pullback bundle* or *induced bundle* of E by f. This relationship is denoted by $F = f^*E$, and represented as in the commutative diagram below, with an induced map of total spaces $\psi(q, e) = e$.

$$\begin{tikzcd} F \arrow[r, "\psi"] \arrow[d, "\tau"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "f"] & M \end{tikzcd} \tag{2.1.1}$$

The pullback bundle is indeed a bundle, as the fiber in F over q, where $f(q) = p$, is given by $F_q = E_p$, and the triviality condition around $q \in N$ is satisfied by $f^{-1}(U_p)$, with $\varphi_F(q, e) = (q, \varphi_E(p, e))$, so $\varphi_F = f^*\varphi_E$.

Definition 2.1.5. Let $E \xrightarrow{\pi} M$ be a $\mathbf{K}$-vector bundle of rank n and $F \xrightarrow{\tau} N$ a $\mathbf{K}$-vector bundle of rank m. A continuous map $\psi : F \to E$ that takes every fiber $\tau^{-1}(\{q\})$ linearly isomorphically to another fiber $\pi^{-1}(\{p\})$ is called a *bundle map*.

Given a bundle map $\psi : F \to E$, a map between base spaces $f : N \to M$ may be defined by letting $f(q) = p$ whenever $\psi(\tau^{-1}(\{q\})) = \pi^{-1}(\{p\})$, so that the diagram (2.1.1), with these new definitions, still commutes. Hence every bundle map induces a pullback bundle.

Definition 2.1.6. Let E be a rank n vector bundle over M with transition functions $g_{\alpha\beta}$, and F a rank m vector bundle also over M with transition functions $h_{\alpha\beta}$.

· *direct sum of vector bundles*: The direct sum, or Whitney sum, of E and F is the bundle $E \oplus F$. If $\pi_1 : E \oplus F \to E$ and $\pi_2 : E \oplus F \to F$ are bundle maps, then the transition functions of $E \oplus F$ are $g_{\alpha\beta} \oplus h_{\alpha\beta} := \begin{bmatrix} \pi_1^* g_{\alpha\beta} & 0 \\ 0 & \pi_2^* h_{\alpha\beta} \end{bmatrix}$.

· *tensor product of vector bundles*: The tensor product of E and F is the bundle $E \otimes F$. The transition functions of $E \otimes F$ are $\pi_1^* g_{\alpha\beta} \otimes \pi_2^* h_{\alpha\beta}$.

· *exterior product of vector bundles*: The exterior product, or wedge product, of E and F is the bundle $E \bigwedge F$. The transition functions of $E \bigwedge F$ are the 2-forms $g_{\alpha\beta} \wedge h_{\alpha\beta}$.

· *underlying real vector bundle*: Given a complex rank n vector bundle E over M, the underlying real vector bundle $E^{\mathbf{R}}$ of E is the real rank $2n$ vector bundle over M whose fibers $E_p^{\mathbf{R}}$ are the associated real vector spaces (given by the restriction of scalars) of E_p.

· *complexification of a vector bundle*: Given a real rank n vector bundle E over M, the complexification of E is the complex rank n vector bundle $E \otimes \underline{\mathbf{C}}$ with fibers $E_p \otimes_{\mathbf{R}} \mathbf{C}$.

Note that the complexification of the underlying real bundle of E is bundle isomorphic to the original bundle E.

Definition 2.1.7. Let M be a differentiable manifold of dimension n. The *complexified tangent bundle* $\pi : T_{\mathbf{C}}M \to M$ is the vector bundle $TM \otimes \underline{\mathbf{C}}$. Equivalently, it consists of

1. the total space $T_{\mathbf{C}}M = \{(p, v_p) \ : \ p \in M, v_p \in T_pM \otimes \mathbf{C}\}$,
2. the projection map $\pi : T_{\mathbf{C}}M \to M$ given by $\pi(p, v_p) = p$,
3. fibers $E_p \cong \mathbf{C}^n$, with
4. local triviality satisfied by induced coordinate charts $U \otimes_{\mathbf{R}} \mathbf{C}$ and induced homeomorphisms $\varphi \otimes \mathbf{C}$.

The *complexified cotangent bundle* $T^*_{\mathbf{C}}M = (T_{\mathbf{C}}M)^* = (T^*M) \otimes \mathbf{C}$ is a vector bundle whose fibers are the dual spaces, so $(T^*_{\mathbf{C}}M)_p = (T_{\mathbf{C}}M)^*_p$.

Note that the complexified tangent bundle is the complexification of the tangent bundle.

2.2 Grassmannian manifolds

In this section we construct complex finite dimensional and infinite dimensional Grassmannian manifolds, since characteristic classes are often first defined on vector bundles over these manifolds. When considering real instead of complex vector bundles, a very simlar analysis may be used.

Definition 2.2.1. Let V be a complex vector space. The *(complex) Grassmannian manifold* $G_k(V)$ is the set of all k-dimensional vector subspaces of V.

Most of the analysis will center around $V = \mathbf{C}^n$ and $V = \mathbf{C}^\infty$. To describe the topology of the Grassmannian, we first introduce the *(complex) Stiefel manifold* $V_k(\mathbf{C}^n) \subset (\mathbf{C}^n)^k$, defined as the set of all ordered k-frames in $\mathbf{C}^n$ (that is, ordered sets of k linearly independent vectors in $\mathbf{C}^n$). The Stiefel manifold is an open subset of $\mathbf{C}^{nk}$ in the Euclidean topology. Let $\sim$ be the equivalence relation on $V_k(\mathbf{C}^n)$ such that $(x_1, \dots, x_k) \sim (y_1, \dots, y_k)$ if and only if $\operatorname{span}\{x_1, \dots, x_k\} = \operatorname{span}\{y_1, \dots, y_k\}$. It follows directly that

$$G_k(\mathbf{C}^n) \cong V_k(\mathbf{C}^n)/\sim, \tag{2.2.1}$$

and we give the Grassmannian the quotient topology of $V_k(\mathbf{C}^n)$. Given this topology, $G_k(\mathbf{C}^n)$ turns out to be a compact manifold.

Proposition 2.2.2. (adapted from Lemma 5.1 in [MS74]) The space $G_k(\mathbf{C}^n)$ is a compact topological manifold.

Proof: First we show this space is Hausdorff (that is, satisfies the conditions of the T_2 axiom). However, instead of showing that $X, Y \in G_k(\mathbf{C}^n)$ have disjoint neighborhoods, we show that there exists a continuous function $f : G_k(\mathbf{C}^n) \to [0,1]$ such that $f(X) = 0$ and $f(Y) = 1$. The existence of such a function satisfies the conditions of the $T_{2^1/_2}$ axiom, from which the Hausdorff property follows immediately. So let $(x_1, \dots, x_k)$ be an ordered basis for X, and define a map $\varphi = \varphi_2 \circ \varphi_1$ by

$$\begin{array}{rrcl} \varphi_1 : & V_k(\mathbf{C}^n) & \to & \bigwedge^k \times \bigwedge^k \\ & (y_1, \dots, y_k) & \mapsto & (x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_k) \end{array}$$

and

$$\begin{array}{rrcl} \varphi_2 : & \bigwedge^k \times \bigwedge^k & \to & \mathbf{R} \\ & (\alpha, \beta) & \mapsto & \min\left\{ \left| \dfrac{\alpha}{|\alpha|} + \dfrac{\beta}{|\beta|} \right|, \left| \dfrac{\alpha}{|\alpha|} - \dfrac{\beta}{|\beta|} \right| \right\}. \end{array}$$

The composition of the two maps is independent of the choice of basis $(x_1, \dots, x_k)$ made at the beginning because any other basis $(x_1', \dots, x_k')$ is related to the first basis through the wedge product, as $x_1' \wedge \cdots \wedge x_k' = \lambda x_1 \wedge \cdots \wedge x_k$ for some scalar λ. In fact, λ is the determinant of the change of basis matrix between these two bases. The arguments are normalized by φ_2 because λ has no effect on the image, up to a sign. Taking the minimum of the two values takes away this effect, and so gives a well-defined map when passing to the quotient (in the context of (2.2.2) below). The composition is continuous as the operations used are all continuous. Then by the universal property of quotients, the map $\widehat{\varphi}$ is continuous and the diagram

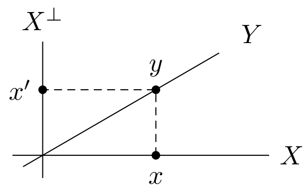
$$\begin{tikzcd} V_k(\mathbf{C}^n) \arrow[r, "\varphi_1"'] \arrow[rr, bend left, "\varphi"] \arrow[d] & \bigwedge^k \times \bigwedge^k \arrow[r, "\varphi_2"'] & \mathbf{R} \\ G_k(\mathbf{C}^n) \arrow[urr, dashed, "\widehat{\varphi}"] & & \end{tikzcd} \tag{2.2.2}$$

commutes. The image of φ in $\mathbf{R}$ is in $[0, \sqrt{2}]$ by the definition of φ_2, and hence is the same for $\widehat{\varphi}$. Set $f = \widehat{\varphi}/c_Y$, for an appropriate normalizing constant $c_Y \in [1, \sqrt{2}]$ (note that $c_Y = \sqrt{2}$ when $Y \subset X^\perp$). The map f fulfills the conditions of the $T_{2^1/_2}$ axiom, so $G_k(\mathbf{C}^n)$ is Hausdorff.

Next we show that $G_k(\mathbf{C}^n)$ is locally Euclidean of dimension $2k(n-k)$, that is, that every $X \in G_k(\mathbf{C}^n)$ has an open neighborhood $U_k \cong \mathbf{C}^{k(n-k)}$. Let $U_X = \{Y \in G_k(\mathbf{C}^n) \ : \ Y \cap X^\perp = \{0\}\}$. Then for each $Y \in U_X$, $Y \oplus X^\perp = \mathbf{C}^n$. Define $T_Y : X \to X^\perp$ for every $Y \in U_X$ by setting

$$T_Y(x) = \mathrm{Proj}_{X^\perp} \left(\mathrm{Proj}_X|_Y\right)^{-1}(x).$$

Note whenever $Y \cap X^\perp = \{0\}$, there is a unique $y \in Y$ such that $x = \mathrm{Proj}_X(y)$, so T_Y is well-defined. The decomposition $Y \oplus X^\perp = \mathbf{C}^n$ for any $Y \in U_X$ is given by $x = y - x'$, and is described in the diagram below.



(2.2.3)

There is a continuous bijection between $Y \in U_X$ and linear transformations from X to $X^\perp$. Let $(x_1, \ldots, x_k)$ be an orthonormal basis of X and $(x'_1, \ldots, x'_{n-k})$ be an orthonormal basis of $X^\perp$. Then the bijection is given by the composition of maps from the diagram

$$\begin{tikzcd}
U_X \arrow[rr, bend right, "Y \mapsto \left(x \mapsto \mathrm{Proj}_{X^\perp} \left(\mathrm{Proj}_X|_Y\right)^{-1}(x)\right)"'] & \pi^{-1}(U_X) \arrow[l, "\pi"'] & \mathrm{Hom}(X, X^\perp). \arrow[l, "{\left[\, x_1+Tx_1 \ \cdots \ x_k+Tx_k \,\right] \longleftarrow\!\shortmid T}"']
\end{tikzcd} \tag{2.2.4}$$

The continuity of $\mathrm{Hom}(X, X^\perp) \to U_X$ is clear, but the reverse is not. To show $Y \mapsto T_Y$ is continuous, we go through several intermediate steps. First, we restrict to orthonormal k-frames in $\pi^{-1}(U_X)$, and show that on that subspace $(y_1, \ldots, y_k) \mapsto T_{\mathrm{span}\{y_1,\ldots,y_k\}}$ is continuous. Let P be the $k \times k$ matrix representing $\mathrm{Proj}_X|_{\mathrm{span}\{y_1,\ldots,y_k\}}$. Note that $P_{ij} = h(y_i, x_j)$, where h is the standard Hermitian inner product, given by $h(z, w) = \sum_{j=1}^n z^j \overline{w}^j$ for $z = (z^1, \ldots, z^n)$ and $w = (w^1, \ldots, w^n)$. So the map $(y_1, \ldots, y_k) \mapsto P$ is continuous.

Recall that the entries of the matrix P^{-1} are rational functions in entries of the matrix P. It follows immediately that the map $(y_1, \ldots, y_k) \mapsto (P)^{-1}$ is continuous. The formula for $(\mathrm{Proj}_X|_{\mathrm{span}\{y_1,\ldots,y_k\}})^{-1}$ is $\sum_{i,j=1}^k (P^{-1})_{ij} h(\ \cdot\ , x_j) y_i$. Finally, the formula for $T_{\mathrm{span}\{y_1,\ldots,y_k\}}$ is then given by

$$\sum_{\ell=1}^{n-k} \sum_{i,j=1}^{k} h(y_i, x'_\ell)(P^{-1})_{ij} h(\ \cdot\ , x_j) x'_\ell,$$

and $(y_1, \ldots, y_k) \mapsto T_{\mathrm{span}\{y_1,\ldots,y_k\}}$ is continuous in $(y_1, \ldots, y_k)$. However, all this was done with an orthonormal basis, so we have only showed continuity on the set of orthonormal k-frames in $\pi^{-1}(U_X)$. This is enough, as U_X has the quotient topology of $V_k/\sim$, for $\sim$ as in equation (2.2.1). Hence we have continuity on all of U_X, and since $X^\perp \cong \mathbf{C}^{n-k}$, it now follows that

$$U_X \cong \mathrm{Hom}(X, X^\perp) \cong \mathrm{Hom}(\mathbf{C}^k, \mathbf{C}^{n-k}) \cong \mathbf{C}^{k(n-k)}.$$

Therefore $G_k(\mathbf{C}^n)$ is locally Euclidean. The Grassmannian $G_k(\mathbf{C}^n)$ is second-countable because $V_k(\mathbf{C}^n)$ is second-countable, and quotienting by an equivalence relation retains second-countability. To see that the Stiefel manifold is second-countable, note that there are countably many tuples $(a_1 + ib_1, \ldots, a_k + ib_k)$, for all a_j and b_j rational. The collection of balls of all positive rational radii centered at these points forms a countable base of topology for $V_k(\mathbf{C}^n)$. Hence $G_k(\mathbf{C}^n)$ is a topological manifold of complex dimension $k(n-k)$.

For compactness, consider the *orthonormal (complex) Stiefel manifold* $V_k^o(\mathbf{C}^n) \subset (\mathbf{C}^n)^k$, defined as the set of all orthonormal ordered k-frames. That is, the set of ordered sets of k orthonormal vectors in $\mathbf{C}^n$, with respect to the standard Hermitian inner product. Note that for any pair of $V, W \in V_k^o(\mathbf{C}^n)$, there is a unitary transformation relating V to W. Indeed, let $V = (v_1, \ldots, v_k)$ and $W = (w_1, \ldots, w_k)$, with $E = (e_1, \ldots, e_n)$ the standard ordered basis. Let $A, B \in U(n, \mathbf{C})$ be unitary matrices that take V, W, respectively, to $(e_1, \ldots, e_k)$. Then

$$V = A^{-1}BW,$$

and $A^{-1}B$ is still unitary. Thus the action is transitive. Applying a unitary transformation to any $V \in V_k^o(\mathbf{C}^n)$, results in some other $V' \in V_k^o(\mathbf{C}^n)$, since the transformation rotates the basis vectors orthogonally. A stabilizer of E is of the form

$$S \;=\; \begin{array}{cl} \left.\begin{bmatrix} 1 & & 0 & 0 \\ & \ddots & & \\ 0 & & 1 & 0 \\ 0 & & 0 & * \end{bmatrix}\right. & \begin{array}{l} \left.\vphantom{\begin{matrix}1\\ \ddots\\ 0\end{matrix}}\right\} k \text{ rows} \\ \} \; n-k \text{ rows} \end{array} \end{array},$$

so the stabilizer subgroup of $V_k^o(\mathbf{C}^n)$ is $U(n-k, \mathbf{C})$. Hence

$$V_k^o(\mathbf{C}^n) \cong U(n, \mathbf{C})/U(n-k, \mathbf{C}). \tag{2.2.5}$$

The space $U(n, \mathbf{C})$ is compact and connected. The quotient of a compact space is still compact, so $V_k^o(\mathbf{C}^n)$ is compact. Finally, take the composition $\iota : V_k^o(\mathbf{C}^n) \hookrightarrow V_k(\mathbf{C}^n) \to G_k(\mathbf{C}^n)$ that takes an ordered k-frame to the subspace it spans to get that $\iota(V_k^o(\mathbf{C}^n)) = G_k(\mathbf{C}^n)$ is compact. ■

In fact, equation (2.2.1) now essentially says that $G_k(\mathbf{C}^n) \cong V_k^o(\mathbf{C}^n)/U(k, \mathbf{C})$, so applying equation (2.2.5) we get that

$$G_k(\mathbf{C}^n) \cong U(n, \mathbf{C})/\big(U(k, \mathbf{C}) \times U(n-k, \mathbf{C})\big).$$

Knowing that $\dim(U(n, \mathbf{C})) = n^2$, we immediately get that $\dim(G_k(\mathbf{C}^n)) = 2k(n-k)$.

The finite-dimensional Grassmannian is used to motivate and define a more important space, which results from taking the direct limit of $G_k(\mathbf{C}^n)$ as $n \to \infty$. This new space is not a topological manifold, since locally it does not look like $\mathbf{C}^n$ for any finite n. Formally, let $\mathbf{C}^\infty := \varinjlim_n [\mathbf{C}^n] = \bigoplus_{n \geqslant 1} \mathbf{C}$.

Definition 2.2.3. The *infinite (complex) Grassmannian manifold* $Gr(k) = G_k(\mathbf{C}^\infty)$ is the set of all k-dimensional vector subspaces of $\mathbf{C}^\infty$.

Writing $G_k(\mathbf{C}^\infty)$ as the union of $G_k(\mathbf{C}^n)$ for all $n \geqslant k$ does not explain the topology on this space, so use the direct limit construction. Take the natural inclusion maps $\iota_{nm} : G_k(\mathbf{C}^n) \to G_k(\mathbf{C}^m)$ for all $m \geqslant n \in \mathbf{Z}_{\geqslant 1}$ and define an equivalence relation $\sim$ such that $X \in G_k(\mathbf{C}^n) \sim Y \in G_k(\mathbf{C}^m)$ if and only if $\iota_{n\ell}(X) = \iota_{m\ell}(Y)$, for $\ell = \max\{n, m\}$. Then

$$G_k(\mathbf{C}^\infty) = \varinjlim_{n \geqslant k} [G_k(\mathbf{C}^n)] := \bigsqcup_{n \geqslant k} G_k(\mathbf{C}^n) \Big/ \sim .$$

From this construction, it follows that $U \subset G_k(\mathbf{C}^\infty)$ is open if and only if $U \cap G_k(\mathbf{C}^n)$ is open for all $n \geqslant k$.

2.3 Tautological bundles

There is a natural type of bundle associated to the Grassmannian, for which $G_k(\mathbf{C}^n)$ is the base space.

Definition 2.3.1. The *tautological complex vector bundle* $\gamma^k(\mathbf{C}^n)$ over $G_k(\mathbf{C}^n)$ consists of

1. the total space $E = \{(X, x) \;:\; x \in X \in G_k(\mathbf{C}^n)\}$, topologized as a subspace of $G_k(\mathbf{C}^n) \times \mathbf{C}^n$,
2. the projection map $\pi : E \to G_k(\mathbf{C}^n)$ defined by $\pi(X, x) = X$,
3. fibers E_X with the vector space structure of X, with

4. local triviality satisfied by taking, for each $X \in G_k(\mathbf{C}^n)$, the same open neighborhood as in the proof of Proposition 2.2.2 above, namely $U_X = \{Y \in G_k(\mathbf{C}^n) \ : \ Y \cap X^\perp = \{0\}\}$ and defining a map φ by

$$\begin{aligned} \varphi :\ \pi^{-1}(U_X) \ &\to \ U_X \times X, \\ (Y, y) \ &\mapsto \ (Y, \mathrm{Proj}_X|_Y\,(y)). \end{aligned}$$

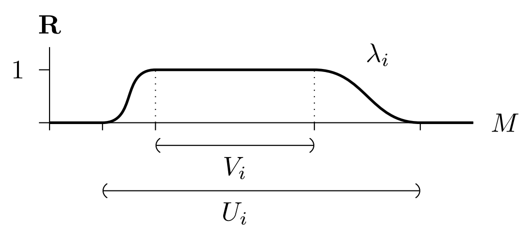
This map is continuous. Since $\varphi^{-1}(Y, x) = (Y, (\mathrm{Proj}_X|_Y)^{-1}\,(x))$ as in diagram (2.2.3), the map φ is indeed a homeomorphism. The continuity of φ^{-1} follows as it's two components are continuous, as described in the discussion following (2.2.4).

Theorem 2.3.2. (adapted from Lemma 5.3 in [MS74]) Let E be a complex rank k bundle over a compact topological manifold M with projection map π. Then there exists a bundle map $\varphi : E \to \gamma^k(\mathbf{C}^N)$ for some $N \in \mathbb{N}$.

Proof: Since the n-dimensional manifold M is compact, there exists a cover $U_1, \ldots, U_r$ of M so that each U_i is contained within some local trivialization of E (so $\pi|_{U_i} : E|_{U_i} \to U_i$ is trivial for all i), and so that there exists a cover $V_1, \ldots, V_r$ of M with $\mathrm{cl}(V_i) \subset U_i$ for all i. To see why this is possible, see Appendix A. Next, construct continuous maps

$$\begin{aligned} \lambda_i :\ M \ &\to \ [0,1], \\ x \ &\mapsto \ \begin{cases} 1 & \text{if } x \in \mathrm{cl}(V_i), \\ 0 & \text{if } x \notin U_i, \\ \alpha_i(x) & \text{else,} \end{cases} \end{aligned}$$

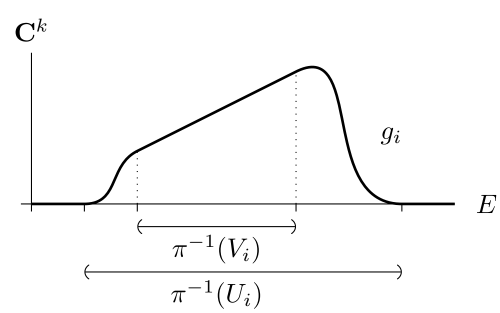
for an appropriate continuous function α_i, for all i. The graph of λ_i is as in the diagram below.



Since each U_i is in some trivialization, there is an associated map $h_i : \pi^{-1}(U_i) \to \mathbf{C}^k$ such that $h_i|_{\pi^{-1}(\{p\})}$ is linear for every $p \in U_i$. Define new maps

$$\begin{aligned} g_i :\ E \ &\to \ \mathbf{C}^k, \\ e \ &\mapsto \ \begin{cases} 0 & \text{if } \pi(e) \notin U_i, \\ \lambda_i(\pi(e)) h_i(e) & \text{else.} \end{cases} \end{aligned}$$

for all i. The graph of g_i is as in the diagram below.



This map g_i is continuous and linear on each fiber $\pi^{-1}(\{p\})$ for every $p \in V_i$. Finally, define a function

$$\begin{array}{rccl} \varphi : & E & \to & (\mathbf{C}^k)^r, \\ & e & \mapsto & (g_1(e), \ldots, g_r(e)). \end{array}$$

For $N = kr$, the map $\varphi : E \to \gamma^k(\mathbf{C}^N)$ is continuous and induces a bundle map $f(e) = ([\varphi(E_{\pi(e)})], \varphi(e))$. Then f is injective on the fibers because because the maps g_i are injective whenever $g_i(\,\cdot\,) = \lambda_i(\pi(\,\cdot\,))h_i(\,\cdot\,)$, since there is at least one h_i injective on the fiber. By linearity and equality of ranks, f is an isomorphism on fibers. ■

Definition 2.3.3. The *universal (complex) vector bundle* γ^k over $G_k(\mathbf{C}^\infty)$ is defined to be the direct limit

$$\gamma^k := \varinjlim_{n \geqslant k} \left[\gamma^k(\mathbf{C}^n)\right].$$

The associated objects are defined similarly as for the tautological complex vector bundle in Definition 2.3.1, except with $\mathbf{C}^\infty$ instead of $\mathbf{C}^n$. However, to prove local triviality and that sets in the direct limit topology are open, a short lemma, given by Lemma 5.4 in [MS74]. There is also a result similar to Theorem 2.3.2 for the universal bundle, proved using similar steps. The interested reader is directed to Section 5.8 in [MS74], where this result is proven.

2.4 Cohomology

At the risk of alienating readers unversed in higher category theory, a very general view of cohomology theories is presented below. The faint of heart may skip ahead to Definition 2.4.3, where the relevant cohomology groups are introduced, and should keep in mind that without a decoration, H^n is any one of H^n_S (singular cohomology group), H^n_{dR} (de Rham cohomology group), or $\check{H}^n$ (Čech cohomology group).

The category $\mathcal{C}$ used below is either Top, which contains topological spaces as objects, or Diff, which contains smooth manifolds as objects. In fact, both Top and Diff are $(\infty,1)$-categories.

Definition 2.4.1. Let X be an object in a category $\mathcal{C}$. Broadly speaking, a *cohomology theory* on X is the $(\infty,0)$-category $\mathcal{C}(X,A)$ of morphisms between X and A, for some other object A in $\mathcal{C}$. Objects in $\mathcal{C}(X,A)$ are *cocycles* $c : X \to A$, and morphisms are *coboundaries* $\delta : c \to c'$. Two cocyles are termed *cohomologous* if there is a coboundary between them.

The object $\mathcal{C}(X,A)$ may be viewed as the hom-space $\mathcal{C}(X,A)$, which is an ∞-groupoid. This definition is a simplification of the vastly more general one in §2.4 of [NSS14] and a slight generalization of the more concrete one in Chapter VII of [ES52]. In our context, there will always be an appropriate object A_n in $\mathcal{C}$ such that there is a sequence of abelian groups $H^n(X;R) = \pi_0(\mathcal{C}(X,A_n))$. For example, when $A_n = K(R,n)$, the Eilenberg–MacLane space with only nontrivial homotopy group π_n being R, we recover singular homology, as

$$\pi_0(\mathrm{Top}(X, K(R,n))) = H^n_S(X;R).$$

For other relevant cohomology theories, a suitable choice of A_n always exists.

Definition 2.4.2. Let A_n be a suitable object in $\mathcal{C}$ for all $n \in \mathbf{Z}_{\geqslant 0}$. If A_n is also an object of an ∞-loop space (which it always will be in our context), then A_n has the structure of some ring R, where we assume the same ring structure for all n. Then $H^n(X;R) := \pi_0(\mathcal{C}(X,A_n))$ is a group called the *nth cohomology group* of X with coefficients in R, and

$$H^*(X;R) := \bigoplus_{n=0}^{\infty} H^n(X;R)$$

is called the *cohomology ring* of X with coefficients in R. Multiplication is defined by the *cup product* $\smile$, for which $\alpha \in H^n(X;R)$ and $\beta \in H^m(X;R)$ imply that $\alpha \smile \beta \in H^{n+m}(X;R)$.

Next we introduce the specific cohomology theories used, with definitions mainly coming from Chapter 5 in [Pra07] and Appendix A in [MS74]. The Čech cohomology theory will be also be used, in Section 4.4, but because of the long setup, it is left until that section.

Definition 2.4.3. Let X be a topological space, R a commutative ring, and $n \in \mathbb{N}$. Define the following terms:

- · *n-simplex*: the smallest convex set in $\mathbf{R}^{n+1}$ containing $n+1$ points not all in a hyperplane, written Δ^n.
- · *n-chain of X*: a map $\sigma^n : \Delta^n \to X$.
- · *nth chain group of X*: the free R-module generated by all σ^n, written $C_n(X)$.
- · *nth cochain group of X*: group of all maps from n-simplices to R, written $C^n(X) := \operatorname{Hom}(C_n, R)$.
- · *cup product*: the map $\smile: C^n(X) \times C^m(X) \to C^{n+m}(X)$.

Definition 2.4.4. Let X be a topological space with cochain groups $C^n = C^n(X)$, and R a commutative ring. The *nth singular cohomology group* of X is the quotient

$$H^n_S(X;R) := \underbrace{\ker(\delta : C^n \to C^{n+1})}_{=:Z^k}/\underbrace{\operatorname{im}(\delta : C^{n-1} \to C^n)}_{=:B^k},$$

where elements of Z^k are called *k-cocycles* and elements of B^k are called *k-coboundaries*. The group operation is the cup product. Given a subspace $Y \subset X$, let $C^n(X,Y) := C^n(X)/C^n(Y)$ be the *nth relative cochain group*. Then the *nth relative cohomology* of X with respect to Y is the quotient

$$H^n_S(X,Y;R) := \ker(\delta : C^n(X,Y) \to C^{n+1}(X,Y))/\operatorname{im}(\delta : C^{n-1}(X,Y) \to C^n(X,Y)).$$

The relative cohomology used most often here is $H^n_S(E, E_0; \mathbf{Z})$, where E_0 represents the set of all non-zero elements in the vector bundle E.

Definition 2.4.5. Let ω be a differential k-form on a manifold M and $\Omega^k(M)$ the set of all differential k-forms on M. Then ω is *closed* if $d\omega = 0$ and *exact* if $\omega = d\eta$ for some other form η on M. Then the *kth de Rahm cohomology group* is

$$H^k_{dR}(M) = \{\omega \in \Omega^k(M) \ : \ \omega \text{ is closed}\} \Big/ \{\omega \in \Omega^k(M) \ : \ \omega \text{ is exact}\}.$$

The implicit ring R in de Rham cohomology theory is $R = \mathbf{R}$.

3 Characteristic classes of vector bundles

In this section we introduce the main characteristic classes that will be discussed in the rest of the paper. Broadly speaking, they are sequences of cohomology classes assigned to vector bundles. If the sequence is finite, then the last element in the sequence is called the *top* class of the vector bundle.

3.1 The Euler class

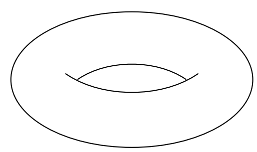
First we present orientation on a vector space and extend it to orientation on a real vector bundle, by consistently orienting the fibers of the bundle. Here, let M be a real topological manifold and $\pi : E \to M$ a real rank n vector bundle.

Definition 3.1.1. An *orientation* of a vector space V is an equivalence class of bases. Two ordered bases are equivalent if and only if the change of basis matrix has positive determinant. A *pre-orientation* on a vector bundle E is a choice of orientation on each fiber E_p. A pre-orientation on E is an *orientation* on E if for each $p \in M$ there exists an open neighborhood $U \ni p$ and a trivialization $\varphi : \pi^{-1}(U) \to U \times \mathbf{R}^n$ such that the restriction $\varphi\big|_{E_p} : E_p \to \{p\} \times \mathbf{R}^n$ preserves orientation, with $\mathbf{R}^n$ given the standard orientation (corresponding to the standard ordered basis). Finally, a bundle for which an orientation exists is called an *orientable bundle.*

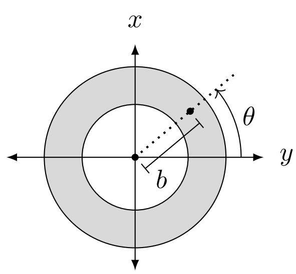
Note there are only two orientation equivalence classes on a vector space. Also observe that an orientation on a vector bundle, if it exists, is continuous (though pre-orientations need not be continuous). Hence E is orientable if and only if along every closed path on E there exists a continuous orientation.

3.1.1 Example: a non-orientable vector bundle on the torus

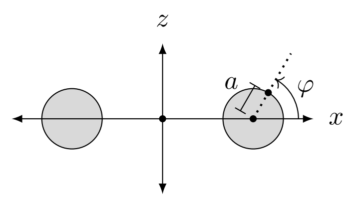
It is a well-known fact that the tangent bundle TM is orientable if and only if M is orientable as a manifold. However, not all vector bundles on an orientable manifold M are orientable. Consider the orientable space $M = T^2$, the two-dimensional torus

$T^2 \quad =$ 

with coordinates (θ, φ) as described in the diagrams below.



on the plane $z = 0$



on the plane $y = 0$

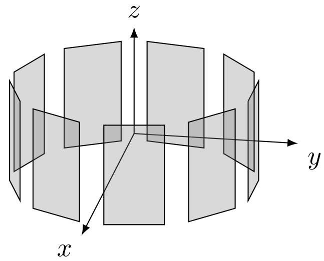
Assume that the inner and outer radii (a and b, respectively) have been fixed. Locally, we may parametrize T^2 by a function f of the two coordinates θ, φ, given by

$$f(\theta, \varphi) = \begin{pmatrix} (b + a\cos(\varphi))\cos(\theta) \\ (b + a\cos(\varphi))\sin(\theta) \\ a\sin(\varphi) \end{pmatrix}.$$

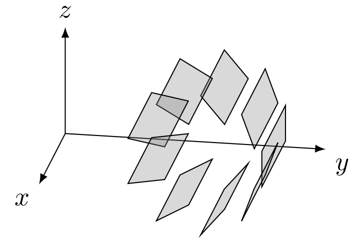
Then the tangent space at (θ, φ) is given by the span of $d_\theta f(\theta, \varphi)$ and $d_\varphi f(\theta, \varphi)$. By slightly modifying the tangent bundle, another real rank 2 vector bundle $\pi : E \to M$ may be defined: take $d_{\theta,\varphi} f(\theta, \varphi/2)$ instead of $d_{\theta,\varphi} f(\theta, \varphi)$ so that the tangent plane at (θ, φ) is the fiber at $(\theta, \varphi/2)$. Since the tangent plane at (θ, φ) is defined irrespective of a and b, set $a = 1$ and $b = 2$. Then the fiber $E_{\theta,\varphi} = \pi^{-1}((\theta, \varphi))$ is generated by the 2-tuple

$$(v_{\theta,\varphi}, w_{\theta,\varphi}) = \left(\begin{pmatrix} -(2 + \cos(\varphi/2))\sin(\theta) \\ (2 + \cos(\varphi/2))\cos(\theta) \\ 0 \end{pmatrix}, \begin{pmatrix} -\sin(\varphi/2)\cos(\theta)/2 \\ -\sin(\varphi/2)\sin(\theta)/2 \\ \cos(\varphi/2)/2 \end{pmatrix} \right). \tag{3.1.1}$$

The vector bundle E is smooth because trigonometric functions are smooth, and because the vector spaces at the endpoints of the two fundamental non-contractible paths (along curves that fix φ or fix θ) are the same. Indeed, on the curve $\varphi = 0$, E gives the tangent planes of T^2, and on the curve $\theta = 0$, E also gives the tangent planes of T^2, but the plane for φ is given at 2φ. Some snapshots (fibers at certain points) of E are given in the diagrams below.



on the curve $\varphi = 0$



on the curve $\theta = 0$

The diagram on the right gives some intuition why the bundle is not orientable. To prove this intuition is correct, we use the notion of a *section* of the vector bundle E, which is a continuous function $s : U \to E$ such that $(\pi \circ s)(u) = u$ for all $u \in U \subset M$.

Suppose that E is orientable, which means that there exists a non-zero section ω of $\bigwedge^2 E$. Let $\gamma_\theta : [0,1] \to T^2$ be a path on T^2, given by $\gamma_\theta(t) = f(\theta, 2\pi t)$ for any fixed θ. Then $\gamma_\theta^* \omega$ is a nowhere-zero section of the pullback bundle $\gamma_\theta^* E$, described in the diagram below.

$$\begin{tikzcd} \gamma_\theta^* E \arrow[d] & E \arrow[d] \\ {[0,1]} \arrow[r, "\gamma"] & T^2 \end{tikzcd}$$

Then $v_{\theta,\varphi} \wedge w_{\theta,\varphi}$ is another nowhere-zero section of $\gamma_\theta^* E$, and so is $v_{\gamma_\theta(t)} \wedge w_{\gamma_\theta(t)}$ for all $t \in [0,1]$. Since the fibers are 1-dimensional, there is always some function χ such that $\chi\omega = \zeta$, for any section ζ. Hence define a map

$$\begin{array}{rccl} F: & [0,1] & \to & \mathbf{R} \setminus \{0\}, \\ & t & \mapsto & \chi_t, \end{array} \qquad \text{where} \qquad \chi_t \omega = v_{\gamma_\theta(t)} \wedge w_{\gamma_\theta(t)}.$$

Next note that

$$\left(v_{\gamma_\theta(0)}, w_{\gamma_\theta(0)}\right) = \left(3\begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) \qquad \text{and} \qquad \left(v_{\gamma_\theta(1)}, w_{\gamma_\theta(1)}\right) = \left(\begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}\right).$$

Since the path $\gamma_{\theta(t)}$ starts and ends at the same point, we have that $F(0) = -F(1)$. However, since F is continuous, it follows that either ω is not a nowhere-zero section or $v_{\gamma_\theta(t)} \wedge w_{\gamma_\theta(t)}$ is not a nowhere-zero section (the latter implying that ω is not a nowhere-zero section). Hence no non-zero section exists on $\bigwedge^2 E$, meaning that E is a non-orientable vector bundle. This completes the example.

Remark 3.1.2. Definition 3.1.1 implies that each fiber E_p of an oriented vector bundle has a "preferred" generator $u_p \in H^n_S(E_p, (E_p)_0; \mathbf{Z})$ in the relative singular cohomology group. The idea is explained more in §9 of [MS74], and follows from representing n-cocycles in $Z^n(E, E_0; \mathbf{Z})$ by a special orientation-preserving embedding of an n-simplex into F. Constructing the n-simplex in a canonical manner gives one of two choices for the orientation of this embedding. Theorem B.3 below or Theorem 9.1 in [MS74] prove that there exists a unique element $u \in H^n_S(E, E_0; \mathbf{Z})$ such that

$$u|_{(E_p,(E_p)_0)} = u_p$$

for all $p \in M$. Then, the natural inclusion $\iota : (E, \emptyset) \to (E, E_0)$ induces a map $\iota^* : H_S^*(E, E_0, \mathbf{Z}) \to H_S^*(E; \mathbf{Z})$ on the cohomology ring. Similarly, the projection map $\pi : E \to M$ induces a map π^* on the respective cohomology rings, which gives a canonical isomorphism $\pi^* : H_S^n(M; \mathbf{Z}) \to H_S^n(E; \mathbf{Z})$ because π is a deformation retract from E to M.

Definition 3.1.3. In the context of the description above, the *Euler class* $e(E)$ of a vector bundle E is

$$e(E) = \left(\pi^*|_{H_S^n}\right)^{-1}(\iota^*(u)). \tag{3.1.2}$$

Theorem 9.1 in [MS74] also proves that $H_S^i(E, E_0; \mathbf{Z}) = 0$ for all $i \neq n$. A basic property of the Euler class is that it is *natural*, which means that whenever $F = f^*E$ for some bundle F over N and map $f : N \to M$, we have $e(F) = f^*(e(F))$. The naturality of the Euler class is not proven here, but the interested reader is refered to Theorems 11.7.11 and 11.7.15 in [AGP02]. The proofs use another class, the Thom class, which we omit here.

3.2 The Chern class

A common definition of the Chern class is the inductive approach in §14 of [MS74], which uses the Euler class (an approach presented in Section 4.1). We use the axioms laid out in §3 in Chapter 17 of [Hus75]. Note that the Euler and Stiefel–Whitney classes are described here in terms of singular cohomology H_S, but the axioms of the Chern class below are given independent of cohomology theory. We do this because the axioms will be checked against different definitions from different cohomology theories.

Remark 3.2.1. Let E be a complex vector bundle over M, R either $\mathbf{Z}$ or $\mathbf{R}$, and H a cohomology theory on M over R. Then there exists an element $c(E) \in H^*(M; R)$, called the *total Chern class of* E, that satisfies the following axioms.

(C_0) For $i \geqslant 0$, there exist elements $c_i(E) \in H^{2i}(E; R)$, called the *Chern classes of* E, such that $c(E) = c_0(E) + c_1(E) + \cdots + c_{\mathrm{rank}(E)}(E)$, with $c_0(E) = 1$ and $c_i(E) = 0$ for $i > \mathrm{rank}(E)$.

(C_1) (Naturality) If $f : N \to M$ is continuous, then $c(f^*E) = f^*(c(E))$.

(C_2) (Whitney product formula) If F is a vector bundle over M, then $c(E \oplus F) = c(E) \smile c(F)$.

(C_3) For the tautological line bundle γ^1 over $G_1(\mathbf{C}^2) = \mathbf{CP}^1$, $c_1(\gamma^1)$ is the negative of the preferred generator of $H^2(\mathbf{CP}^1; R)$.

In some texts, the opposite convention for (C_3) is used. That is, $c_1(\gamma^1)$ is just the preferred generator of $H^2(\mathbf{CP}^1; R)$. In differential and algebraic geometry, the convention of taking the negative generator is standard, so we keep with that convention here. For more on the conventions, see the remark after the same axioms in §7 of Chapter 23 in [May99].

As described in §3 of Chapter 17 of [Hus75], the last axiom (C_3) may be replaced by a condition that $c_1(\gamma^1(\mathbf{C}^\infty))$ be the negative of the preferred generator of $H^2(\mathbf{CP}^\infty; R)$. The two axioms are equivalent, by considering the inclusion map $\mathbf{CP}^1 \hookrightarrow \mathbf{CP}^\infty$ and applying axiom (C_1). An analogous result holds for the Stiefel–Whitney class.

The definitions given in the succeeding sections are shown to conform to these axioms. Then, with the theorems presented in §5, these axioms uniqely define the Chern classes.

3.3 The Stiefel–Whitney class

The Stiefel–Whitney classes of a vector bundle may be viewed as the real analogues of the Chern classes. They are included to show the similarities when considering real instead of complex vector bundles, with the setup as in §4 of [MS74].

Remark 3.3.1. Let E be a real vector bundle over M. Then there exists an element $w(E) \in H^*_S(M; \mathbf{Z}/2\mathbf{Z})$, called the *total Stiefel–Whitney class of* E, that satisfies the following axioms.

(W_0) For $i \geqslant 0$, there exist elements $w_i(E) \in H^i_S(E; \mathbf{Z}/2\mathbf{Z})$, called the *Stiefel–Whitney classes of* E, such that $w(E) = w_0(E) + w_1(E) + \cdots + w_{\mathrm{rank}(E)}(E)$, with $w_0(E) = 1$ and $w_i(E) = 0$ for $i > \mathrm{rank}(E)$.

(W_1) (Naturality) If $f : N \to M$ is continuous, then $w(f^*E) = f^*(w(E))$.

(W_2) (Whitney product formula) If F is a vector bundle over M, then $w(E \oplus F) = w(E) \smile w(F)$.

(W_3) For the real tautological line bundle γ^1 over $G_1(\mathbf{R}^2) = \mathbf{RP}^1$, which is defined in the same way as the complex bundle, $w_1(\gamma^1)$ is the generator of $H^1_S(\mathbf{RP}^1; \mathbf{Z}/2\mathbf{Z})$.

4 Existence of characteristic classes

In this section we prove, in four different ways, that the Chern classes exist. Some of the approaches may be used to show that the Stiefel–Whitney classes exist.

A common proof of existence (given in §20 of [BT82], §1 in Chapter 17 of [Hus75], and §4.D of [Hat02]) is to use the Leray–Hirsch theorem on the projectivization of a vector bundle. Although the theorem is mentioned in §5.2 to help prove uniqueness, a proper proof of the theorem is omitted, so it is not presented here as a tool to show existence.

4.1 Existence 1: The Gysin sequence

This is Chern's original approach, and mostly follows Milnor and Stasheff from §14 of [MS74]. The cohomology theory used is singular cohomology.

To begin, fix a Hermitian metric h on the complex rank n vector bundle $\pi : E \to M$. Over E_0 associate another bundle $\pi_0 : F \to E_0$, where as before $E_0 = \{e \in E \ : \ e \neq 0\}$ is the set of all non-zero vectors of E. The relation between these bundles is demonstrated in the diagram

$$\begin{tikzcd} E \arrow[dd, "\pi"'] & & \\ & E_0 \arrow[ul, hook', "\iota"'] \arrow[dl, "\pi_0 = \pi \circ \iota"] & F \arrow[l, "\pi_1"'] \\ M & & \end{tikzcd} \qquad \text{with} \qquad \begin{aligned} E_0 &= \{(p, v_p) \ : \ p \in M, v_p \in E_p \text{ is non-zero}\}, \\ F &= \{(q, v_q) \ : \ q = (p, v_p) \in E_0, v_q \in (\mathbf{C}v_p)^\perp\}, \\ F_q &= (\mathbf{C}\pi_2(q))^\perp, \end{aligned}$$

where $\pi_2((p, v_p)) = v_p$ is the projection map onto the second component of elements in E. The map ι is the inclusion map. As $\mathbf{C}v_p$ is the complex vector space spanned by some non-zero v_p, we have that $(\mathbf{C}v_p)^\perp \cong \{x \in E_p \ : \ h(v_p, x) = 0\}$, giving $F \to E_0$ a $\mathbf{C}^{n-1}$-bundle structure. Next, we construct an exact sequence associated to the underlying real bundle of E.

Theorem 4.1.1. (adapted from Theorem 12.2 in [MS74]) Every vector bundle E gives rise to an associated long exact sequence, called the *Gysin sequence*, given by

$$\cdots \longrightarrow H^{j-1}_S(E_0; \mathbf{Z}) \longrightarrow H^{j-2n}_S(M; \mathbf{Z}) \xrightarrow{e} H^j_S(M; \mathbf{Z}) \xrightarrow{\pi_0^*} H^j_S(E_0; \mathbf{Z}) \longrightarrow H^{j-2n+1}_S(M; \mathbf{Z}) \xrightarrow{e} \cdots, \tag{4.1.1}$$

where $e : v \mapsto v \smile e$ is the cup product with the Euler class e of $E^{\mathbf{R}}$.

Proof: Start with the long exact sequence for relative homology (described in §3.1 of [Hat02]), given by

$$\cdots \longrightarrow H^{j-1}_S(E_0; \mathbf{Z}) \xrightarrow{\delta} H^j_S(E, E_0; \mathbf{Z}) \longrightarrow H^j_S(E; \mathbf{Z}) \longrightarrow H^j_S(E_0; \mathbf{Z}) \xrightarrow{\delta} H^{j+1}_S(E, E_0; \mathbf{Z}) \longrightarrow \cdots. \tag{4.1.2}$$

Consider E as a real oriented bundle, as described in Definition 2.1.3. Since E has rank n, the underlying real bundle $E^{\mathbf{R}}$ has rank $2n$. For every $j \in \mathbf{Z}$, the Thom isomorphism theorem (see §10 in [MS74]) states that the map $\cdot \smile u : H_S^{j-2n}(E;\mathbf{Z}) \to H_S^j(E, E_0;\mathbf{Z})$ is an isomorphism, where $u \in H_S^{2n}(E, E_0;\mathbf{Z})$ is the unique element whose restriction to $(E_p, (E_p)_0)$ for every $p \in M$ gives the preferred generator u_{E_p} (this term is precisely defined in Definition B.2). Hence we have an isomorphism

$$\begin{array}{rrcl} \alpha : & H_S^{j-2n}(E;\mathbf{Z}) & \to & H_S^j(E, E_0;\mathbf{Z}), \\ & v & \mapsto & v \smile u, \end{array}$$

for u as in Remark 3.1.2. Recall from Definition 2.4.2 that $\smile$ takes p-cocycles and q-cocycles to $(p+q)$-cocycles. Via the restriction from (E, E_0) to $(E, \emptyset)$, the map α induces another map

$$\begin{array}{rrcl} \beta := \alpha\big|_E : & H_S^{j-2n}(E;\mathbf{Z}) & \to & H_S^j(E;\mathbf{Z}), \\ & v & \mapsto & (v \smile u)|_E = v \smile (u|_E). \end{array}$$

Applying α to groups of the sequence (4.1.2) gives a new sequence

$$\cdots \longrightarrow H_S^{j-1}(E_0;\mathbf{Z}) \longrightarrow H_S^{j-2n}(E;\mathbf{Z}) \xrightarrow{\beta} H_S^j(E;\mathbf{Z}) \longrightarrow H_S^j(E_0;\mathbf{Z}) \longrightarrow H_S^{j-2n+1}(E;\mathbf{Z}) \xrightarrow{\beta} \cdots,$$

This sequence is still exact because α is an isomorphism. Next, consider the cohomology ring map $\pi^* : H_S^*(M;\mathbf{Z}) \to H_S^*(E;\mathbf{Z})$, which is an isomorphism because E deformation retracts to M and cohomology is a homotopy invariant. Definition 3.1.3 defines the Euler class exactly in terms of this isomorphism, so letting e represent the map $v \mapsto v \smile e(E^{\mathbf{R}})$, which is just the composition $(\pi^*)^{-1} \circ \beta$, the sequence attains the desired form of (4.1.1). ■

The whole process of the proof above is described in the diagram below.

$$\begin{tikzcd}
\cdots \arrow[r] & H_S^{j-1}(E_0;\mathbf{Z}) \arrow[r, "\delta"] & H_S^j(E, E_0;\mathbf{Z}) \arrow[r] & H_S^j(E;\mathbf{Z}) \arrow[r] & H_S^j(E_0;\mathbf{Z}) \arrow[r, "\delta"] & H_S^{j+1}(E, E_0;\mathbf{Z}) \arrow[r] & \cdots \\
\cdots \arrow[r, "\pi_0^*"] & H_S^{j-1}(E_0;\mathbf{Z}) \arrow[u, equal] \arrow[r] & H_S^{j-2n}(M;\mathbf{Z}) \arrow[u, "\alpha \circ \pi^*"'] \arrow[r, "e"] & H_S^j(M;\mathbf{Z}) \arrow[u, "\pi^*"'] \arrow[r, "\pi_0^*"] & H_S^j(E_0;\mathbf{Z}) \arrow[u, equal] \arrow[r] & H_S^{j-2n+1}(M;\mathbf{Z}) \arrow[u, "\alpha \circ \pi^*"'] \arrow[r] & \cdots
\end{tikzcd}$$

Observe that for $0 \leqslant j < 2n$, every third group of the bottom level, which is (4.1.1), vanishes, as $H_S^{j-2n}(M;\mathbf{Z}) \cong 0$, since a negative index of a cohomology group means the group is trivial. It follows that $H_S^j(M;\mathbf{Z}) \cong H_S^j(E_0;\mathbf{Z})$, so π_0^* is an isomorphism for all such j. We now have enough information to describe an inductive construction of the Chern classes.

Definition 4.1.2. Let E be a complex rank n bundle over M and $\pi_1 : F \to E_0$ as in the beginning of this section. For every non-negative integer j, define the *jth Chern class* of E to be

$$c_j(E) := \begin{cases} (\pi_0^*)^{-1}(c_j(F)) & \text{if } j < n, \\ e\left(E^{\mathbf{R}}\right) & \text{if } j = n, \\ 0 & \text{if } j > n. \end{cases} \tag{4.1.3}$$

Hence simply by knowing the Euler class, we may determine the Chern class by reducing it to a smaller case through the rank $(n-1)$ bundle $F \to E_0$.

Proposition 4.1.3. The axiomatic properties of the Chern classes are satisfied by the above definition.

Proof: First consider axiom (C_0). By the definition of (4.1.3) and Definition 3.1.3, the top Chern class is

$$c_n(E) = e(E^{\mathbf{R}}) = \left(\pi^*|_{H_S^{2n}}\right)^{-1} (\iota^*(u)) \in H_S^{2n}(M; \mathbf{Z}),$$

so it lies in the correct space. For the other cases, first let $j = n-1$. Since $F \xrightarrow{\pi_1} E_0$ has rank $n-1$, $c_{n-1}(F)$ is the top Chern class, so

$$c_{n-1}(E) = (\pi_0^*)^{-1}(c_{n-1}(F)) = (\pi_0^*)^{-1}\left(e\left(F^{\mathbf{R}}\right)\right) = (\pi_0^*)^{-1} \underbrace{\left(\left(\pi_1^*|_{H_S^{2n-2}}\right)^{-1} \underbrace{(\iota^*(u))}_{\in\ H_S^{2n-2}(F;\mathbf{Z})}\right)}_{\in\ H_S^{2n-2}(E_0;\mathbf{Z})} \in H_S^{2n-2}(M; \mathbf{Z}),$$

and the $(j-1)$th Chern class also lies in the correct space. The process is analogous for all $0 < j < n-1$. Finally, consider $j = 0$. After $n-1$ applications of the definition in (4.1.3), we finally get some complex rank 0 bundle G. Further, the map ι from the definition of the Euler class of equation (3.1.2) becomes the identity, since $E_p = (E_p)_0$. Having rank 0 also means the total space is the base space, so the map π^* in equation (3.1.2) is also the identity, meaning that $e(G^{\mathbf{R}}) = u$. As $G^{\mathbf{R}}$ takes its top Chern class from $H_S^0(G; \mathbf{Z})$, and the only non-zero cohomology group (the 0th one) gives the number of connected components of E (and M is assumed to be connected, so G is connected as well), we have that $H_S^0(G^{\mathbf{R}}; \mathbf{Z}) = \mathbf{Z}$. Next, from Remark 3.1.2 and Definition 3.1.1, we may make a choice as to which generator u is, so we choose $u = 1$. Hence

$$c_0(E) = (\pi_0^*)^{-1}(c_0(G)) = (\pi_0^*)^{-1}\left(e\left(G^{\mathbf{R}}\right)\right) = (\pi_0^*)^{-1}(1) = 1,$$

so the conditions of axiom (C_0) are satisfied. For axiom (C_1), let $f : N \to M$ be smooth and $F = f^*E$. Then (C_1) for Chern classes below the top class follows by considering the indicated part of the commutative diagram below, as the maps π_0^* and τ_0^* are isomorphisms, formed by two copies of the sequence (4.1.1). The map φ is the map between the total spaces of E and F, determined by the bundle map f, as in Definition 2.1.4.

$$\begin{tikzcd}
\cdots \arrow[r] & H_S^{j-1}(E_0;\mathbf{Z}) \arrow[r] \arrow[d, "\varphi^*|_{E_0}"] & H_S^{j-2n}(M;\mathbf{Z}) \arrow[r, "\beta"] \arrow[d, "f^*"] & H_S^{j}(M;\mathbf{Z}) \arrow[r, "\pi_0^*"] \arrow[d, "f^*"] & H_S^{j}(E_0;\mathbf{Z}) \arrow[r] \arrow[d, "\varphi^*|_{E_0}"] & H_S^{j-2n+1}(M;\mathbf{Z}) \arrow[r, "\beta"] \arrow[d, "f^*"] & \cdots. \\
\cdots \arrow[r] & H_S^{j-1}(F_0;\mathbf{Z}) \arrow[r] & H_S^{j-2n}(N;\mathbf{Z}) \arrow[r] & H_S^{j}(N;\mathbf{Z}) \arrow[r, "\tau_0^*"] & H_S^{j}(F_0;\mathbf{Z}) \arrow[r] & H_S^{j-2n+1}(N;\mathbf{Z}) \arrow[r] & \cdots.
\end{tikzcd}$$

In the case of the top dimension, the result follows from the naturality of the Euler class. Next, axiom (C_3) is a direct consequence of the definition in equation (4.1.3) for $E = \gamma^1(\mathbf{C}^2)$ and $j = n = 1$, since the non-zero Euler class generates the space $H_S^2(\mathbf{CP}^1; \mathbf{Z})$ by Remark 3.1.2. The last one, axiom (C_2), which states that $c(E \oplus F) = c(E) \smile c(F)$, requires some polynomial algebra which is not introduced here. The interested reader is directed to Lemma 14.8 in §14 of [MS74]. ∎

4.2 Existence 2: The CW structure of the Grassmannians

Here we construct $G_k(\mathbf{C}^\infty)$ as a CW-complex using Schubert cells and direct limits so that the singular cohomology may be computed. This result shows the existence of Chern classes on the tautological k-plane bundle γ^k over $G_k(\mathbf{C}^\infty)$, which then may be extended, by Theorem 2.3.2 and naturality, to Chern classes on arbitrary bundles. The approach takes mainly from §5 in Chapter 1 of [GH94] and §6 of [MS74].

Definition 4.2.1. An ordered collection of subspaces $F = (V_0, V_1, \ldots, V_m)$ of $\mathbf{C}^n$ is called a *flag* of $\mathbf{C}^n$ if

$$\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_m = \mathbf{C}^n.$$

If $m < n$, then F is called a *partial flag* of $\mathbf{C}^n$, and if $m = n$, then F is called a *complete flag* of $\mathbf{C}^n$.

Definition 4.2.2. A *Schubert symbol* $\sigma = (\sigma_1, \ldots, \sigma_k)$ is an ordered k-tuple of strictly increasing, non-negative integers. To every Schubert symbol σ and flag F associate a *Schubert cell*

$$e(\sigma, F) := \{X \in G_k(\mathbf{C}^n) \ : \ \dim(X \cap V_{\sigma_i}) = i \text{ for all } i = 1, \ldots, k\}. \tag{4.2.1}$$

When there is no confusion about the flag, we omit F and write $e(\sigma)$. The closure $\overline{e}(\sigma) := \mathrm{cl}(e(\sigma))$ is called the *Schubert variety.*

Let $(x_1, \ldots, x_n)$ be the standard ordered orthonormal basis for $\mathbf{C}^n$ and write $V_i = \mathrm{span}\{x_1, \ldots, x_i\}$. With $V_0 := \{0\}$, the collection $(V_0, V_1, \ldots, V_n)$ is a flag for $\mathbf{C}^n$ and will be the only flag used from here on. Next, an *open n-cell* in this context may be thought of as $\mathbf{R}^n$, so $\mathbf{C}^n$ is an open $2n$-cell.

The implied cell structure of $e(\sigma)$ is described in the succeeding lemma. In our context, the only relevant Schubert symbols are k-subsequences of $(1, 2, \ldots, n)$. The fact that $\overline{e}(\sigma)$ is a variety is not proved here, and the interested reader is directed to Chapter I of [Rya87] or the briefer (but more relevant) §5 in Chapter 1 of [GH94].

4.2.1 Example: the cell structure of $G_2(\mathbf{C}^4)$

To motivate theorems later in this section, consider the example of $G_2(\mathbf{C}^4)$. There are six Schubert symbols for this space, namely

$$\sigma^1 = (1,2), \qquad \sigma^2 = (1,3), \qquad \sigma^3 = (1,4), \qquad \sigma^4 = (2,3), \qquad \sigma^5 = (2,4), \qquad \sigma^6 = (3,4).$$

The associated Schubert cells, all subsets of $G_2(\mathbf{C}^4)$, are

$$\begin{aligned}
e(\sigma^1) &= \{X \ : \ \dim(X \cap V_1) = 1, \ \dim(X \cap V_2) = 2, \ \dim(X \cap V_3) = 2, \ \dim(X \cap V_4) = 2\},\\
e(\sigma^2) &= \{X \ : \ \dim(X \cap V_1) = 1, \ \dim(X \cap V_2) = 1, \ \dim(X \cap V_3) = 2, \ \dim(X \cap V_4) = 2\},\\
e(\sigma^3) &= \{X \ : \ \dim(X \cap V_1) = 1, \ \dim(X \cap V_2) = 1, \ \dim(X \cap V_3) = 1, \ \dim(X \cap V_4) = 2\},\\
e(\sigma^4) &= \{X \ : \ \dim(X \cap V_1) = 0, \ \dim(X \cap V_2) = 1, \ \dim(X \cap V_3) = 2, \ \dim(X \cap V_4) = 2\},\\
e(\sigma^5) &= \{X \ : \ \dim(X \cap V_1) = 0, \ \dim(X \cap V_2) = 1, \ \dim(X \cap V_3) = 1, \ \dim(X \cap V_4) = 2\},\\
e(\sigma^6) &= \{X \ : \ \dim(X \cap V_1) = 0, \ \dim(X \cap V_2) = 0, \ \dim(X \cap V_3) = 1, \ \dim(X \cap V_4) = 2\}.
\end{aligned}$$

The totality of this list implies that as sets, $G_2(\mathbf{C}^4) = \bigcup_{i=1}^6 e(\sigma^i)$, where the sets in the union intersect trivially. As an example, we now take σ^5 and construct a special basis for arbitrary X contained in it. First, since $\dim(X \cap V_2) = 1$, fix $v_1 \in X \cap V_2$ so that $\langle v_1, x_2 \rangle = 1$. Such a v_1 must exist because $x_1 \notin X$, as $X \cap V_1 = \{0\}$. However, as $X \cap V_2 \neq \{0\}$, for some $a_2 \neq 0$ the equation $a_1 x_1 + a_2 x_2 = 0$ holds. Set $v_1 = \frac{a_1}{a_2} x_1 + x_2$, so

$$v_1 = (\ * \ , \ 1 \ , \ 0 \ , \ 0 \ ).$$

The first coordinate could be anything, since $V_2 = \mathrm{span}\{(1,0,0,0), (0,1,0,0)\}$, but the last two coordinates must be zero. Next, as $X \cap V_4$ is 2-dimensional, and $v_1 \in X \cap V_4$, choose a vector v_2 such that (v_1, v_2) generates $X \cap V_4$, with $\langle v_2, x_2 \rangle = 0$ and $\langle v_2, x_4 \rangle = 1$. The first condition may be guaranteed by translating by v_1 and the second condition may be guaranteed by scaling. Then

$$v_2 = (\ * \ , \ 0 \ , \ * \ , \ 1 \ ),$$

with the first and third coordinates taking any values, as above. By a slight abuse of notation, $e(\sigma^5)$ is

$$\left\{ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{bmatrix} \ : \ * \in \mathbf{C} \right\}, \tag{4.2.2}$$

the elements of which all have rank 2. The cell is not the described set, but actually homeomorphic to the union of the rowspaces of the matrices in the set. There is a homeomorphism between $\{[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}]\}$ and $\mathbf{C}^3$, given by allowing the three entries $*$ to be the the three defining components of $\mathbf{C}^3$. Next, we begin the more difficult task of showing that $e(\sigma^5)$ is a cell in the larger CW-complex of $G_2(\mathbf{C}^4)$.

The dimension of the cell $e(\sigma^5)$ is 6, because 5 of the 8 entries in the matrix in (4.2.2) are specified, while the other three are free to vary, so $e(\sigma^5) \cong \mathbf{C}^3 \cong D^6$. We may similarly construct the matrices and determine dimension for all the Schubert cells:

$$\begin{array}{r|cccccc} & \sigma^1=(1,2) & \sigma^2=(1,3) & \sigma^3=(1,4) & \sigma^4=(2,3) & \sigma^5=(2,4) & \sigma^6=(3,4) \\ \hline \text{matrix} & \left[\begin{smallmatrix}1&0&0&0\\0&1&0&0\end{smallmatrix}\right] & \left[\begin{smallmatrix}1&0&0&0\\0&*&1&0\end{smallmatrix}\right] & \left[\begin{smallmatrix}1&0&0&0\\0&*&*&1\end{smallmatrix}\right] & \left[\begin{smallmatrix}*&1&0&0\\*&0&1&0\end{smallmatrix}\right] & \left[\begin{smallmatrix}*&1&0&0\\*&0&*&1\end{smallmatrix}\right] & \left[\begin{smallmatrix}*&*&1&0\\*&*&0&1\end{smallmatrix}\right] \\ \text{dimension} & 0 & 2 & 4 & 4 & 6 & 8 \end{array} \tag{4.2.3}$$

Note the correspondence between the elements of each 2-tuple and the position of the columns $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ in the associated matrix. To be part of a CW-complex (see [Pra06], §3 in Chapter 3, for a complete description), the space must have all of its closed cells intersecting finitely many open cells (the closure finite "C" condition) and all intersections of closed sets with other closed sets must also be closed (the weak topology "W" condition). Moreover, every point in the boundary of $e(\sigma^5)$ must be contained in some smaller dimensional cell $e(\sigma^i)$.

The satisfaction of the C and W conditions is left to the general case below. To find the boundary, first we find the closure. Since $*$ can be anything, set the one in row 2, column 3 to n and multiply this matrix by an elementary matrix to get

$$\begin{bmatrix}1 & 0\\ 0 & 1/n\end{bmatrix}\begin{bmatrix}* & 1 & 0 & 0\\ * & 0 & n & 1\end{bmatrix} = \begin{bmatrix}* & 1 & 0 & 0\\ * & 0 & 1 & 1/n\end{bmatrix}.$$

Note that the resulting matrix is contained in $e(\sigma^5)$ for all n. In the limit, we have

$$\operatorname{cl}(e(\sigma^5)) \supset \lim_{n\to\infty}\left[\begin{bmatrix}* & 1 & 0 & 0\\ * & 0 & 1 & 1/n\end{bmatrix}\right] = \begin{bmatrix}* & 1 & 0 & 0\\ * & 0 & 1 & 0\end{bmatrix} \cong e(\sigma^4).$$

Next, perform a similar elementary matrix operation on σ^5 and specify two $*$ elements for

$$\begin{bmatrix}1 & 0\\ * & 1\end{bmatrix}\begin{bmatrix}* & 1 & 0 & 0\\ * & 0 & * & 1\end{bmatrix} = \begin{bmatrix}* & 1 & 0 & 0\\ * & * & * & 1\end{bmatrix} \xrightarrow{*_{1,1}\to n,\ *_{2,1}\to 0} \begin{bmatrix}n & 1 & 0 & 0\\ 0 & * & * & 1\end{bmatrix}.$$

Finally, multiply by a similar elementary matrix and take the limit, so

$$\operatorname{cl}(e(\sigma^5)) \supset \lim_{n\to\infty}\left[\begin{bmatrix}1/n & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}n & 1 & 0 & 0\\ 0 & * & * & 1\end{bmatrix}\right] = \lim_{n\to\infty}\left[\begin{bmatrix}1 & 1/n & 0 & 0\\ 0 & * & * & 1\end{bmatrix}\right] = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & * & * & 1\end{bmatrix} \cong e(\sigma^3).$$

Similar operations may be applied to σ^5 to get $e(\sigma^2)$ and $e(\sigma^1)$ in the closure. However it is not possible to get $e(\sigma^6)$ this way, because the minor of the last two columns of $e(\sigma^5)$ will always have rank at most 1, and in $e(\sigma^6)$ they have rank 2 (and elementary matrix operations preserve rank). The boundary then is the closure, the five cells $e(\sigma^i)$ for $i = 1, \ldots, 5$, minus the interior, which is just the cell $e(\sigma^5)$ itself. That is,

$$\operatorname{bd}(e(\sigma^5)) = e(\sigma^1) \cup e(\sigma^2) \cup e(\sigma^3) \cup e(\sigma^4).$$

Using the table from (4.2.3), $e(\sigma^1)$ is a 0-cell, $e(\sigma^2)$ is a 2-cell, and $e(\sigma^2)$ and $e(\sigma^3)$ are 4-cells. Since these are strictly smaller in dimension than the 6-cell $e(\sigma^5)$, part of the CW-complex structure is satisfied. However, it still remains to describe the gluing maps between $e(\sigma^5)$ and the lower-dimensional cells, which is left to the next section.

This completes the example, and we now generalize all of the statements made above.

4.2.2 Generalization

Lemma 4.2.3. The space $e(\sigma)$ is an open cell of dimension $2\sum_{i=1}^{k}(\sigma_i - i)$, for $|\sigma| = k$.

Proof: From the description in (4.2.1) and letting T be the set of all ordered k-subsets of $(1,\dots,n)$ (so every $t \in T$ is a Schubert symbol), it follows that $G_k(\mathbf{C}^n) = \bigcup_{\sigma\in T} e(\sigma)$ as a set. It remains to describe the $e(\sigma)$, which will be done by choosing a special basis for an arbitrary $X \in e(\sigma)$.

First, let $v_1 \in X \cap V_{\sigma_1}$ be the normalized generator of $X \cap V_{\sigma_1}$ (which is a 1-dimensional space), so that $\langle v_1, x_{\sigma_1}\rangle = 1$. Since the x_is are in the standard basis, it follows that

$$v_1 = \Big(\underbrace{*,\dots,*}_{\sigma_1 - 1 \text{ elements}},\ 1,\ \underbrace{0,\dots,0}_{n-\sigma_1 \text{ elements}} \Big).$$

Next, take $v_2 \in X \cap V_{\sigma_2}$ so that v_1, v_2 generate $X \cap V_{\sigma_2}$ with the conditions that $\langle v_2, x_{\sigma_2}\rangle = 1$ and $\langle v_2, x_{\sigma_1}\rangle = 0$. The first condition may be guaranteed by an appropriate factor and the second by an appropriate translation by v_1. For similar reasons as above,

$$v_2 = \Big(\overbrace{\underbrace{*,\dots,*}_{\sigma_1 - 1 \text{ elements}},\ 0,\ \underbrace{*,\dots,*}_{\sigma_2-\sigma_1 - 1 \text{ elements}}}^{\sigma_2 - 1 \text{ elements}},\ 1,\ \underbrace{0,\dots,0}_{n-\sigma_2 \text{ elements}} \Big).$$

To make the construction completely clear, lastly consider $v_3 \in X \cap V_{\sigma_3}$. This element is chosen so that v_1, v_2, v_3 generate $X \cap V_{\sigma_3}$ with the conditions that $\langle v_3, x_{\sigma_3}\rangle = 1$ and $\langle v_3, x_{\sigma_2}\rangle = \langle v_3, x_{\sigma_1}\rangle = 0$. These conditions are guaranteed similarly as for v_2. Then v_3 may be described as

$$v_3 = \Big(\overbrace{\underbrace{*,\dots,*}_{\sigma_1 - 1 \text{ elements}},\ 0,\ \underbrace{*,\dots,*}_{\sigma_2-\sigma_1 - 1 \text{ elements}},\ 0,\ \underbrace{*,\dots,*}_{\sigma_3-\sigma_2 - 1 \text{ elements}}}^{\sigma_3 - 1 \text{ elements}},\ 1,\ \underbrace{0,\dots,0}_{n-\sigma_3 \text{ elements}} \Big).$$

Continue in this manner, defining v_i as the generator that completes the orthonormal generating set $\{v_1,\dots,v_{i-1}\}$ of $X \cap V_{\sigma_i}$, with the condition that $\langle v_i, x_{\sigma_j}\rangle = \left\{\begin{smallmatrix} 1 & \text{if } j=i, \\ 0 & \text{if } j<i. \end{smallmatrix}\right.$ Hence any X is the rowspace of the matrix

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ \vdots \\ v_k \end{bmatrix} = \begin{bmatrix}
* \cdots * & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
* \cdots * & 0 & * \cdots * & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
* \cdots * & 0 & * \cdots * & 0 & * \cdots * & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots \quad \vdots & \vdots & \vdots \quad \vdots & \vdots & \vdots \quad \vdots & 0 & * & \cdots & * & 0 & 0 & \cdots & 0 \\
\vdots \quad \vdots & \vdots & \vdots \quad \vdots & \vdots & \vdots \quad \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
* \cdots * & 0 & * \cdots * & 0 & * \cdots * & 0 & * & \cdots & * & 1 & 0 & \cdots & 0
\end{bmatrix}. \tag{4.2.4}$$

This matrix has rank k by the construction, so it describes a k-plane in $\mathbf{C}^n$, i.e. an element of $G_k(\mathbf{C}^n)$. This means that we have a bijection between certain k-planes and matrices of the same form as the matrix above. Finally, note that v_i has $i + (n - \sigma_i)$ fixed elements, so in general the matrix in (4.2.4) has

$$\sum_{i=1}^{k}(i + n - \sigma_i) = kn - \sum_{i=1}^{k}(\sigma_i - i)$$

fixed elements. Since M is a $k \times n$ matrix with entries in $\mathbf{C}$, there exists a homeomorphism giving

$$e(\sigma) \cong \mathbf{C}^{kn-(kn-\sum(\sigma_i - i))} = \mathbf{C}^{\sum(\sigma_i - i)}. \tag{4.2.5}$$

■

Theorem 4.2.4. (appears as Theorem 6.4 in [MS74] and Proposition 1.17 in [Hat09]) The spaces $G_k(\mathbf{C}^n)$ and $G_k(\mathbf{C}^\infty)$ are CW-complexes containing only the $e(\sigma)$ as cells.

Proof: For n finite, we first need to show that each point in the boundary of a 2ℓ-cell lies within a $2m$-cell (since $\mathbf{C}$ is a 2-cell, all the $e(\sigma)$ cells are even-dimensional), for some $m < 2\ell$. Generalizing from the intuition in §4.2.1, the closure of a Schubert cell is given by

$$\mathrm{cl}(e(\sigma)) = \{X \ : \ \dim(X \cap V_{\sigma_i}) \geqslant i\} = \bigcup_{\tau_i \leqslant \sigma_i \ \forall \ i} e(\tau),$$

and the boundary is $\mathrm{bd}(e(\sigma)) = \mathrm{cl}(e(\sigma)) \setminus e(\sigma)$. It follows that for every $e(\tau) \in \mathrm{bd}(e(\sigma))$, there is an index j such that $\tau_j < \sigma_j$. Since for all other indices i, $\tau_i \leqslant \sigma_i$, and all else being equal, a matrix of the form (4.2.4) with a standard basis column vector in row τ_j has strictly more fixed elements than a matrix with the same vector in column σ_j.

To show the CW-structure, it is enough to describe a map $q : B \to G_k(\mathbf{C}^n)$ from a ball B of dimension $\dim(e(\sigma))$ such that $q|_{\mathrm{int}(B)} : \mathrm{int}(B) \to e(\sigma)$ is a homeomorphism onto its image. This is done by applying rotations and scalings to the matrix in (4.2.4) to get orthonormal rows. This operation is described in detail in the proof of Proposition 1.17 in [Hat09]. Although this source only considers the real case, the complex case follows by considering the real and imaginary parts of each complex coordinate as separate coordinates of the real case. That is, for every row in the matrix (4.2.4), add another row that is exactly the same, except the fixed 1 is now 0 (as the repeated row represents the imaginary part and $\mathrm{im}(1) = 0$), and then apply the procedure of [Hat09].

For the infinite case, $G_k(\mathbf{C}^\infty)$ is constructed as the direct limit of the sequence $X^0 \hookrightarrow X^1 \hookrightarrow X^2 \hookrightarrow \cdots$ with each finite skeleton constructed as above. The closure finite condition is satisfied in the direct limit, because the definition of the Schubert cells and (4.2.4) give that intersections occur only in smaller dimensions than the dimension of the cell. Hence the property is inherited from the finite case. The weak topology condition follows similarly. ■

Now that we have the CW-complex structure for $G_k(\mathbf{C}^\infty)$, we may calculate the singular cohomology. We need some combinatorics for this, so recall the q-binomial ceofficients, given by

$$\binom{n}{k}_q = \prod_{i=1}^{k} \frac{1 - q^{n-i+1}}{1 - q^i},$$

with the case $q = 1$ recovering the usual binomial coefficients. Note that $\prod_{i=1}^{k}(1 - q^{n-i+1})$ has factors $1 - q^i$ for all $i = 1, \ldots, k$, so the usual binomial coefficients are well-defined.

Lemma 4.2.5. The space $G_k(\mathbf{C}^n)$ has $\binom{n}{n-k}$ cells, of which

$$p(n, k, r) := [q^r]\binom{n}{n-k}_q = [q^r]\binom{n}{k}_q$$

have dimension $2r$.

The proof of this lemma is omitted, and the reader is directed to [Sta12], Propositions 1.7.2 and 1.7.3.

Example 4.2.6. Consider again the case $G_2(\mathbf{C}^4)$. Then

$$\prod_{i=1}^{2} \frac{1-q^{4-i+1}}{1-q^i} = \frac{1-q^4}{1-q} \cdot \frac{1-q^3}{1-q^2} = \frac{(1-q)^2(1+q)(1+q^2)(1+q+q^2)}{(1-q)^2(1+q)} = 1+q+2q^2+q^3+q^4.$$

This matches exactly the calculated cell sizes in (4.2.3).

Proposition 4.2.7. As a $\mathbf{Z}$-module, $H_S^*(G_k(\mathbf{C}^n);\mathbf{Z}) \cong \mathbf{Z}^{\binom{n}{k}}$.

Proof: To calculate the cohomology, first we need the homology, which may be calculated from the sequence of cells that make up $G_k(\mathbf{C}^n)$. In this case, setting $p(n,k,r) = [q^r]\binom{n}{k}_q$, we have the chain groups forming a sequence

$$\begin{array}{ccccccccccccc}
C_{2k(n-k)} & \longrightarrow & C_{2k(n-k)-1} & \longrightarrow & C_{2k(n-k)-2} & \longrightarrow & \cdots & \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 \longrightarrow 0. \\
\| & & \| & & \| & & & & \| & & \| & & \| \\
\mathbf{Z}^{p(n,k,k(n-k))} & & 0 & & \mathbf{Z}^{p(n,k,k(n-k)-1)} & & & & \mathbf{Z}^{p(n,k,2)} & & 0 & & \mathbf{Z}
\end{array}$$

From Lemma 4.2.3 we know that there are no cells of odd dimension in $G_k(\mathbf{C}^n)$, so $C_i = 0$ for i odd. From cellular homology theory, the homology of the above chain complex is the homology of the Grassmannian. That is, for $i = 0, 1, \ldots, k(n-k)$,

$$(H_S)_i(G_k(\mathbf{C}^n)) = \ker(C_{i+1} \to C_i)/\mathrm{im}(C_i \to C_{i-1}) = \begin{cases} \mathbf{Z}^{p(n,k,i/2)} & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd.} \end{cases} \tag{4.2.6}$$

Then, by the universal coefficient theorem (see [Pra07], §4 in Chapter 4) over the ring $\mathbf{Z}$,

$$H_S^i(G_k(\mathbf{C}^n)) = \mathrm{Hom}((H_S)_i(G_k(\mathbf{C}^n)), \mathbf{Z}) \oplus \mathrm{Ext}(H_{i-1}(G_k(\mathbf{C}^n)), \mathbf{Z}).$$

The group Ext is zero because over $\mathbf{Z}$, it contains only the torsion part of its argument, and the groups 0 and $\mathbf{Z}^j$ for any integer j are torsion-free. The group Hom with coefficients in $\mathbf{Z}$ represents the free part of the group it acts on, so $\mathrm{Hom}((H_S)_i(G_k(\mathbf{C}^n)), \mathbf{Z}) \cong \mathbf{Z}^{p(n,k,i/2)}$ for i even, and is 0 otherwise. Hence

$$H_S^i(G_k(\mathbf{C}^n);\mathbf{Z}) = (H_S)_i(G_k(\mathbf{C}^n)) = \begin{cases} \mathbf{Z}^{p(n,k,i/2)} & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

Finally, summing up over all the dimensions, recalling that $\sum p(n,k,i/2) = \binom{n}{k}$, and applying Lemma 4.2.5, we get that

$$H_S^*(G_k(\mathbf{C}^n);\mathbf{Z}) \cong \bigoplus_{i=0}^{2k(n-k)} \mathbf{Z}^{p(n,k,i/2)} = \mathbf{Z}^{\binom{n}{k}}.$$

■

In the infinite case, there is an integer sequence $\{\omega_{k,r}\}_{r=0}^{\infty}$ that the dimensions of the $2r$-cells of $H_S^*(G_k(\mathbf{C}^n);\mathbf{Z})$ reach in the limit as $n \to \infty$, although we do not prove that the limit exists. This generating function of this integer sequence is $\prod_{i=1}^{k}(1-x^i)^{-1}$, meaning that

$$\omega_{k,r} = [q^r] \prod_{i=1}^{k} \frac{1}{1-q^i}. \tag{4.2.7}$$

Theorem 4.2.8. As a ring, $H_S^*(G_k(\mathbf{C}^\infty);\mathbf{Z}) \cong \mathbf{Z}[a_1, \ldots, a_k]$, and the a_i are algebraically independent.

Proof: (Sketch) This is a proof done by induction. The first step is to apply the Gysin sequence (4.1.1) for $M = G_1(\mathbf{C}^\infty) = \mathbf{CP}^\infty$ over a line bundle, and set a_1 to be the preferred generator of $H^2_S(M;\mathbf{Z})$. Then a_1^i follows as the only generator of $H^{2i}_S(M;\mathbf{Z})$, for $i \leqslant k$. In the inductive case, apply the Gysin sequence to $M = G_n(\mathbf{C}^\infty)$ to get a homomorphism $H^{i+2n}_S(E_0;\mathbf{Z}) \to H^{i+2n}_S(G_{n-1}(\mathbf{C}^\infty);\mathbf{Z}) \to H^{i+2n}_S(G_n(\mathbf{C}^\infty);\mathbf{Z})$, which will define a_k. The inductive hypothesis is enough to get that there are no relations among all the generators. ■

For a complete proof, the interested reader is directed to Theorems 14.4 and 14.5 of [MS74], where the approach follows the steps described. The case for $k = 1$ is given for $M = G_1(\mathbf{C}^m)$, and because of the nice properties of limits and the Grassmannians, the result extends to $G_1(\mathbf{C}^\infty)$. It should be noted that for $G_k(\mathbf{C}^m)$, where $k > 1$, the generators a_i are not algebraically independent, so the proof must pass to ∞ in the base case.

Definition 4.2.9. In the context of Theorem 4.2.8, the *ith Chern class* of $\gamma^k(\mathbf{C}^\infty)$, for $1 \leqslant i \leqslant k$, is $c_i(\gamma^k) := -a_i$, and let $c_0(\gamma^k) = 1$. For an arbitrary $\mathbf{C}^k$-bundle $\pi : E \to M$, the *ith Chern class* of E is $c_i(E) = f^*(c_i(\gamma^k))$, where f is induced by a bundle map from E into $\gamma^k(\mathbf{C}^n)$, which exists by Theorem 2.3.2.

Note that the bundle map is unique up to homotopy, hence the induced maps on cohomology groups are the same for different bundle maps (see Proposition 1.5 in [Hat09]).

Proposition 4.2.10. The axiomatic properties of the Chern classes are satisfied by the above definition.

Proof: Axiom (C_0) follows directly from the definition above. For axiom (C_1), let $f : N \to M$ be continuous and $F = f^*E$, with both E and F rank k bundles. Let $h : M \to G_k(\mathbf{C}^\infty)$ be conitunous so that $h^*(\gamma^k) = E$, and define $g = h \circ f : N \to G_k(\mathbf{C}^\infty)$ so that $g^*(\gamma^k) = F$ also a bundle map. Then

$$f^*(c(E)) = f^*(h^*(c(\gamma^k))) = (h \circ f)^*(c(\gamma^k)) = g^*(c(\gamma^k)) = c(F).$$

For axiom (C_2), the proof is left incomplete, with only the setup given. Suppose that E is a rank k bundle and F is a rank ℓ bundle, with $g : M \to G_k(\mathbf{C}^\infty)$ and $h : M \to G_\ell(\mathbf{C}^\infty)$ continuous such that $g^*(\gamma^k) = E$ and $h^*(\gamma^\ell) = F$. Since $E \oplus F$ is a rank $k + \ell$ bundle, there is a continuous map $t : M \to G_{k+\ell}(\mathbf{C}^\infty)$ such that $t^*(\gamma^{k+\ell}) = E \oplus F$. It is possible to show, the most difficult being the second equality, that

$$c(E \oplus F) = t^*(c(\gamma^{k+\ell})) = g^*(c(\gamma^k)) \smile h^*(c(\gamma^\ell)) = c(E) \smile c(F).$$

Finally, the conditions of axiom (C_3) are satisfied by the construction of c_1 in the proof of Theorem 4.2.8, as

$$H^*_S(G_1(\mathbf{C}^2);\mathbf{Z}) = \underset{\mathbf{Z}}{\underset{\parallel}{H^0_S(G_1(\mathbf{C}^2);\mathbf{Z})}} \oplus \underset{0}{\underset{\parallel}{H^1_S(G_1(\mathbf{C}^2);\mathbf{Z})}} \oplus \underset{\mathbf{Z}}{\underset{\parallel}{H^2_S(G_1(\mathbf{C}^2);\mathbf{Z})}} \cong \mathbf{Z}[x],$$

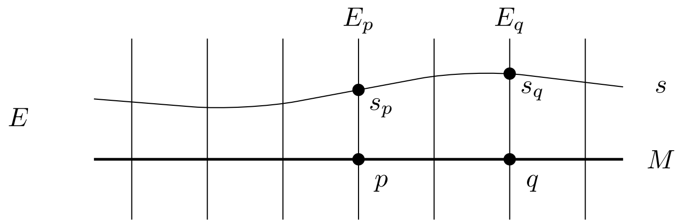
where $c_0 \in H^0_S$ is always 1, so x must be the (preferred) generator of H^2_S. Therefore $c_1 = -x$. ■

4.3 Existence 3: Connections and curvature

Here we define the Chern class by means of differential geometry, introducing connections and curvature in the process. The cohomology theory used is de Rham cohomology. A basic knowledge of smooth manifolds is assumed. The approach follows the constructions found in Chapter 4.2 of [Huy05] and Chapter 3.3 of [Wel80] with slightly adjusted notation.

4.3.1 Forms, connections, and curvature

Definition 4.3.1. Let M be a smooth manifold and $\pi : E \to M$ a $\mathbf{K}$-vector bundle of rank r. A (*smooth*) *section* of E is a smooth map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$ for all $p \in M$. That is, $\pi(s(p)) = \pi(s_p) = p$, so $s_p \in E_p$.



Define $\Gamma(E)$ to be the space of smooth sections of E. It is an infinite-dimensional **K**-vector space: for $s_1, s_2 \in \Gamma(E)$, $(s_1 + s_2)_p = (s_1)_p + (s_2)_p$, and for $\lambda \in \mathbf{K}$, $(\lambda s)_p = \lambda s_p$.

Note that $\Gamma(E)$ is a C^∞-module, that is, if $f \in C^\infty(M)$ and $s \in \Gamma(E)$, then $(fs)_p = f(p)s_p$. Moreover, the manifold M is diffeomorphic to the zero section $0 : M \to E$ (where $0(p) = 0_p \in E_p$). In general, a section can be thought of as a cross-section of all the fibers of E over M.

Definition 4.3.2. Let E be a vector bundle over a smooth manifold M. For any $m \in \mathbf{Z}_{\geqslant 0}$, an *$E$-valued differential form of degree m*, or an *E-valued m-form*, is a section of the vector bundle $\bigwedge^m(T^*M) \otimes E$. That is, an element

$$\omega \in \mathcal{A}^m(M; E) := \Gamma(\textstyle\bigwedge^m(T^*M) \otimes E).$$

In local coordinates, we write $\omega = \alpha_{i_1 \cdots i_m} dx^{i_1} \wedge \cdots \wedge dx^{i_m}$ in Einstein notation, for $\alpha_{i_1 \cdots i_m} \in E$. Note that $\mathcal{A}^0(M; E)$ is the space of smooth sections of E. When the context supplies the details, we sometimes omit E and just write $\mathcal{A}^m(E)$. Sometimes we write $\mathcal{A}^m(M)$ to represent the simpler space $\Gamma(\bigwedge^m(T^*M))$.

Definition 4.3.3. A *connection* on a vector bundle E over M is a **K**-linear homomorphism $\nabla : \mathcal{A}^0(M; E) \to \mathcal{A}^1(M; E)$ that satisfies the Leibniz rule

$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla(s)$$

for any $f \in C^\infty(M)$ and $s \in \mathcal{A}^0(M; E)$.

A section $s \in \mathcal{A}^0(M; E)$ is called *parallel* with respect to ∇ on E if and only if $\nabla s = 0$. A special connection is $\nabla = d$, the usual derivative, which is called the *trivial connection.*

Proposition 4.3.4. Let E_2, E_2 be bundles over M with sections s_1, s_2 and connections ∇_1, ∇_2 respectively. Then $\nabla_1 \otimes \nabla_2$, given by $(\nabla_1 \otimes \nabla_2)(s_1 \otimes s_2) = (\nabla_1 s_1) \otimes s_2 + s_1 \otimes (\nabla_2 s_2)$, is a connection on $E_1 \otimes E_2$.

Proof: Let $f \in C^\infty(M)$, $s_1 \in \Gamma(E_1)$, and $s_2 \in \Gamma(E_2)$, so $s_1 \otimes s_2 \in \Gamma(E_1 \otimes E_2)$. Then

$$\begin{aligned}
(\nabla_1 \otimes \nabla_2)(f \cdot (s_1 \otimes s_2)) &= (\nabla_1 \otimes \nabla_2)((f \cdot s_1) \otimes s_2) \\
&= (\nabla_1(f \cdot s_1)) \otimes s_2 + (f \cdot s_1) \otimes (\nabla_2 s_2) && \text{(by definition)} \\
&= (df \otimes s_1 + f \cdot \nabla_1 s_1) \otimes s_2 + (f \cdot s_1) \otimes \nabla_2 s_2 && \text{(as } \nabla_1 \text{ is a connection)} \\
&= df \otimes s_1 \otimes s_2 + f \cdot \nabla_1 s_1 \otimes s_2 + (f \cdot s_1) \otimes \nabla_2 s_2 \\
&= df \otimes s_1 \otimes s_2 + f \cdot (\nabla_1 s_1 \otimes s_2 + s_1 \otimes \nabla_2 s_2) \\
&= df \otimes (s_1 \otimes s_2) + f \cdot (\nabla_1 \otimes \nabla_2)(s_1 \otimes s_2),
\end{aligned}$$

so the Leibniz rule is satisfied. For **K**-linearity, let f be multiplication by $k \in \mathbf{K}$. Then the first term in the last line above disappears, as $dk = 0$ for any constant k, and we are left with

$$(\nabla_1 \otimes \nabla_2)(k \cdot (s_1 \otimes s_2)) = k \cdot (\nabla_1 \otimes \nabla_2)(s_1 \otimes s_2).$$

Additivity of this operator follows from the additivity of ∇, since

$$\begin{aligned}
(\nabla_1 \otimes \nabla_2)((s_1 + s_2) \otimes s) &= (\nabla_1(s_1 + s_2)) \otimes s + (s_1 + s_2) \otimes (\nabla_2 s) \\
&= (\nabla_1 s_1 \otimes s + s_1 \otimes (\nabla_2 s)) + (\nabla_1 s_2 \otimes s + s_2 \otimes \nabla_2 s) \\
&= (\nabla_1 \otimes \nabla_2)(s_1 \otimes s) + (\nabla_1 \otimes \nabla_2)(s_2 \otimes s),
\end{aligned}$$

and similarly in the second factor of $s_1 \otimes s_2$, so $\mathbf{K}$-linearity is satisfied. Hence $\nabla_1 \otimes \nabla_2$ is a connection on $E_1 \otimes E_2$. ∎

Definition 4.3.5. Let ∇ be a connection on E. Let $\omega \in \mathcal{A}^k(M)$ be a k-form, so $d\omega \in \mathcal{A}^{k+1}(M)$, and let $s \in \Gamma(E)$ be a section, so $\nabla s \in \Gamma(T^*M \otimes E)$. Define a $\mathbf{K}$-linear operator d^∇ by

$$\begin{aligned} d^\nabla : \Omega^k(E) &\to \Omega^{k+1}(E), \\ \omega \otimes s &\mapsto d\omega \otimes s + (-1)^k \omega \wedge \nabla s. \end{aligned}$$

For $k = 0$ this definition corresponds to the definition of the connection ∇, and the tensor product becomes multiplication. We must check that d^∇ is well-defined. For $f \in C^\infty(M)$, we have that $(f\omega) \otimes s = \omega \otimes (fs)$, so expanding both sides by the definition of d^∇, we have that

$$d^\nabla((f\omega) \otimes s) = d(f\omega) \otimes s + (-1)^k f\omega \wedge \nabla s = (df) \wedge \omega \otimes s + f d\omega \otimes s + (-1)^k f\omega \wedge \nabla s$$

and

$$d^\nabla(\omega \otimes (fs)) = d\omega \otimes (fs) + (-1)^k \omega \wedge \nabla(fs) = f(d\omega) \otimes s + (-1)^k \omega \wedge (df \otimes s + f\nabla s).$$

Hence d^∇ is well-defined.

Definition 4.3.6. Let $\pi : E \to M$ be a vector bundle with associated connection ∇. Then the *curvature* F_∇ of E is the map

$$\begin{aligned} F_\nabla : \mathcal{A}^0(M; E) &\to \mathcal{A}^2(M; E), \\ s &\mapsto (d^\nabla \circ \nabla)(s). \end{aligned}$$

Unlike ∇, the curvature is not a differential operator, as it is $\mathcal{A}^0(M; E)$-linear. Indeed,

$$\begin{aligned} F_\nabla(f \cdot s) &= (d^\nabla \circ \nabla)(f \cdot s) \\ &= d^\nabla(df \otimes s + f \cdot \nabla(s)) \\ &= d^\nabla(df \otimes s) + d^\nabla(f \cdot \nabla(s)) \\ &= d(df) \otimes s - df \wedge \nabla(s) + df \wedge \nabla(s) + f \cdot d^\nabla(\nabla(s)) \\ &= 0 + 0 + f \cdot F_\nabla(s) \\ &= f \cdot F_\nabla(s). \end{aligned}$$

Next, note that given a trivialization domain $U \subset M$ (so $E|_U$ is trivial) with a local frame $(e_1, \ldots, e_r)$, there is always the trivial connection $d : \mathcal{A}^0(U; E|_U) \to \mathcal{A}^1(U; E|_U)$. Moreover, any other connection on $E|_U$ is just d scaled by a matrix of one forms. In other words, any connection ∇ on $E|_U$ is of the form $\nabla = d + A$ for $A \in \mathcal{A}^1(\mathrm{End}(E))$. It follows that for such ∇, we have $\nabla e_i = A_i^j e_j$. Here, and further, we use the Einstein notation, which means that whenever a term has two repeated indices, one in the subscript and one in the superscript, there is an implied sum over that index.

Lemma 4.3.7. Any connection is locally of the form $\nabla = d + A$ for A a matrix of one forms and d the trivial connection.

Proof: Let $(e_1, \ldots, e_r)$ be a local frame corresponding to a local trivialization. An E-valued 0-form is a finite sum of terms of the form $\omega \otimes s$, for $\omega \in \mathcal{A}^0(M; E)$, $s = s^i e_i \in \Gamma(E)$ and s^i smooth $\mathbf{K}$-valued functions. Then $\omega \otimes s = \omega \otimes (s^i e_i) = (s^i \omega) \otimes e_i$, so apply the connection ∇ to $\omega \otimes s$, recalling from the discussion after Definition 4.3.6, that $\nabla e_i = A_i^j e_j$. Hence

$$\begin{aligned} \nabla(\omega \otimes s) &= \nabla(s^i \omega \otimes e_i) \\ &= d(s^i \omega) \otimes e_i + (s^i \omega) \wedge \nabla e_i \\ &= d(s^i \omega) \otimes e_j + s^i \omega \wedge A_i^j e_j \\ &= (d(s^j \omega) + (A_i^j s^i) \wedge \omega) e_j \\ &= (d + A \wedge \cdot)(\omega \otimes s). \end{aligned}$$

In other words, in a fixed local trivialization, $\nabla = d + A\wedge\cdot$. ■

Remark 4.3.8. Note in the proof above we could have used $\omega \in \mathcal{A}^m(M)$, since no particular property of $\mathcal{A}^0$ was used, so the same result holds for d^∇. Hence locally, for $s \in \Gamma(E)$, we may write

$$\begin{aligned} F_\nabla(s) &= (d + A\wedge\cdot)(d + A\wedge\cdot)(s) \\ &= (d + A\wedge\cdot)(ds + As) \\ &= d^2 s + d(As) + A\wedge(ds) + A\wedge(As) \\ &= (dA)s + (A\wedge A)s \\ &= (dA + A\wedge A)(s), \end{aligned}$$

so $F_\nabla = dA + A\wedge A$. Let us now describe a way of relating $A = A_\alpha$ and $\widetilde{A} = A_\beta$. Let $(e_1, \dots, e_r)$ be the local trivialization on U_α and $(\widetilde{e}_1, \dots, \widetilde{e}_r)$ the local trivialization on U_β. First, write $\tilde{e}_i = e_j g_i^j$, for g_i^j the change of basis matrix, that is, a smooth map $g_i^j : U \to GL(r, \mathbf{K})$. Then $\nabla\tilde{e}_i = \tilde{A}_i^j \tilde{e}_j$, so

$$\begin{aligned} \nabla\tilde{e}_i &= \nabla(e_j g_i^j) \\ &= \tilde{A}_i^j e_k g_j^k, \end{aligned} \qquad \text{and} \qquad \begin{aligned} \nabla(e_j g_i^j) &= \nabla(g_i^j e_j) \\ &= dg_i^j \otimes e_j + g_i^j \nabla e_j \\ &= dg_i^k \otimes e_k + g_i^j A_j^k \otimes e_k \\ &= \tilde{A}_i^j g_j^k \otimes e_k. \end{aligned}$$

Hence $\tilde{A}_i^j g_j^k = dg_i^k + g_i^j A_j^k$. Multiply both sides by $(g^{-1})_k^\ell$ and sum over k to get that

$$\tilde{A}_i^\ell = (g^{-1})_k^\ell A_j^k g_i^j + (g^{-1})_k^\ell dg_i^k, \qquad \text{or} \qquad \tilde{A} = g^{-1}Ag + g^{-1}dg.$$

Now suppose $\{U_\alpha\}_{\alpha\in I}$ covers M with transition maps $g_{\alpha\beta}$ for E and connection matrices A_α of ∇ with respect to the local trivializations $(\varphi_\alpha, U_\alpha)$. Then A_α is a map $U_\alpha \to GL(r, \mathbf{K})$, and the relation between A_α and A_β is given by

$$A_\beta = g_{\beta\alpha}^{-1} A_\alpha g_{\beta\alpha} + g_{\beta\alpha}^{-1} dg_{\beta\alpha}. \tag{4.3.1}$$

4.3.2 Polarized polynomials and the Chern–Weil theorem

Here $\pi : E \to M$ is a rank n complex vector bundle with a connection ∇. We begin with a slight digression about homogeneous polynomials.

Definition 4.3.9. Given a complex vector space V, let $P : V^{\times k} \to \mathbf{C}$ be a k-multilinear symmetric map. The *polarization* of P is the map

$$\begin{aligned} \widetilde{P} : V &\to \mathbf{C}, \\ A &\mapsto P(A, \dots, A). \end{aligned}$$

This map is a homogeneous polynomial of degree k, so $\widetilde{P}(\lambda A) = \lambda^k \widetilde{P}(A)$. When $V = GL(n, \mathbf{C})$, the space of complex $n \times n$ matrices, the map P is called *invariant* whenever, for $A, B, B_1, \dots, B_k \in GL(n, \mathbf{C})$,

$$P(AB_1A^{-1}, \dots, AB_kA^{-1}) = P(B_1, \dots, B_k),$$

and similarly $\widetilde{P}$ is called invariant when $\widetilde{P}(ABA^{-1}) = \widetilde{P}(B)$.

To define terms associated to the Chern class, first recall that the determinant of an element in $GL(n, \mathbf{C})$ is a homogeneous polynomial of degree n. Taking $A \in GL(n, \mathbf{C})$, define degree k homogeneous polynomials

σ_k in the entries of A by the sum

$$\begin{aligned}
\det(I+A) &= \sum_{s\in S_n} \operatorname{sgn}(s) \prod_{i=1}^{n} (I+A)_{i,s_i} \\
&= \sum_{s\in S_n} \operatorname{sgn}(s) \prod_{i=1}^{n} \left(\delta_{i,s_i} + A_{i,s_i}\right) \\
&= \sigma_0(A) + \sigma_1(A) + \cdots + \sigma_n(A).
\end{aligned} \tag{4.3.2}$$

The first term $\sigma_0(A)$ is always 1, as for every $s \in S_n$ only the identity permutation has $i = s_i$ for all i. This permutation contributes $(1+A_{1,1})\cdots(1+A_{n,n}) = 1 +$ (no terms without an $A_{j,j}$ factor) to the sum. The other terms are then grouped by degree to give the stated result. Note that all of the σ_k are invariant, since the determinant is invariant.

Definition 4.3.10. Define the *kth Chern form* of a vector bundle E to be

$$c_k(E,\nabla) = \sigma_k\left(\frac{iF_\nabla}{2\pi}\right) \in \mathcal{A}^{2k}(M;E).$$

Definition 4.3.11. The *kth Chern class* of E is the cohomology class

$$c_k(E) = [c_k(E,\nabla)] \in H^{2k}_{dR}(M).$$

The *total Chern class* of E is the formal sum

$$c(E) = c_0(E) + c_1(E) + \cdots + c_n(E).$$

To show that the above definition is well defined, we need to show that for any two connections ∇ and $\widetilde{\nabla}$ on E, $[c_k(E,\nabla)] = [c_k(E,\widetilde{\nabla})]$, and that the form is indeed closed (that is, $c_k(E) \in H^{2k}_{dR}(M)$). A slight generalization of these statements is known as the Chern–Weil theorem.

Theorem 4.3.12. [Chern, Weil]
Let ∇ be any connection on a vector bundle E over M. Then

1. (appears as Corollary 4.4.5 in [Huy05]) for any k-multilinear symmetric invariant map $P : GL(n,\mathbf{K})^{\oplus k} \to \mathbf{K}$, the $\mathbf{K}$-valued $2k$-form $\widetilde{P}(aF_\nabla)$ is closed for all $a \in \mathbf{K}$, and

2. (appears as Lemma 4.4.6 in [Huy05]) if $\nabla, \widetilde{\nabla}$ are two connections on E, then $[\widetilde{P}(aF_\nabla)] = [\tilde{P}(aF_{\widetilde{\nabla}})]$.

Both statements require some other results, which use d^∇ as an operator on $\mathcal{A}^*(\operatorname{End}(E))$. The algebra structure on $\mathcal{A}^*(\operatorname{End}(E))$ is defined by taking $\omega \otimes T \in \mathcal{A}^k(\operatorname{End}(E))$ and $\eta \otimes S \in \mathcal{A}^\ell(E)$ and writing

$$(\omega \otimes T) \wedge (\eta \otimes S) := (\omega \wedge \eta) \otimes TS.$$

in $\mathcal{A}^k(M)$ — in $\Gamma(\operatorname{End}(E))$
in $\mathcal{A}^\ell(M)$ — in $\mathcal{A}^{k+\ell}(M)$
in $\Gamma(\operatorname{End}(E))$

The proof of the theorem also requires several lemmas, two of which are given in Appendix C, and the Bianchi identity, which states that $d^\nabla F_\nabla = 0$ (Lemma 4.3.5 in [Huy05]). The fact that for any $B \in \mathcal{A}^1(\operatorname{End}(E))$, the curvature of the connection $\nabla + B$ is locally given by $F_{\nabla+B} = F_\nabla + d^\nabla B + B\wedge B$ is a straightforward exercise (also Lemma 4.3.4 in [Huy05]), following from the fact that $\nabla + B = d + A + B$ locally whenever $\nabla = d + A$.

Proof: For **1.**, since $\widetilde{P}(aF_\nabla) = P(aF_\nabla,\ldots,aF_\nabla)$ by definition and $d(\tilde{P}(aF_\nabla)) = d(P(aF_\nabla,\ldots,aF_\nabla)) = 0$ by Lemma C.2 and the Bianchi identity, the result follows.

For statement **2.**, we know $\nabla^1 = \nabla^0 + B$ for some global $B \in \mathcal{A}^1(\mathrm{End}(E))$. Define $\nabla_t = \nabla^0 + tB$ for $t \in [0,1]$, so $\nabla_0 = \nabla^0$ and $\nabla_1 = \nabla^1$. Let $P(t) = \widetilde{P}(F_{\nabla_t})$. We need to show that $P(1) - P(0)$ is exact, so let

$$F_t := F_{\nabla_t} = F_{\nabla^0 + tB} = F_{\nabla^0} + td^{\nabla^0}B + t^2 B \wedge B.$$

Using the usual commutator on 0-forms, define a new bracket operator $[\,\cdot\,,\,\cdot\,]$ on $\mathrm{End}(E)$-valued forms. For $\omega \in \mathcal{A}^k(M)$, $\eta \in \mathcal{A}^\ell(M)$ and $T, S \in \Gamma(\mathrm{End}(E))$, let

$$\begin{aligned} [\,\cdot\,,\,\cdot\,]:\ \mathcal{A}^k(\mathrm{End}(E)) \times \mathcal{A}^\ell(\mathrm{End}(E)) &\ \to\ \mathcal{A}^{k+\ell}(\mathrm{End}(E)), \\ (\omega \otimes T, \eta \otimes S) &\ \mapsto\ (\omega \wedge \eta) \otimes [T, S]. \end{aligned}$$

By writing $B = dx^i \otimes B_i$ in some local coordinates $(x^1, \ldots, x^n)$ and matrices B_i, we then find that

$$\begin{aligned} [B, B] &= [dx^i \otimes B_i, dx^j \otimes B_j] \\ &= dx^i \wedge dx^j [B_i, B_j] \\ &= dx^i \wedge dx^j (B_i B_j - B_j B_i) \\ &= 2dx^i \wedge dx^j B_i B_j \\ &= 2B \wedge B. \end{aligned}$$

Hence $F_t = F_{\nabla^0} + td^{\nabla^0}B + \frac{t^2}{2}[B, B]$. Then

$$\frac{d}{dt}F_t = d^{\nabla^0}B + t[B, B] = \left(d^{\nabla^0} + [tB, \cdot]\right) B.$$

Given $\nabla_t = \nabla^0 + tB$, let $B = \beta_i \otimes B^i$ and $C = \gamma_j \otimes C^j$ for $\beta_i \in \mathcal{A}^1(M)$ and $\gamma_j \in \mathcal{A}^k(M)$. When $k = 1$, we have that for a section s,

$$\begin{aligned} (\nabla_t C)(s) &= \nabla_t(Cs) - C \wedge (\nabla_t s) \\ &= (d + tB\wedge)(Cs) - C \wedge (d + tB)s \\ &= dC \wedge s + C \wedge ds + tB \wedge Cs - C \wedge ds - C \wedge tBs \\ &= (dC + tBC - CtB)s. \end{aligned}$$

For general k, the negative sign becomes $(-1)^k$, so

$$\begin{aligned} d^{\nabla_t}C &= d^{\nabla_t}(\gamma_j \otimes C^j) \\ &= d\gamma_j \otimes C^j + (-1)^k \gamma_j \wedge \nabla_t C^j \\ &= d\gamma_j \otimes C^j + (-1)^k \gamma_j \left(dC^j + tB \wedge C^j - C^j \wedge tB\right) \\ &= d^{\nabla_0}\left(\gamma_j \otimes C^j\right) + (-1)^{2k} tB \wedge \gamma_j \otimes C^j - (-1)^k \gamma_j \otimes C^j \wedge tB \\ &= d^{\nabla_0}C + [tB, C]. \end{aligned}$$

Hence $d^{\nabla_t} = d^{\nabla^0} + [tB, \cdot]$ on $\mathcal{A}^*(\mathrm{End}(E))$, and so $\frac{d}{dt}F_t = d^{\nabla_t}B$. Let

$$\underbrace{TP(\nabla^1, \nabla^0)}_{\in\ \mathcal{A}^{2k-1}_{\mathbf{K}}(M)} := k\int_0^1 P(F_t, \ldots, F_t, B)dt,$$

where the integrand is a $(2k-1)$-form. It remains to show that $d(TP(\nabla^1, \nabla^0)) = P(1) - P(0)$. Note

$$\begin{aligned} P(1) - P(0) &= \int_0^1 \left(\frac{d}{dt}P(t)\right) dt \\ &= \int_0^1 \left(P\left(\frac{d}{dt}F_t, \ldots, F_t\right) + \cdots + P\left(F_t, \ldots, \frac{d}{dt}F_t\right)\right) dt \\ &= k\int_0^1 P(F_t, \ldots, F_t, d^{\nabla^t}B)dt, \end{aligned}$$

with the last equality following since P is symmetric on $\mathcal{A}^{2m}(\mathrm{End}(E))$ for any $m \in \mathbf{Z}$. Finally, using Lemma C.2 and the Bianchi identity, we see that

$$d(TP(\nabla^1, \nabla^0)) = k \int_0^1 d(P(F_t, \dots, F_t, B))dt = k \int_0^1 P(F_t, \dots, F_t, d^{\nabla^t} B)dt,$$

as desired. ■

4.3.3 Line bundles and axiom satisfaction

We begin with some observations about the tensor products of line bundles, the second of which is necessary to show that the current definition of Chern classes satisfies the conditions of the Chern class axioms.

Proposition 4.3.13. For two line bundles L, L' over M, $c_1(L \otimes L') = c_1(L) + c_1(L')$. In particular, $c_1(L^{\otimes k}) = kc_1(L)$ for any $k \in \mathbf{Z}$.

Proof: Let ∇ be a connection on L and ∇' a connection on L' (the existence of connections is a well-known result). Let $\widetilde{\nabla} := \nabla \otimes I_{L'} + I_L \otimes \nabla'$ be a connection on $L \otimes L'$. Proposition 4.3.4 shows that $\widetilde{\nabla}$ indeed is a connection, and the Chern–Weil theorem gives that any connection may be used for the definition of the Chern class. The first Chern class of $L \otimes L'$ is then

$$\begin{aligned}
c_1(L \otimes L') &= \left[\sigma_1\left(\frac{iF_{\widetilde{\nabla}}}{2\pi}\right)\right] \\
&= \left[\mathrm{trace}\left(\frac{iF_{\nabla \otimes I_{L'} + I_L \otimes \nabla'}}{2\pi}\right)\right] \\
&= \left[\frac{i}{2\pi}\mathrm{trace}\left(F_\nabla \otimes I_{L'} + I_L \otimes F_{\nabla'}\right)\right] \\
&= \left[\frac{i}{2\pi}\left(\mathrm{trace}\left(F_\nabla\right)\mathrm{trace}\left(I_{L'}\right) + \mathrm{trace}\left(I_L\right)\mathrm{trace}\left(F_{\nabla'}\right)\right)\right] \\
&= \left[\frac{i}{2\pi}\left(\mathrm{trace}\left(F_\nabla\right) + \mathrm{trace}\left(F_{\nabla'}\right)\right)\right] \\
&= \left[\sigma_1\left(\frac{iF_\nabla}{2\pi}\right) + \sigma_1\left(\frac{iF_{\nabla'}}{2\pi}\right)\right] \\
&= c_1(L) + c_1(L').
\end{aligned}$$

Induction is used to show that $c_1(L^{\otimes k}) = kc_1(L)$. The case $k = 1$ is a tautology, so with the induction hypothesis, the result for $k > 0$ follows immediately by letting $L' = L^{\otimes k-1}$. Finally, since $L \otimes L^*$ is trivial, the first Chern class of $L \otimes L^*$ is 0. This means that $c_1(L) = -c_1(L^*)$, so having the result for $k > 0$ for L^* means having the result for $k < 0$ for L. ■

Proposition 4.3.14. For the kth tensor product of the tautological line bundle γ^1 over $G_1(\mathbf{C}^2) = \mathbf{CP}^1$, the first Chern class $c_1((\gamma^1)^{\otimes k})$ is k times the negative of the preferred generator of $H^2_{dR}(\mathbf{CP}^1; \mathbf{R})$.

Proof: To get a connection ∇ on the tautological bundle $\gamma^1(\mathbf{C}^2)$, use two charts $U_0 = \{z \neq \infty\}$ and $U_1 = \{w = \frac{1}{z} \neq \infty\}$ to cover $\mathbf{CP}^1$. Using observations from Proposition 4.3.13 and §6.1 below, the transition function from U_0 to U_1 is z^{-1} on γ^1, and is $g = z^{-k}$ on $(\gamma^1)^{\otimes k}$. The connection ∇ has locally-defined matrices (in this case 1×1 matrices) A_0, A_1 on U_0, U_1, respectively. We claim that

$$\nabla_0 = \frac{k\overline{z}dz}{1 + z\overline{z}} \quad \text{on } U_0 \qquad , \qquad \nabla_1 = \frac{k\overline{w}dw}{1 + w\overline{w}} \quad \text{on } U_1$$

are local expressions for a globally defined ∇ (this is in fact called the *Chern connection*, as in Example 4.2.16 ii) of [Huy05]). The compatibility condition that must be satisfied, given by equation (4.3.1), is $A_1 = g^{-1}A_0 g + g^{-1}dg$. In this case we have

$$\begin{aligned}
g^{-1}A_0 g + g^{-1}dg &= z^k\left(\frac{k\overline{z}dz}{1+z\overline{z}}\right)z^{-k} + z^k d(z^{-k}) \\
&= \frac{k\overline{z}dz}{1+z\overline{z}} - kz^k z^{-k-1}dz \\
&= \frac{-k\overline{w}^{-1}w^{-2}dw}{1+(w\overline{w})^{-1}} + kww^{-2}dw \\
&= \frac{-kdw}{w(1+w\overline{w})} + \frac{kdw}{w} \\
&= \frac{-kdw + kdw + kw\overline{w}dw}{w(1+w\overline{w})} \\
&= \frac{k\overline{w}dw}{1+w\overline{w}} \\
&= A_1
\end{aligned}$$

as desired. The next step is to find the curvature, for which we apply the equation described in Remark 4.3.8. Since $A = A_0$ on U_0 and U_0 covers all of $\mathbf{CP}^1$ except for one point, the answer is the same if we just integrate over U_0 using A_0. Then

$$\begin{aligned}
F_\nabla^0 &= dA_0 + A_0 \wedge A_0 \\
&= d\left(\frac{k\overline{z}}{1+z\overline{z}}\right)\wedge dz + \underbrace{(\cdots)}_{\text{0-form}}\ dz\wedge dz \\
&= \frac{k(1+z\overline{z})d\overline{z}\wedge dz - k\overline{z}\,(\overline{z}dz + zd\overline{z})\wedge dz}{(1+z\overline{z})^2} \\
&= \frac{-kdz\wedge d\overline{z}}{(1+z\overline{z})^2}.
\end{aligned}$$

Rewriting, the curvature is

$$F_\nabla^0 = \frac{-kd(x+iy)\wedge d(x-iy)}{(1+(x+iy)(x-iy))^2} = \frac{2ikdx\wedge dy}{(1+x^2+y^2)^2}\ , \qquad \text{and} \qquad c_1((\gamma^1)^{\otimes k}) = \frac{-kdx\wedge dy}{\pi(1+x^2+y^2)^2}$$

is the first Chern form. To compare the equivalence class of $c_1((\gamma^1)^{\otimes k})$ with the equivalence class of the preferred generator, we integrate this form over the manifold[1]. The integral of the preferred class over $\mathbf{CP}^1$

[1] This is called the *Chern number* of the manifold, and integration over the manifold is evaluation of the *fundamental class*, which exists because $\mathbf{CP}^1$ is orientable and compact.

is 1, and the integral of $c_1((\gamma^1)^{\otimes k})$ is

$$\begin{aligned}
\int_{\mathbf{CP}^1} c_1((\gamma^1)^{\otimes k}) &= \frac{-k}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dx\wedge dy}{(1+x^2+y^2)^2} \\
&= \frac{-k}{\pi}\int_0^{\infty}\int_0^{2\pi}\frac{rd\theta\wedge dr}{(1+r^2)^2} \\
&= -2k\int_0^{\infty}\frac{rdr}{(1+r^2)^2} \\
&= -k\int_1^{\infty}\frac{du}{u^2} \\
&= -k\left.\frac{-1}{u}\right|_{u=1}^{u=\infty} \\
&= -k.
\end{aligned}$$

Since -1 is the value of the integral of the negative of the preferred generator, it follows that $c_1((\gamma^1)^{\otimes k})$ is k times the negative of the preferred generator of $H^2_{dR}(\mathbf{CP}^1;\mathbf{R})$, as desired. ■

Proposition 4.3.15. The axiomatic properties of the Chern classes are satisfied by the above definition.

Proof: Axiom (C_0) follows by Definition 4.3.11 and by noting that first, $c_0(E) = 1$ since the degree 0 homogeneous polynomial is 1. Second, $c_m(E) = 0$ for all $m > n$ whenever E is a rank n, since $\sigma_m = 0$. For axiom (C_1), suppose that $f : N \to M$ is a smooth map of base spaces and ∇ is a connection on E. Following Example 4.2.6 v) of [Huy05] and Lemma 3 in Appendix C of [MS74], there is a *pullback connection* $f^*\nabla$ on f^*E. To define it, consider a trivialization domain $U_i \subset M$, on which we write $\nabla = d + A_i$ for some $n \times n$ matrix A_i of one forms, as we showed above. Then on the trivialization domain $f^{-1}(U_i)$, we have that $(f^*\nabla)|_{f^{-1}(U_i)} = d + f^*A_i$. Hence on $f^{-1}(U_i)$, applying the result of Remark 4.3.8,

$$F_{(f^*\nabla)|_{f^{-1}(U_i)}} = F_{d+f^*A_i} = d(f^*A_i) + (f^*A_i)\wedge(f^*A_i) = f^*(dA_i + A_i\wedge A_i) = f^*(F_{\nabla|_{U_i}}).$$

The local definitions glue to a global definition of $f^*\nabla$, so the curvature of a pullback connection is the pullback of the curvature of the original connection globally. Hence

$$c_k(f^*E, f^*\nabla) = \sigma_k\left(\frac{iF_{f^*\nabla}}{2\pi}\right) = \sigma_k\left(\frac{if^*F_\nabla}{2\pi}\right) = f^*\sigma_k\left(\frac{iF_\nabla}{2\pi}\right) = f^*c_k(E,\nabla),$$

where the third equality follows as $\widetilde{P}(f^*A) = f^*\widetilde{P}(A)$ for any homogeneous polynomial $\widetilde{P}$. For axiom (C_2), the sum of line bundles $E \oplus G$ (here G is chosen as the other bundle, instead of F, to lessen confusion with the curvature F) has connection $\nabla_E \oplus \nabla_G$, and curvature $F_{\nabla_E} \oplus F_{\nabla_G}$, which follows directly from the definition of the curvature. Then applying the definitions above,

$$\begin{aligned}
c(E\oplus G) &= \det\left(I_{E\oplus G} + \frac{iF_{\nabla_E}\oplus F_{\nabla_G}}{2\pi}\right) \\
&= \det\left(\left(I_E + \frac{iF_{\nabla_E}}{2\pi}\right)\oplus\left(I_G + \frac{iF_{\nabla_G}}{2\pi}\right)\right) \\
&= \det\left(I_E + \frac{iF_{\nabla_E}}{2\pi}\right)\det\left(I_G + \frac{iF_{\nabla_G}}{2\pi}\right) \\
&= c(E) \smile c(G).
\end{aligned}$$

Axiom (C_3) follows from the case $k = 1$ of Proposition 4.3.14. ■

4.4 Existence 4: Sheaf theory

Unlike the previous section showing existence of the whole Chern class, here we only show existence of the first Chern class, in Čech cohomology. This is enough, however, to define the whole Chern class, as the splitting principle from Section 5.3 shows. We follow Chapter 5 of [Pra07], Chapter 2 of [Har77], and Chapter 1 of [Bre97] for the main definitions. The notes [Vak13] are recommended for an excellent introduction to the wider concept of algebraic geometry.

4.4.1 Sheaves, presheaves, and other definitions

Definition 4.4.1. Let X be a topological space. Then $\mathcal{F}$ is a *presheaf* on X if

i. for every $U \subset X$, there is an abelian group $\mathcal{F}(U)$, with $\mathcal{F}(\emptyset) = \{0\}$,

ii. for every $V \subset U \subset X$, there is a homomorphism $\text{res}_{U\to V} : \mathcal{F}(U) \to \mathcal{F}(V)$ such that $\text{res}_{U\to U} = \text{id}_U$, and

iii. for every $W \subset V \subset U$, the composition $\text{res}_{V\to W} \circ \text{res}_{U\to V}$ is equal to $\text{res}_{U\to W}$.

A presheaf $\mathcal{F}$ is a *sheaf* on X, if whenever $\{V_i\}_{i\in I}$ an open covering of U,

i. if $f \in \mathcal{F}(U)$ and $\text{res}_{U\to V_i}(f) = 0$ for all $i \in I$, then $f = 0$, and

ii. if $f_k \in \mathcal{F}(V_k)$ and $\text{res}_{V_i\to V_i\cap V_j}(f_i) = \text{res}_{V_j\to V_i\cap V_j}(f_j)$ for all $i, j \in I$, then there exists $f \in \mathcal{F}(U)$ such that $\text{res}_{U\to V_k}(f) = f_k$ for all $k \in I$.

Inclusion of open sets $U \subset X$ creates a poset of open sets containing $p \in X$ (with the natural restriction maps). For $q, r \in \mathcal{F}_p$ with $q \in \mathcal{F}(U)$ and $r \in \mathcal{F}(V)$, write $q \sim r$ if there exists $W \subset U \cap V$ with $p \in W$ and $\text{res}_{U\to W}(q) = \text{res}_{V\to W}(r)$, which is an equivalence relation. Then the equivalence class $[q]$ is called the *germ* of q at $p \in U$. The *stalk* of a presheaf $\mathcal{F}$ at $p \in X$ is defined as the direct limit

$$\mathcal{F}_p := \varinjlim_{U\ni p} \Big[\mathcal{F}(U)\Big] = \{q \in \mathcal{F}(U) \ : \ U \subset X,\ p \in U\}/\sim .$$

The stalk $\mathcal{F}_p$ is also called the *set of germs* of $\mathcal{F}$ at p.

Definition 4.4.2. Suppose that $\mathcal{F}$ is a presheaf on a topological space X. The *sheafification* of $\mathcal{F}$ is a sheaf $\mathcal{F}^+$ on X such that for all $U \subset X$

$$\mathcal{F}^+(U) := \left\{ f : U \to \bigsqcup_{p\in V} \mathcal{F}_p \ : \ \begin{array}{l} f(p) \in \mathcal{F}_p \text{ and for all } p, \text{ there is some open } V \ni p \text{ and} \\ t \in \mathcal{F}(V) \text{ such that } f = t \text{ as functions from } V \text{ to } \bigsqcup_{r\in V} \mathcal{F}_r \end{array} \right\}.$$

Definition 4.4.3. Let X be a smooth manifold. For every $U \subset X$, let $\mathcal{A}(U)$ be the group of all complex-valued differentiable functions on U, with function addition as the group action. The restriction maps $\text{res}_{U\to V}$ are defined naturally, by a restriction $f|_V$ to a subset. Then $\mathcal{A}$ is called the *sheaf of germs of differentiable functions* on X. Analogously, $\mathcal{A}^*$ is called the *sheaf of germs of non-zero differentiable functions on* X, where the group operation is function multiplication.

Example 4.4.4. Consider the presheaf $\mathcal{F}$ for which $\mathcal{F}(U) = G$ for all non-empty $U \subset X$, and $\mathcal{F}(\emptyset) = 0$. This sheaf is called the *constant presheaf.* The sheafification of such an $\mathcal{F}$ is called the *constant sheaf*, and has the property that $\mathcal{F}_p = G$ at all the stalks.

For instance, when $G = \mathbf{Z}$, the sheafification is denoted by $\underline{\mathbf{Z}}$, and $\underline{\mathbf{Z}}_p = \mathbf{Z}$ for all p. Therefore $\underline{\mathbf{Z}}$ is called the *sheaf of locally constant integer-valued functions.*

Definition 4.4.5. Let $\mathcal{F}, \mathcal{G}$ be sheaves on a topological space X. A *morphism of sheaves* $\psi : \mathcal{F} \to \mathcal{G}$ is a collection of homomorphisms

$$\left\{ f_U : \mathcal{F}(U) \to \mathcal{G}(U) \ : \ \text{res}^{\mathcal{G}}_{U\to V} \circ f_U = f_V \circ \text{res}^{\mathcal{F}}_{U\to V} \right\},$$

or equivalently, maps f_U for all $U \subset X$ and $V \subset U$ such that the diagram below commutes.

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "f_U"] \arrow[d, "\mathrm{res}^{\mathcal{F}}_{U\to V}"'] & \mathcal{G}(U) \arrow[d, "\mathrm{res}^{\mathcal{G}}_{U\to V}"] \\
\mathcal{F}(V) \arrow[r, "f_V"'] & \mathcal{G}(V)
\end{tikzcd}$$

An *sequence of sheaves* is an ordered collection $\{\mathcal{F}_i\}$ of sheaves and morphisms $\psi_i : \mathcal{F}_i \to \mathcal{F}_{i+1}$ for all i. The sequence is called *exact* if the associated sequence of stalks is exact. Sequences of sheaves are written as

$$\cdots \xrightarrow{\psi_{i-1}} \mathcal{F}_i \xrightarrow{\psi_i} \mathcal{F}_{i+1} \xrightarrow{\psi_{i+1}} \cdots .$$

Note that once we have a morphism of sheaves, we may turn it into a morphism of stalks via the direct limit. We also take it as a definition that a sequence of sheaves is exact if and only if it is exact on the stalks, although exactness for sheaves may be defined with the image sheaf and kernel sheaf. For more on this statement, the interested reader is directed to the discussion after Definition 2.1 in Chapter I of [Bre97].

4.4.2 Sequences and cohomology

The next proposition follows an example from §2.5 of [Hir66], using what is sometimes called the *exponential sheaf sequence*.

Proposition 4.4.6. Let $X \ni p$ be a smooth manifold, and define maps on stalks

$$\begin{array}{rccl} \alpha_p : & \underline{\mathbf{Z}}_p & \to & \mathcal{A}_p, \\ & f & \mapsto & f, \end{array} \qquad \text{and} \qquad \begin{array}{rccl} \beta_p : & \mathcal{A}_p & \to & \mathcal{A}_p^*, \\ & f & \mapsto & e^{2\pi i f}. \end{array}$$

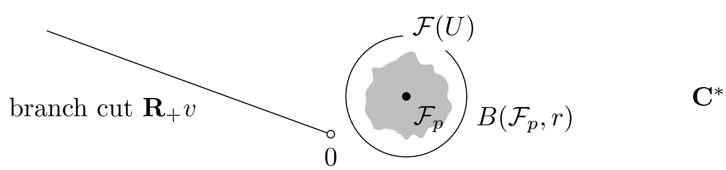
For α and β the analogous maps on the whole sheaves, the following sequence is exact:

$$0 \longrightarrow \underline{\mathbf{Z}} \xrightarrow{\alpha} \mathcal{A} \xrightarrow{\beta} \mathcal{A}^* \longrightarrow 0. \tag{4.4.1}$$

Proof: A sequence of sheaves is exact if and only if it is exact on the stalks, so we restrict ourselves to some $p \in X$. To be exact at $\underline{\mathbf{Z}}_p$, it must be that $\ker(\alpha_p)$ is trivial, or α_p is injective on U. It is the inclusion map, which is injective by definition, so we have exactness at $\underline{\mathbf{Z}}$.

For exactness at $\mathcal{A}_p$, note that β_p is a group homomorphism as addition in $\mathcal{A}_p$ becomes multiplication of the images in $\mathcal{A}_p^*$. Also note that the identity element $f = 0$ gets taken to the identity element $g = 1 = e^{2\pi i 0}$. The kernel of the map β_p is $\{[f] \in \mathcal{A}_p \ : \ e^{2\pi i g} = 1 \text{ for some } g \in [f]\}$, and whenever g takes values in the integers, it satisfies $e^{2\pi i g} = 1$. Hence $\ker(\beta_p) \cong \mathbf{Z} \cong \mathrm{im}(\alpha_p)$, and the sequence is exact at $\mathcal{A}_p$.

Finally, for exactness at $\mathcal{A}_p^*$, the map β must be surjective. Take $g \in \mathcal{A}_p^*$, for which $\beta_p : \log(g)/(2\pi i) \mapsto g$ for a given branch of the complex logarithm. The function β_p is well-defined, because we may assume without loss of generality that U is small enough so that $\mathcal{F}(U)$ lies in a small ball around $\mathcal{F}_p$ in $\mathbf{C}^*$, and so a branch of the complex logarithm may be fixed, as in the diagram below.



Hence β_p is surjective, and the sequence is exact at $\mathcal{A}_p^*$, so the whole sequence is exact. ■

Now we introduce the notion of Čech cohomology. Sheaf cohomology is a slightly more general concept, but the two are equivalent in most settings (and in the main setting presented here, that of a paracompact Hausdorff space).

Definition 4.4.7. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be a cover of a topological space X, and $\mathcal{F}$ a presheaf on X. Let k be any non-negative integer. Then

· a *k-cochain* is a map c^k acting on $\mathcal{U}$ that assigns to every collection of $k+1$ sets $(U_{\alpha_0}, \dots, U_{\alpha_k})$ an element in $\mathcal{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_k})$;

· under addition, the group of all k-cochains is denoted by $C^k(\mathcal{U}; \mathcal{F})$;

· the *coboundary operator* $\delta : C^k(\mathcal{U}; \mathcal{F}) \to C^{k+1}(\mathcal{U}; \mathcal{F})$ is a map defined by

$$\left(\delta c^k\right)\left(U_{\alpha_0}, \dots, U_{\alpha_{k+1}}\right) = \sum_{i=0}^{k+1} (-1)^i \, c^k \left(U_{\alpha_0}, \dots, \widehat{U_{\alpha_i}}, \dots, U_{\alpha_{k+1}}\right) \Big|_{U_{\alpha_0} \cap \cdots \cap U_{\alpha_{k+1}}} \tag{4.4.2}$$

with the α_i distinct. The notation $\widehat{U_{\alpha_i}}$ means that the argument U_{α_i} is omitted. When the domain of δ is unclear, write $\delta^k : C^k \to C^{k+1}$. This map is a homomorphism of cochains, and due to the coefficient $(-1)^i$, has the property that $\delta\delta = 0$. Then the *kth Čech cohomology group of $\mathcal{U}$* is

$$\check{H}^k(\mathcal{U}; \mathcal{F}) := \ker(\delta : C^k \to C^{k+1}) / \mathrm{im}(\delta : C^{k-1} \to C^k)$$

for Z^k the group of *k-cocycles* and B^k the group of *k-coboundaries* as in Definition 2.4.4. To define these cohomology groups over just X, we take the direct limit of covers $\mathcal{U}$ of X, with a partial ordering given by refinements of covers. That is, $\mathcal{U} < \mathcal{V}$ if and only if $\mathcal{U} = \{U_\alpha\}$ is a refinement of $\mathcal{V} = \{V_\beta\}$, meaning that $U_\alpha \subset V_{\beta(\alpha)}$ for some appropriate $\beta(\alpha)$. Hence define the *kth Čech cohomology group of X*

$$\check{H}^k(X; \mathcal{F}) := \varinjlim_{\mathcal{U} \text{ covers } X} \left[\check{H}^k(\mathcal{U}; \mathcal{F})\right].$$

Remark 4.4.8. The direct limit may be difficult to calculate sometimes, but fortunately there is a special type of cover $\mathcal{U}$ that eases the calculation. A cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of X is called a *Leray cover* of 1st order of X if U_α is contractible for all $\alpha \in I$ (if X is a manifold, such a cover always exists). If $\mathcal{U}$ is a Leray cover of X, then $\check{H}^1(\mathcal{U}; \mathcal{F}) \cong \check{H}^1(X; \mathcal{F})$, as in Theorem 12.8 in [For91]. There is a more general definition for a Leray cover of kth order that relates $\check{H}^k(\mathcal{U}; \mathcal{F})$ and $\check{H}^k(X; \mathcal{F})$, but that definition is beyond the scope of this paper.

There is a theorem in homological algebra that every short exact sequence induces a long exact sequence on cohomology (see, for example, Theorem 2.16 of [Hat02] or §6 of Chapter 5 of [Bre93]). A similar approach works for sheaves, in the sense that every short exact sequence on sheaves induces a long exact sequence on the Čech cohomology of the sheaves (see, for example, §3 of Chapter 0 of [GH94]). Continuing with the same sequence from (4.4.1), there exists a long exact sequence of sheaves

$$\begin{tikzcd} 0 \arrow[r] & \check{H}^0(X; \underline{\mathbf{Z}}) \arrow[r, "\check{\alpha}"] & \check{H}^0(X; \mathcal{A}) \arrow[r, "\check{\beta}"] & \check{H}^0(X; \mathcal{A}^*) \arrow[dll, "\rho"] & \\ & \check{H}^1(X; \underline{\mathbf{Z}}) \arrow[r, "\check{\alpha}"] & \check{H}^1(X; \mathcal{A}) \arrow[r, "\check{\beta}"] & \check{H}^1(X; \mathcal{A}^*) \arrow[r] & \cdots . \end{tikzcd} \tag{4.4.3}$$

4.4.3 The Chern class of a complex line bundle

This section uses the ideas of §2.11 of Chapter 1 of [Hir66]. To describe in more detail the properties and usefulness of these objects, consider the smooth manifold $X = \mathbf{CP}^1 = G_1(\mathbf{C}^2)$. Then using the same short exact sequence as in (4.4.1) to get the same long exact sequence as in (4.4.3), consider the following section of it:

$$\cdots \longrightarrow \check{H}^1(\mathbf{CP}^1;\mathcal{A}) \longrightarrow \check{H}^1(\mathbf{CP}^1;\mathcal{A}^*) \xrightarrow{\rho} \check{H}^2(\mathbf{CP}^1;\underline{\mathbf{Z}}) \longrightarrow \check{H}^2(\mathbf{CP}^1;\mathcal{A}) \longrightarrow \cdots \tag{4.4.4}$$

Let's calculate the two cohomology groups. First, de Rham's theorem (stated in Section 5.4, but not proved) states that in this particular case, Čech cohomology is isomorphic to singular cohomology, so

$$\check{H}^2(\mathbf{CP}^1;\underline{\mathbf{Z}}) \cong H^2_S(\mathbf{CP}^1;\mathbf{Z}) \cong H^2_S(\mathbf{S}^2;\mathbf{Z}) = \mathbf{Z}.$$

To calculate the other group, use the open cover $\mathcal{U} = \{U_0, U_1\}$ of $\mathbf{CP}^1$, given by $U_0 = \mathbf{CP}^1 \setminus \{0\}$ and $U_1 = \mathbf{CP}^1 \setminus \{1\}$. Since $\mathbf{CP}^1$ is $\mathbf{S}^2$ topologically, $U_0 \cong \mathbf{C}$ can be thought of as the sphere with the south pole removed, and $U_2 \cong \mathbf{C}$ as the sphere with the north pole removed. This cover is a Leray cover of 1st order for $\check{H}^1$. We calculate $\check{H}^1(\mathbf{CP}^1;\mathcal{A}^*)$ indirectly, by proving that the two groups $\check{H}^i(\mathbf{CP}^1;\mathcal{A})$, for $i = 1,2$ in (4.4.4) vanish, so ρ must be an isomorphism. Since $C^2 = 0$, it must be that

$$Z^1 = \ker(\delta : C^1 \to C^2) = C^1. \tag{4.4.5}$$

To show that $\check{H}^1(\mathbf{CP}^1;\mathcal{A}) = 0$, we have to show that $\mathrm{im}(\delta : C^0 \to C^1) = C^1$. So take $\check{f} \in C^0$ and $\check{g} \in C^1$. The inclusion $\mathrm{im}(\delta : C^0 \to C^1) \subset C^1$ is clear, so it remains to show the other inclusion. Let $\eta_\alpha : \mathcal{A}(\mathbf{CP}^1) \to [0,1]$ be a partition of unity subordinate to $\{U_0, U_1\}$, so $\eta_0(f) + \eta_1(f) = 1$ on all of $\mathcal{A}(\mathbf{CP}^1)$, but $\eta_\alpha(f) = 0$ whenever $\mathrm{supp}(f) \subset U_\alpha^c$, for $\alpha = 0,1$. Consider new functions $d^i :\ C^i \to C^{i-1}$ for $i = 1,2$ defined by

$$\left(d^1\check{g}\right)(U_\alpha) = \underbrace{\left(\eta_0\check{g}\right)(U_0, U_\alpha)}_{\text{non-zero only on } U_0\cap U_\alpha} + \underbrace{\left(\eta_1\check{g}\right)(U_1, U_\alpha)}_{\text{non-zero only on } U_1\cap U_\alpha}$$

and

$$\left(d^2\check{h}\right)(U_\alpha, U_\beta) = \underbrace{\left(\eta_0\check{h}\right)(U_0, U_\alpha, U_\beta)}_{\text{non-zero only on } U_0\cap U_\alpha\cap U_\beta} + \underbrace{\left(\eta_1\check{h}\right)(U_1, U_\alpha, U_\beta)}_{\text{non-zero only on } U_1\cap U_\alpha\cap U_\beta}.$$

Next observe that

$$\begin{aligned}
\left(d^2 \circ \delta^1\right)(\check{g})(U_\alpha, U_\beta) &= \left(\eta_0\delta^1\check{g}\right)(U_0, U_\alpha, U_\beta) + \left(\eta_1\delta^1\check{g}\right)(U_1, U_\alpha, U_\beta) \\
&= \left(\eta_0\check{g}\right)(U_\alpha, U_\beta) - \left(\eta_0\check{g}\right)(U_0, U_\beta) + \left(\eta_0\check{g}\right)(U_0, U_\alpha) \\
&\qquad + \left(\eta_1\check{g}\right)(U_\alpha, U_\beta) - \left(\eta_1\check{g}\right)(U_1, U_\beta) + \left(\eta_1\check{g}\right)(U_1, U_\alpha)
\end{aligned}$$

and

$$\begin{aligned}
\left(\delta^0 \circ d^1\right)(\check{g})(U_\alpha, U_\beta) &= \left(d^1\check{g}\right)(U_\beta) - \left(d^1\check{g}\right)(U_\alpha) \\
&= \left(\eta_0\check{g}\right)(U_0, U_\beta) + \left(\eta_1\check{g}\right)(U_1, U_\beta) - \left(\eta_0\check{g}\right)(U_0, U_\alpha) - \left(\eta_1\check{g}\right)(U_1, U_\alpha),
\end{aligned}$$

and adding the two together, four terms cancel, leaving

$$\left(d^2 \circ \delta^1 + \delta^0 \circ d^1\right)(\check{g})(U_\alpha, U_\beta) = \left(\eta_0\check{g}\right)(U_\alpha, U_\beta) + \left(\eta_1\check{g}\right)(U_\alpha, U_\beta) = \check{g}(U_\alpha, U_\beta), \tag{4.4.6}$$

which means that $d^2 \circ \delta^1 + \delta^0 \circ d^1 = \mathrm{id}$. If $\check{g}$ is a 1-cocycle, then $d^1\check{g}$ is a 0-cochain. By (4.4.5), $\delta^1\check{g} = 0$, so by the calculated result (4.4.6), we have that $\delta^0(d^1\check{g}) = \check{g}$, meaning that $\check{g}$ is a 1-coboundary. Hence $C^1 \subset \mathrm{im}(\delta^0)$, so $C^1 = \mathrm{im}(\delta^0)$, meaning that $\check{H}^1(\mathbf{CP}^1;\mathcal{A}) = 0$. An identical argument works for $\check{H}^2$, but with larger indices. The general argument of showing that every k-cochain in $\check{H}^k$ is a k-coboundary is described in Appendix D, and works for any sheaf over a topological space with such a partition of unity (such a sheaf is called a *fine* sheaf, and note that $\underline{\mathbf{Z}}$ and $\mathcal{A}^*$ are not fine sheaves).

Hence the two outer groups in (4.4.4) vanish, so ρ is an isomorphism, meaning that $\check{H}^1(\mathbf{CP}^1;\mathcal{A}^*) \cong \mathbf{Z}$. This completes the example, and next we show that there is a natural equivalence between (equivalence classes of) complex line bundles over $\mathbf{CP}^1$ and elements of $H^1(M;\mathcal{A}^*)$.

Proposition 4.4.9. Let X be a topological manifold. Then $\check{H}^1(X;\mathcal{A})$ is isomorphic to the space of isomorphism classes of complex line bundles on X.

Proof: Since X is a topological manifold, it has a Leray cover. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ cover X such that for any complex line bundle L over X, L is trivial over each U_α, in the sense of Definition 2.1.1. If the U_α are contractible, this cover has the desired property. Note that a vector bundle may be completely described by $\mathcal{U}$ and transition functions, as in the discussion after Definition 2.1.1. By the domain and range, these maps satisfy the definition of being 1-cochains, and in this case, also 1-cocycles.

Let L, L' be two line bundles with respective local homeomorphisms φ_α, ψ_α and transition functions $g_{\alpha\beta}$, $g'_{\alpha\beta}$. Then L and L' are isomorphic if and only if there is a smooth map $T\colon L \to L'$ with $T_\alpha = \psi_\alpha T \varphi_\alpha^{-1}$ such that

$$T_\beta = g_{\beta\alpha} T_\alpha g'_{\alpha\beta} \tag{4.4.7}$$

for all $\alpha, \beta \in I$. Since φ_α is a map $\pi^{-1}(U_\alpha) \to U_\alpha \times \mathbf{C}$, it may be viewed as a 1-cochain (or rather, it's inverse). Since C^1 (and all cochain groups) are abelian, condition (4.4.7) says that the difference of two line bundles is

$$g_{\beta\alpha}\left(g'_{\beta\alpha}\right)^{-1} = T_\beta\left(T_\alpha\right)^{-1} = \left(\psi_\beta T \varphi_\beta^{-1}\right)\left(\psi_\alpha T \varphi_\alpha^{-1}\right)^{-1} = \psi_\beta T \varphi_\beta^{-1}\varphi_\alpha T^{-1}\psi_\alpha^{-1} = \underbrace{\left(\psi_\beta\psi_\alpha^{-1}\right)}_{\text{1-coboundary}}\underbrace{\left(\varphi_\beta\varphi_\alpha^{-1}\right)^{-1}}_{\text{1-coboundary}},$$

which is a 1-coboundary. Hence two line bundles are isomorphic if and only if their difference is a 1-coboundary, which is the condition of equivalence of 1-cocycles. ■

Definition 4.4.10. For the isomorphism ρ as defined by (4.4.4), the *first Chern class* of a complex line bundle L is $c_1(L) := \rho(L)$.

Since all the Chern classes are not defined here, the axioms of the Chern class cannot all be checked. However, we may still at least show axiom (C_3), which only concerns the first Chern class. By Theorem 2.3.2, there is always a bundle map from a complex line bundle E into the tautological bundle γ^1, which gives a ring structure to $\check{H}^1(\mathbf{CP}^1;\mathcal{A}^*)$, with preferred generator the dual of the tautological bundle. Since ρ is an isomorphism, it takes the identity to the identity, so γ^1 is mapped to the negative of the preferred generator of $\check{H}^2(\mathbf{CP}^1;\underline{\mathbf{Z}})$.

5 Uniqueness of characteristic classes

Here we show that an arbitrary vector bundle may be pulled back to a Whitney sum of line bundles, allowing the definitions of the Chern class in §4.2 and §4.4 to be extended to arbitrary rank n bundles. To do this, we will need the Leray–Hirsch theorem, a strong statement that we do not prove. The reader is referred to the theorem and proof in other texts, among them Theorem 4D.1 of [Hat02], Theorem 5.11 of [BT82], and Theorem 1.1 of Chapter 17 of [Hus75].

5.1 The projectivization of a vector bundle

Recall the following definitions, mostly from Section 2, for V a vector space over $\mathbf{K}$ (for $\mathbf{K}$ either $\mathbf{R}$ or $\mathbf{C}$):

· *projectivization* of V: the space $\mathbf{P}(V) = \{v \in V \ : \ \lambda_1 v \sim \lambda_2 v,\ \lambda_1,\lambda_2 \in \mathbf{K}\setminus\{0\}\}$.
· *line bundle*: a vector bundle of rank 1.
· *tautological line bundle*: the bundle $\pi : E = \{(x,y) \in \mathbf{P}(\mathbf{K}^n)\times\mathbf{K}^{n+1} \ : \ y \in x\} \to \mathbf{P}(\mathbf{K}^n) \cong G_1(\mathbf{K}^n)$.
· *induced bundle*: for $\pi : E \to M$, a bundle $\pi' : E' = f^*E \to M'$ induced by the map $f : M' \to M$.

Definition 5.1.1. Let $\pi_i : E_i \to M_i$ be vector bundles, $\psi_i : E_i \to E_{i+1}$ bundle maps (or morphisms), in the sense of Definition 2.1.5, and $f_i : M_i \to M_{i+1}$ the associated maps of base spaces, for $i \in I$. Then the diagram below commutes, and the top row (5.1.1) is called a *sequence of vector bundles.*

$$
\begin{tikzcd}
\cdots \arrow[r, "\psi_{i-2}"] & E_{i-1} \arrow[r, "\psi_{i-1}"] \arrow[d, "\pi_{i-1}"'] & E_i \arrow[r, "\psi_i"] \arrow[d, "\pi_i"'] & E_{i+1} \arrow[r, "\psi_{i+1}"] \arrow[d, "\pi_{i+1}"'] & \cdots \\
\cdots \arrow[r, "f_{i-2}"] & M_{i-1} \arrow[r, "f_{i-1}"] & M_i \arrow[r, "f_i"] & M_{i+1} \arrow[r, "f_{i+1}"] & \cdots
\end{tikzcd}
\tag{5.1.1}
$$

The sequence is called *exact* if $\operatorname{im}(\psi_i) = \ker(\psi_{i+1})$ for all i.

Given a complex vector bundle $\pi : E \to M$ of rank n, we would like to construct a space M' and a map $f : M' \to M$ such that the pullback bundle f^*E is a direct sum of line bundles. Such a structure may be created by successive decompositions to the *projectivization* of a vector bundle, with each decomposition reducing the rank of a vector bundle by 1. Let us define this process first by comparing the two bundles.

<u>complex rank n bundle</u>:

$$
\begin{tikzcd}
E \arrow[d, "\pi"'] \\
M
\end{tikzcd}
\quad
\begin{array}{l}
\text{fiber at } p \in M\text{: } E_p, \\
\text{trivialization at } p \in U \subset M\text{: } (U, \varphi).
\end{array}
$$

<u>projectivization of complex rank n bundle</u>:

$$
\begin{tikzcd}
\mathbf{P}(E) \arrow[d, "f"'] \\
M
\end{tikzcd}
\quad
\begin{array}{l}
\text{fiber at } p \in M\text{: } \mathbf{P}(E_p) \cong \mathbf{CP}^{n-1}, \\
\text{trivialization at } p \in U \subset M\text{: } (U, \psi), \\
\qquad \text{with } \psi : f^{-1}(U) \cong \mathbf{P}(\varphi^{-1}(U)) \to U \times \mathbf{CP}^{n-1}.
\end{array}
$$

Using f as a map between manifolds, we can define the pullback bundle f^*E over $\mathbf{P}(E)$, with total space given by $f^*E = \{(x, y) \in \mathbf{P}(E) \times E \ : \ f(x) = \pi(y)\}$. There are two other natural vector bundles related to E over $\mathbf{P}(E)$.

Definition 5.1.2. Let E be a complex rank n vector bundle over M. In the context of the definition above, the *universal subbundle* S of E, or *tautological bundle*, is given by

$$
\begin{array}{rcl}
s: \ S = \{(x, y) \in f^*E \ : \ y \in x\} & \to & \mathbf{P}(E), \\
(x, y) & \mapsto & x.
\end{array}
$$

The fiber at $x \in \mathbf{P}(E)$ is simply x. When the base of S is ambiguous, write $S(E)$ for the same definition. By choosing a Hermitian metric on M, the *universal quotient bundle* $Q := S^\perp$ of E is given by

$$q : Q \to \mathbf{P}(E).$$

The bundle S is a rank 1 subspace of E, hence Q is a complex bundle of rank $n-1$, as f^*E has rank n. The definition of Q gives the split short exact sequence

$$
0 \longrightarrow S \xrightarrow{\ \iota\ } f^*E \xrightarrow{\ \rho\ } Q \longrightarrow 0\ ,
\tag{5.1.2}
$$

which is called the *tautological exact sequence.*

This sequence is split by construction, with ι the natural inclusion map and ρ the natural projection map, and thus $f^*E \cong S \oplus S^\perp$. All this may be summarized in the diagram of vector bundles below.

$$
\begin{tikzcd}
E \arrow[dd, "\pi"'] & 0 \arrow[r] & S \arrow[r, "\iota"] \arrow[dr, dashed, "s"'] & f^*E \arrow[r, "\rho"] \arrow[d] & Q \arrow[r] \arrow[dl, dashed] & 0 \\
 & & & \mathbf{P}(E) \arrow[dlll, "f"'] & & \\
M & & & & &
\end{tikzcd}
\tag{5.1.3}
$$

5.2 The Leray–Hirsch theorem

To prove the main theorem of this section, the splitting principle, we need the following result. The statement uses the more general notion of a *fiber bundle*, which is just a vector bundle, except the condition of $\pi^{-1}(p)$ having the structure of a vector space is removed. Note also the following theorem is given independent of cohomology theory.

Theorem 5.2.1. [Leray, Hirsch]
Let E be a fiber bundle over M paracompact, with projection map π. If there exist $v_1, \ldots, v_n \in H^*(E;\mathbf{Z})$ such that for every $p \in M$ the elements $v_1|_{\pi^{-1}(p)}, \ldots, v_n|_{\pi^{-1}(p)}$ generate $H^*(\pi^{-1}(p);\mathbf{Z})$, then

$$H^*(E;\mathbf{Z}) \cong H^*(M;\mathbf{Z}) \otimes H^*(\pi^{-1}(p);\mathbf{Z}). \tag{5.2.1}$$

That is, $H^*(E;\mathbf{Z})$ is freely generated by $v_1, \ldots, v_n$ over $H^*(M;\mathbf{Z})$.

We proceed as in §20 of [BT82], beginning with the assumption that the first Chern class c_1 of a line bundle has already been defined.

From the tautological short exact sequence (5.1.2), let $x = c_1(S)$, the first Chern class of the universal subbundle of E. Since S is a vector bundle over $\mathbf{P}(E)$, $x \in H^2(\mathbf{P}(E))$ by axiom (C_0). Next, since $\mathbf{P}(E)_p = \mathbf{P}(E_p)$ by definition, it follows directly that $S(E)|_{\mathbf{P}(E)_p} = S(E_p)$. That is, the vector bundle $S(E)$ restricted to a fiber $\mathbf{P}(E)_p$ is the universal subbundle of the line bundle (since E_p has the structure of a vector space) E_p. Then by axiom (C_1),

$$c_1(S(E_p)) = c_1(S(E))|_{\mathbf{P}(E)_p}$$

if we view the restriction map $|_{\mathbf{P}(E)_p}$ as a bundle map from $\mathbf{P}(E)$ to $\mathbf{P}(E_p)$. In the context of diagram (5.1.3), $\mathbf{P}(E)_p = f^{-1}(p)$, it follows that $x|_{f^{-1}(p)}$ generates $H^*(\mathbf{P}(E)_p)$. Since $H^*(\mathbf{CP}^n;\mathbf{Z}) = \mathbf{Z}[x]/(x^{n+1})$, as a vector space $H^*(\mathbf{P}(E)_p)$ is generated by $x^0, \ldots, x^{n-1}$, so we may apply Leray–Hirsch to get that $x^n \in H^*(\mathbf{P}(E))$ may be expressed uniquely as a linear combination of the x^is, with coefficients in $H^*(M)$.

Definition 5.2.2. In the context of the discussion above, the elements $c_i(E) \in H^{2i}(M)$ such that

$$x^n = c_n(E) + c_{n-1}(E)x + \cdots + c_1(E)x^{n-1} \tag{5.2.2}$$

are called the *Chern classes* of E, with c_i the *ith Chern class* of E. The sum $1 + c_1 + \cdots + c_n$ is called the *total Chern class* of E.

We finish this section with a statement about the ring structure of $H^*(\mathbf{P}(E))$. Because of the equality (5.2.2) and the Leray–Hirsch isomorphism (5.2.1), it follows immediately that

$$H^*(\mathbf{P}(E)) = H^*(M)[x]/(c_n(E) + c_{n-1}(E)x + \cdots + c_1(E)x^{n-1} - x^n).$$

The negative sign in front of x^n is not too pleasing, so some authors instead let $x = c_1(S^*) = -c_1(S)$, using the dual bundle from the very beginning, giving reason for the choice of negative generator in Section 3.2. As a result, in the other convention half of the Chern classes, the odd-indexed ones, have the opposite sign. In the interests of keeping the constructive steps clear, this complication in this section so far was avoided. Herein the more prevalent notation is adopted, yielding the new equation

$$H^*(\mathbf{P}(E)) = H^*(M)[x]/(x^n + c_1(E)x^{n-1} + \cdots + c_{n-1}(E)x + c_n(E)). \tag{5.2.3}$$

5.3 The splitting principle

Here we develop a definition of the Chern classes that is independent of cohomology theory.

Definition 5.3.1. Given a rank n vector bundle $\pi : E \to M$, a manifold $F(E)$ is called a *split manifold* for E if there exists a map $f : F(E) \to E$ such that the pullback bundle f^*E decomposes as a sum of line bundles (i.e. $f^*E \cong E_1 \oplus \cdots \oplus E_n$, where E_i is of rank 1 for all i), and the induced map on cohomology groups, $f^* : H^*(M) \to H^*(F(E))$ is injective. In this case, the map f is called the *splitting map*.

Now we show the existence of a split manifold for any rank n vector bundle. First we construct the manifold, without checking the second condition of cohomological group injectivity. This will be done directly afterward.

If E has rank 1, then $F(E) = M$. So suppose that E has rank 2 and take the projectivization of E. Consider the bundles mentioned above on $\mathbf{P}(E)$:

$$\begin{tikzcd}
E \arrow[d, "\pi"'] & S_1 \arrow[dr] & f_1^*E \arrow[d] & Q_1 \arrow[dl] \\
M & & \mathbf{P}(E) \arrow[ll, "f_1"] &
\end{tikzcd}
\qquad
\begin{aligned}
\operatorname{rank}(S_1) &= 1,\\
\operatorname{rank}(Q_1) &= n-1,\\
\operatorname{rank}(f_1^*E) &= n.
\end{aligned}$$

Note that $f_1^*E \cong S_1 \oplus Q_1$ so $\mathbf{P}(E)$ is a split manifold for E with splitting map $\pi_1 = f_1$ and $f_1^*E \cong S_1 \oplus Q_1$, a direct sum of line bundles. Now suppose that E has rank 3. Then we take the only bundle that is not 1-dimensional, Q_1, which is 2-dimensional by (5.1.2), and repeat the process on it:

$$\begin{tikzcd}
E \arrow[d, "\pi"'] & S_1 \arrow[dr] & f_1^*E \arrow[d] & Q_1 \arrow[dl] & S_2 \arrow[dr] & f_2^*Q_1 \arrow[d] & Q_2 \arrow[dl] \\
M & & \mathbf{P}(E) \arrow[ll, "f_1"] & & & \mathbf{P}(Q_1) \arrow[lll, "f_2"] &
\end{tikzcd}
\qquad
\begin{aligned}
\operatorname{rank}(S_2) &= 1,\\
\operatorname{rank}(Q_2) &= n-2,\\
\operatorname{rank}(f_2^*Q_1) &= n-1,\\
\operatorname{rank}(\pi_2^*E) &= n.
\end{aligned}$$

The space $\mathbf{P}(Q_1)$ is a split manifold for E, with splitting map $\pi_2 = f_1 \circ f_2$. As E has rank 3, Q_1 has rank 2, and, consequently, so does $f_2^*Q_1$. The space π_2^*E pulled back from E decomposes as the sum of line bundles as

$$\pi_2^*E \cong (f_1 \circ f_2)^*E \cong f_2^*f_1^*E \cong f_2^*(S_1 \oplus Q_1) \cong f_2^*S_1 \oplus f_2^*Q_1 \cong f_2^*S_1 \oplus S_2 \oplus Q_2. \tag{5.3.1}$$

The bundle S_1 is 1-dimensional, so $f_2^*S_1$ is as well. The bundle S_2 is 1-dimensional by (5.1.2), and Q_2 is 1-dimensional as (5.1.2) is exact. For thoroughness, we do one more level, and suppose that E has rank 4. So the process must be repeated one last time:

$$\begin{tikzcd}
E \arrow[d, "\pi"'] & S_1 \arrow[dr] & f_1^*E \arrow[d] & Q_1 \arrow[dl] & S_2 \arrow[dr] & f_2^*Q_1 \arrow[d] & Q_2 \arrow[dl] & S_3 \arrow[dr] & f_3^*Q_2 \arrow[d] & Q_3 \arrow[dl] \\
M & & \mathbf{P}(E) \arrow[ll, "f_1"] & & & \mathbf{P}(Q_1) \arrow[lll, "f_2"] & & & \mathbf{P}(Q_2) \arrow[lll, "f_3"] &
\end{tikzcd}
\qquad
\begin{aligned}
\operatorname{rank}(S_3) &= 1,\\
\operatorname{rank}(Q_3) &= n-3,\\
\operatorname{rank}(f_3^*Q_2) &= n-2,\\
\operatorname{rank}(\pi_3^*E) &= n.
\end{aligned}$$

A split manifold for E is $\mathbf{P}(Q_2)$ with splitting map $\pi_3 = f_1 \circ f_2 \circ f_3$. The decomposition of π_3^*E into a sum of line bundles is given by

$$\begin{aligned}
\pi_3^*E &\cong (f_1 \circ f_2 \circ f_3)^*E\\
&\cong f_3^*f_2^*f_1^*E\\
&\cong f_3^*(f_2^*S_1 \oplus S_2 \oplus Q_2)\\
&\cong f_3^*f_2^*S_1 \oplus f_3^*S_2 \oplus f_3^*Q_2\\
&\cong f_3^*f_2^*S_1 \oplus f_3^*S_2 \oplus S_3 \oplus Q_3.
\end{aligned}$$

Above we used (5.3.1) to get the third equivalence. For reasons similar to those given previously, each term in the final direct sum is a 1-dimensional bundle. We now generalize the process and let E have rank n:

$$\begin{tikzcd}
E \arrow[d, "\pi"'] & S_1 \arrow[dr] & f_1^*E \arrow[d] & Q_1 \arrow[dl] & S_2 \arrow[dr] & f_2^*Q_2 \arrow[d] & Q_2 \arrow[dl] & S_3 \arrow[dr] & f_3^*Q_2 \arrow[d] & Q_3 \arrow[dl] & & S_{n-1} \arrow[dr] & f_{n-1}^*Q_{n-2} \arrow[d] & Q_{n-1} \arrow[dl] \\
M & & \mathbf{P}(E) \arrow[ll, "f_1"] & & & \mathbf{P}(Q_1) \arrow[lll, "f_2"] & & & \mathbf{P}(Q_2) \arrow[lll, "f_3"] & & \cdots \arrow[ll, "f_4"] & & \mathbf{P}(Q_{n-2}) \arrow[ll, "f_{n-1}"] &
\end{tikzcd}$$

Then a split manifold for E is $\mathbf{P}(Q_{n-2})$ with splitting map $\pi_{n-1} = f_1 \circ f_2 \circ \cdots \circ f_{n-1}$, and the decomposition of the total space given by

$$\pi_{n-1}^* E \cong f_{n-1}^* \cdots f_2^* S_1 \oplus f_{n-1}^* \cdots f_3^* S_2 \oplus \cdots \oplus f_{n-1}^* S_{n-2} \oplus S_{n-1} \oplus Q_{n-1}. \tag{5.3.2}$$

Each of the n terms above is a line bundle. Having constructed the manifold in general, we now check the assertion that the associated map on cohomology groups, π_i^*, is injective at each step.

· If E has rank 1, then $\pi_0^* : H^*(M) \to H^*(F(E)) = H^*(M)$ is the identity map, which is injective.

· If E has rank 2, then apply equation (5.2.3) (also equation (20.7) in [BT82]). We then let $c_i(E)$ be the ith Chern class of E and $x_1 = c_1(S_1^*)$, where S_1^* is the dual bundle of S_1, giving

$$H^*(F(E)) = H^*(M)[x_1]/(x_1^2 + c_1(E)x_1 + c_2(E)).$$

Then $H^*(M)$ embeds in $H^*(F(E))$ by the isomorphism (5.2.1), which is just f_1^*. Indeed, this isomorphism is the justification for the embedding of all the cohomology classes of M into those of $F(E)$.

· If E has rank 3, then again applying (5.2.3), with $x_1 = c_1(S_2^*)$ and $x_2 = f_2^* c_1(S_1^*)$,

$$H^*(F(E)) = H^*(M)[x_1, x_2] \Big/ \begin{pmatrix} x_1^3 + c_1(E)x_1^2 + c_2(E)x_1 + c_3(E), \\ x_2^2 + c_1(Q_1)x_2 + c_2(Q_1) \end{pmatrix}.$$

· If E has rank 4, then similarly with $x_1 = c_1(S_3^*)$, $x_2 = f_3^* c_1(S_2^*)$, and $x_3 = f_3^* f_2^* c_1(S_1^*)$,

$$H^*(F(E)) = H^*(M)[x_1, x_2, x_3] \Bigg/ \begin{pmatrix} x_1^4 + c_1(E)x_1^3 + c_2(E)x_1^2 + c_3(E)x_1 + c_4(E), \\ x_2^3 + c_1(Q_1)x_2^2 + c_2(Q_1)x_2 + c_3(Q_1), \\ x_3^2 + c_1(Q_2)x_3 + c_2(Q_2) \end{pmatrix}.$$

· If E has rank n, then similarly with $x_1 = c_1(S_{n-1}^*)$, $\ldots$, and $x_{n-1} = f_{n-1}^* f_{n-2}^* \cdots f_2^* c_1(S_1^*)$,

$$H^*(F(E)) = H^*(M)[x_1, x_2, \ldots, x_{n-1}] \Bigg/ \begin{pmatrix} x_1^n + c_1(E)x_1^{n-1} + \cdots + c_{n-1}(E)x_1 + c_n(E), \\ x_2^{n-1} + c_1(Q_1)x_2^{n-2} + \cdots + c_{n-2}(Q_1)x_2 + c_{n-1}(Q_1), \\ \vdots \\ x_{n-2}^3 + c_1(Q_{n-3})x_{n-2}^2 + c_2(Q_{n-3})x_{n-2} + c_3(Q_{n-3}), \\ x_{n-1}^2 + c_1(Q_{n-2})x_{n-1} + c_2(Q_{n-2}) \end{pmatrix}.$$

Note that if E has rank 1, then $E \cong S$, so Q a rank 0 vector space, and taking the projectivization of Q gives a rank 0 bundle on the empty set (and it does not make sense to say that $\mathbf{P}(Q)$ is the split manifold here). So for a rank n bundle, we let $\mathbf{P}(Q_m) = \mathbf{P}(Q_{n-2})$ for all $m \geqslant n-1$, so that later calculations are easier. That is, we say that if E is a rank n bundle, then taking the projectivization $n-1$ times of the appropriate spaces or greater than $n-1$ times gives the same split manifold. So if $E_1, \ldots, E_m$ are vector bundles of different rank all over M, then there exists a splitting map $\sigma : F(E_\bullet) \to M$. Within this setting, consider the following theorem.

Theorem 5.3.2. [The Splitting principle]
Let $E_1, \ldots, E_m$ be rank $r_1, \ldots, r_m$ vector bundles, respectively, all over M, and $F(E_\bullet)$ a split manifold of all the bundles E_i with splitting map σ. If $P(\phi_1(c(E_1), \ldots, c(E_m)), \ldots, \phi_k(c(E_1), \ldots, c(E_m)))$ is a polynomial expression about the Chern classes of the vector bundles, for ϕ_i a direct sum, direct product, tensor product, etc, then

$$P(\phi_1(c(\sigma^* E_1), \ldots, c(\sigma^* E_m)), \ldots, \phi_k(c(\sigma^* E_1), \ldots, c(\sigma^* E_m))) = 0$$

implies

$$P(\phi_1(c(E_1), \ldots, c(E_m)), \ldots, \phi_k(c(E_1), \ldots, c(E_m))) = 0.$$

That is, to show that a polynomial identity is true on the Chern classes of the E_i, it is sufficient to show that the identity is true on the Chern classes of the respective direct sums of line bundles of the E_i, over the split manifold $F(E_\bullet)$.

Proof: We assume the naturality of the Chern classes (axiom (C_0) in §3.2). Hence

$$\begin{aligned}0 &= P(\phi_1(c(\sigma^*E_1),\ldots,c(\sigma^*E_m)),\ldots,\phi_k(c(\sigma^*E_1),\ldots,c(\sigma^*E_m)))\\ &= P(\phi_1(\sigma^*c(E_1),\ldots,\sigma^*c(E_m)),\ldots,\phi_k(\sigma^*c(E_1),\ldots,\sigma^*c(E_m)))\\ &= P(\sigma^*\phi_1(c(E_1),\ldots,c(E_m)),\ldots,\sigma^*\phi_k(c(E_1),\ldots,c(E_m)))\\ &= \sigma^*P(\phi_1(c(E_1),\ldots,c(E_m)),\ldots,\phi_k(c(E_1),\ldots,c(E_m))).\end{aligned}$$

The second equality uses naturality of the Chern classes, as $c(\sigma^*E_i) = \sigma^*(c(E_i))$. Next, since σ^* distributes over the ϕ_i and P is a polynomial, we get the next equalities. Finally, since σ^* is injective, as shown above, it must be that

$$P(\phi_1(c(E_1),\ldots,c(E_m)),\ldots,\phi_k(c(E_1),\ldots,c(E_m))) = 0.$$

■

This theorem finally gives us uniqueness of the Chern class. Indeed, suppose that there are two natural transformations $c, c' : E \to H^*(M)$ satisfying the axioms of §3.2. Since the first Chern class over the tautological line bundle is determined by axiom (C_3), it follows that $c_1(\gamma^1) = c_1'(\gamma^1)$, and moreover, by the $\mathbf{C}^\infty$ version of Theorem 2.3.2, $c(L) = c'(L)$ for any line bundle L. Let $\sigma : F(E) \to E$ be the splitting map of E, so then applying axioms (C_1) and (C_2),

$$\begin{aligned}\sigma^*c(E) &= c(\sigma^*E)\\ &= c(L_1 \oplus \cdots \oplus L_m)\\ &= c(L_1) \smile \cdots \smile c(L_m)\\ &= c'(L_1) \smile \cdots \smile c'(L_m)\\ &= c'(L_1 \oplus \cdots \oplus L_m)\\ &= c'(\sigma^*E)\\ &= \sigma^*c'(E).\end{aligned}$$

Hence $c(\sigma^*E) - c'(\sigma^*E) = 0$, and by the splitting principle, it follows that $c(E) - c'(E) = 0$, so $c(E) = c'(E)$. Therefore the axioms of §3.2 uniquely determine the Chern class.

5.4 Equivalence of cohomology theories

So far we showed that when defined in a general cohomology theory, there is only one definition of the Chern classes. Another way would have been to use the Leray–Hirsch theorem and the splitting principle in a particular cohomology theory, and apply two theorems of de Rham cohomology (which are only stated here). For both, let M be an n-dimensional closed manifold.

Theorem 5.4.1. [de Rham]

$$H^*_{dR}(M) \cong H^*_S(M; \mathbf{R}).$$

The theorem also asserts that cup products of cochains correspond to wedge products of forms. The interested reader is directed to §3 in Chapter 5 of [Pra07] for a proof, which goes through a triangulation of M, and by constructing special maps, gives the desired equality on simplices.

Theorem 5.4.2. [Čech, de Rham]

$$H^*_{dR}(M) \cong \check{H}^*(M; \mathcal{K}).$$

This theorem uses the constant sheaf $\mathcal{K}$, with $\mathcal{K}(U) = \mathbf{R}$ for all open sets $U \subset M$. A complete proof is given by Theorem 8.9 in [BT82], which uses the generalized Mayer–Vietoris principle and some diagram chasing.

6 Examples

In this section we will compute the characteristic classes of certain complex vector bundles. A general convention is, when given a manifold M, to write $c(M)$ for the Chern class of the tangent bundle $c(TM)$.

6.1 The sphere S^2

As there is an isomorphism between S^2 and $\mathbf{CP}^1 = \{[z_0 : z_1] \in \mathbf{C}^2 \ : \ z_i \neq 0 \text{ for some } i\}$, so we calculate the Chern classes of the tangent bundle of $\mathbf{CP}^1$. The two charts that cover $\mathbf{CP}^1$ are $U_0 = \{z_0 \neq 0\}$ and $U_1 = \{z_1 \neq 0\}$, with maps $[z_0 : z_1] \mapsto w_0 = \frac{z_1}{z_0}$ and $[z_0 : z_1] \mapsto w_1 = \frac{z_0}{z_1}$. On $U_0 \cap U_1$, we may write $w_1 = \frac{1}{w_0}$, hence $dw_1 = -dw_0/w_0^2$. The vector fields $\frac{\partial}{\partial w_0}$ and $\frac{\partial}{\partial w_1}$ span the respective tangent spaces, and we claim that the transition function from U_0 to U_1 is $-w_0^2$. This claim follows as

$$dw_1 = -\frac{dw_0}{w_0^2} \quad \Longrightarrow \quad dw_1\left(\frac{\partial}{\partial w_1}\right) = -\frac{dw_0}{w_0^2}\left(\frac{\partial}{\partial w_1}\right) \quad \Longrightarrow \quad dw_0\left(\frac{\partial}{\partial w_1}\right) = -w_0^2 \quad \Longrightarrow \quad \frac{\partial}{\partial w_1} = -w_0^2\frac{\partial}{\partial w_0}.$$

However, if we choose to start with $-\frac{\partial}{\partial w_0}$ instead of $\frac{\partial}{\partial w_0}$, then the transition function is w_0^2. Next, note that $\mathbf{CP}^1 = G_1(\mathbf{C}^2)$, the base space for the tautological bundle $\gamma^1 = \gamma^1(\mathbf{C}^2)$. Recall that γ^1 has total space

$$\begin{aligned} E &= \left\{(X, x) \ : \ x \in X \in G_1(\mathbf{C}^2)\right\} \\ &= \{((z_0, z_1), (y_0, y_1)) \ : \ \exists\, \lambda \in \mathbf{C} \text{ s.t. } (y_0, y_1) = \lambda(z_0, z_1)\} \end{aligned}$$

with projection map $\pi(X, x) = X$. The same two charts U_0, U_1 trivialize γ^1, except the spanning sections here are $(z_0, z_1) \mapsto (1, \frac{z_1}{z_0})$ and $(z_0, z_1) \mapsto (\frac{z_0}{z_1}, 1)$. Using $w_0 = \frac{z_1}{z_0}$ and $w_1 = \frac{z_0}{z_1}$ as before, it follows immediately that the transition function from U_0 to U_1 is $1/w_0$, as

$$(w_1, 1) = \left(\frac{1}{w_0}, \frac{w_0}{w_0}\right) = \frac{1}{w_0}\,(1, w_0)\,.$$

Next, if E is a vector bundle with transition functions $g_{\alpha\beta}$, and F a vector bundle with transition functions $h_{\alpha\beta} = g_{\alpha\beta}^{-1}$, then $F = E^*$, the dual bundle of E. Since $(1/z_0)^{-2} = z_0^2$, it follows that the tangent bundle of $\mathbf{CP}^1$ is the tensor product of twice the dual of the tautological bundle γ^1. In other words,

$$TS^2 \cong T\mathbf{CP}^1 \cong (\gamma^1)^* \otimes (\gamma^1)^*.$$

Proposition 4.2.7 gives that the only non-zero homology groups of $G_1(\mathbf{C}^2)$ are $H^0 = H^2 = \mathbf{Z}$, so the ring may be given by $H_S^*(G_1(\mathbf{C}^2); \mathbf{Z}) = \mathbf{Z}[x]/(x^2)$, with $c_1(\gamma^1) := -x$. To relate the Chern class of a bundle to the Chern class of the dual bundle, recall Proposition 4.3.13, which gives that $c_1((\gamma^1)^*) = -c_1(\gamma^1) = x$. Finally, apply Proposition 4.3.13 to get that

$$c_1(S^2) = c_1((\gamma^1)^* \otimes (\gamma^1)^*) = 2c_1((\gamma^1)^*) = 2x.$$

By the Chern class axioms, $c_0(S^2) = 1$, so $c(S^2) = 1 + 2x$.

6.2 The projective space $\mathbf{CP}^n$

Recall that $\mathbf{CP}^n \cong G_1(\mathbf{C}^{n+1})$, which is a smooth complex n-dimensional manifold, so the tangent bundle is a rank n complex vector bundle. Let $S = \gamma^1(\mathbf{C}^{n+1})$, the tautological line bundle over $\mathbf{CP}^n$. Then S is a subbundle of the trivial bundle $\underline{\mathbf{C}}^{n+1}$ over $\mathbf{CP}^n$. Consider the vector bundle diagram

$$\begin{tikzcd} S \arrow[dr] & \subset & \underline{\mathbf{C}}^{n+1} \arrow[d] & = & \mathbf{CP}^n \times \mathbf{C}^{n+1} \\ & & \mathbf{CP}^n & & \end{tikzcd}$$

and note that $S = \{(\ell, v) \; : \; v \in \ell\} \subset \{(\ell, v) \; : \; \ell \in \mathbf{CP}^n, v \in \mathbf{C}^{n+1}\} = \mathbf{CP}^n \times \mathbf{C}^{n+1}$. It follows immediately that $\underline{\mathbf{C}}^{n+1} \cong S \oplus S^\perp$ as vector bundles. Next, recall from the proof of Theorem 2.2.2 that for a line ℓ in $\mathbf{C}^{n+1}$, there is a neighborhood $U_\ell \cong \mathrm{Hom}(\ell, \ell^\perp) \cong \mathrm{Hom}(S_\ell, S_\ell^\perp) \cong \mathrm{Hom}(S, S^\perp)_\ell$. The tangent space of U_ℓ at ℓ is the same as the tangent space of $\mathbf{CP}^n$ at ℓ, so

$$T_\ell \mathbf{CP}^n = T_\ell U_\ell \cong T_\ell \left(\mathrm{Hom}(S, S^\perp)_\ell\right) = \mathrm{Hom}(S, S^\perp)_\ell.$$

Since $T_\ell U_\ell = T_\ell U_{\ell'}$ for any other $\ell' \in U_\ell$, the neighborhoods are compatible. Hence

$$T\mathbf{CP}^n \cong \mathrm{Hom}(S, S^\perp) \cong S^* \otimes S^\perp$$

from the definition of $\mathrm{Hom}(V, W) \cong \mathrm{Hom}(V, \mathbf{C}) \otimes W$ for vector spaces V, W over $\mathbf{C}$. Next, tensor $\underline{\mathbf{C}}^{n+1} \cong S \oplus S^\perp$ on both sides with S^* to get

$$S^* \otimes \underline{\mathbf{C}}^{n+1} = S^* \otimes (\underline{\mathbf{C}} \oplus \cdots \oplus \underline{\mathbf{C}}) = S^* \oplus \cdots \oplus S^* = (S^*)^{\oplus n+1}$$

on the left and

$$S^* \otimes \left(S \oplus S^\perp\right) = (S^* \otimes S) \oplus \left(S^* \otimes S^\perp\right) \cong \underline{\mathbf{C}} \oplus T\mathbf{CP}^n$$

on the right. Hence $T\mathbf{CP}^n \oplus \underline{\mathbf{C}} \cong (\gamma^1(\mathbf{C}^n)^*)^{\oplus n+1}$. Then by the Whitney product formula,

$$\begin{aligned}
c(\mathbf{CP}^n) &:= c(T\mathbf{CP}^n) \\
&= c(T\mathbf{CP}^n) \smile [1] \\
&= c(T\mathbf{CP}^n) \smile c(\underline{\mathbf{C}}) \\
&= c(T\mathbf{CP}^n \oplus \underline{\mathbf{C}}) \\
&= c((\gamma^1(\mathbf{C}^n)^*)^{\oplus n+1}) \\
&= c(\gamma^1(\mathbf{C}^n)^*)^{\smile n+1} \\
&= (1+x)^{n+1}.
\end{aligned}$$

The last line comes from the observations of the previous example. This result agrees with the case $n = 1$ above, since $c(\mathbf{CP}^1) = (1+x)^2 = 1 + 2x + x^2$ and $x^2 = 0$.

6.3 The projective space $\mathbf{CP}^n \times \mathbf{CP}^m$

For the cartesian product of manifolds, the tangent bundle is described by $T(M \times N) = \pi_M^* TM \oplus \pi_N^* TN$, where $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$. Using the previous example, let $c(\mathbf{CP}^n) = (1+x)^{n+1}$ and $c(\mathbf{CP}^m) = (1+y)^{m+1}$. By the Whitney sum formula and naturality, $c(\mathbf{CP}^n \times \mathbf{CP}^m) = \pi_1^* c(\mathbf{CP}^n) \smile \pi_2^* c(\mathbf{CP}^m)$, so

$$c_k(\mathbf{CP}^n \times \mathbf{CP}^m) = \sum_{i=0}^{k} \pi_1^* c_i(\mathbf{CP}^n) \smile \pi_2^* c_{k-i}(\mathbf{CP}^m) = \sum_{i=0}^{k} \binom{n+1}{i} \binom{m+1}{k-i} \pi_1^* x^i \smile \pi_2^* y^{k-i}.$$

Since $H_S^*(\mathbf{CP}^n \times \mathbf{CP}^m; \mathbf{Z}) = \mathbf{Z}[x, y]/(x^{n+1}, y^{m+1})$, the Künneth formula says that $\pi_1^* x^i$ is, up to isomorphism, the element x^i (and analogously for y^i). Hence the expression above is in the same cohomology class as $\sum_{i=0}^{k} \binom{n+1}{i} \binom{m+1}{k-i} x^i \smile y^{k-i}$.

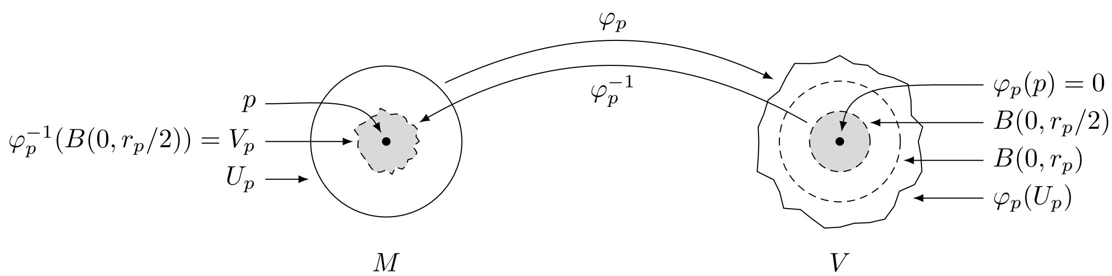
Appendices

A Locally finite covers of compact spaces

Here we prove a lemma for the proof of Theorem 2.3.2, regarding nested sets of closed and open covers.

Lemma A.1. Let M be a compact topological n-manifold. Then there exists a cover $\mathcal{U} = U_1, \ldots, U_r$ of M such that there are open sets $V_i \subset U_i$ for all i with $\mathrm{cl}(V_i) \subset U_i$ and $\bigcup_i V_i = M$.

Proof: Since M is compact, for all $p \in M$ there exists a set $U_p \subset M$ such that $\varphi_p(U_p) \subset \mathbf{R}^n$ for some homeomorphism $\varphi_p : U_p \to \varphi_p(U_p) \subset \mathbf{R}^n$. Without loss of generality (by translation), assume that $\varphi_p(p) = 0$. Since φ_p is a homeomorphism, there is some $r_p > 0$ such that $B(0, r_p) \subset \varphi_p(U_p)$, for $B(x, r)$ the open ball centered at x with radius r. Let $V_p = \varphi_p^{-1}(B(0, r_p/2))$, so then $\mathrm{cl}(V_p) \subset U_p$ as desired. This process is described in the diagram below.



Since M is compact, there exists a finite subcover $\mathcal{V} = \{V_1, \ldots, V_r\}$, such that $\mathcal{V}$ still covers M. Then, letting U_i be the associated set of V_i in the construction above, we have a cover $\mathcal{U} = \{U_1, \ldots, U_r\}$ that covers M and satisfies the conditions. ■

In the proof of Theorem 2.3.2, the sets U_i and V_i need to be trivialization domains. This is possible because given U_i, we may assume without loss of generality, by shrinking U_i, that the bundle trivializes over U_i.

B Orientation of a vector space

Consider the set $\mathbf{R}^n_0$ of all the non-zero vectors in $\mathbf{R}^n$ and the relative singular cohomology groups $H^i_S(\mathbf{R}^n, \mathbf{R}^n_0; \mathbf{Z})$, as in Definition 2.4.4. As in §3.1 of [Hat02] and §4.1 above, begin with the long exact sequence of the pair $(\mathbf{R}^n, \mathbf{R}^n_0)$, given by

$$\cdots \longrightarrow H^{j-1}_S(\mathbf{R}^n_0; \mathbf{Z}) \xrightarrow{\delta} H^j_S(\mathbf{R}^n, \mathbf{R}^n_0; \mathbf{Z}) \longrightarrow H^j_S(\mathbf{R}^n; \mathbf{Z}) \longrightarrow H^j_S(\mathbf{R}^n_0; \mathbf{Z}) \xrightarrow{\delta} H^{j+1}_S(\mathbf{R}^n, \mathbf{R}^n_0; \mathbf{Z}) \longrightarrow \cdots,$$

which starts with $0 \to H^0_S(\mathbf{R}^n, \mathbf{R}^n_0; \mathbf{Z}) \to \cdots$. For $i > 0$, the group $H^i_S(\mathbf{R}^n; \mathbf{Z})$ is trivial, and since $\mathbf{R}^n_0$ is homotopy equivalent to S^{n-1}, for $i > 1$ it follows that

$$H^i_S(\mathbf{R}^n, \mathbf{R}^n_0; \mathbf{Z}) \cong H^{i-1}_S(\mathbf{R}^n_0; \mathbf{Z}) \cong H^{i-1}_S(S^{n-1}; \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } i = n, \\ 0 & \text{else.} \end{cases}$$

Next, consider the exact sequence

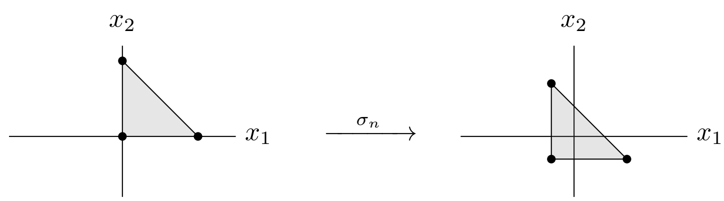
$$\begin{array}{ccccccccc} 0 \longrightarrow H^0_S(\mathbf{R}^n, \mathbf{R}^n_0; \mathbf{Z}) \longrightarrow & H^0_S(\mathbf{R}^n; \mathbf{Z}) & \xrightarrow{\alpha} & H^0_S(\mathbf{R}^n_0; \mathbf{Z}) & \xrightarrow{\delta} H^1_S(\mathbf{R}^n, \mathbf{R}^n_0; \mathbf{Z}) \longrightarrow 0. \\ & \parallel & & \parallel & \\ & \mathbf{Z} \text{ for all } n & & \mathbf{Z} \text{ for } n = 0, n \geqslant 2 & \end{array}$$

When $n = 0$, the map α is the identity, so since the sequence is exact, $H^0_S(\mathbf{R}^n, \mathbf{R}^n_0; \mathbf{Z}) = 0$ and $H^0_S(\mathbf{R}^0_0; \mathbf{Z}) = 0$. When $n = 1$, $H^0_S(\mathbf{R}^1_0; \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}$ and $\alpha(x) = (x, x)$, so the map α is an injection. Again applying properties of an exact sequence, we have $H^0_S(\mathbf{R}^1, \mathbf{R}^1_0; \mathbf{Z}) = 0$ and $H^1_S(\mathbf{R}^n, \mathbf{R}^n_0; \mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}/\mathbf{Z} \cong \mathbf{Z}$. Therefore

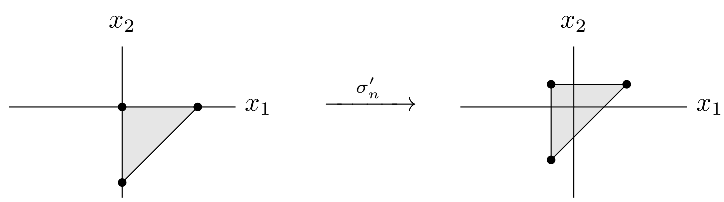
$$H^i_S(\mathbf{R}^n, \mathbf{R}^n_0; \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } i = n, \\ 0 & \text{else.} \end{cases}$$

Theorem B.1. A choice of orientation of $\mathbf{R}^n$ corresponds to a choice of generator of $H^n_S(\mathbf{R}^n, \mathbf{R}^n_0; \mathbf{Z})$.

Proof: Let $(e_1, \ldots, e_n)$ be the standard ordered basis of $\mathbf{R}^n$. Let Δ_n be the simplex defined by the ordered list of vertices $(0, e_1, \ldots, e_n)$. Let $\sigma_n : \Delta_n \to \mathbf{R}^n$ be the map that acts on Δ_n only by translation, moving the centroid (or barycenter) of Δ_n to the origin of $\mathbf{R}^n$. For example, in the case of $n = 2$:



Since the boundary $\partial(\sigma_n \Delta_n)$ is contained in $\mathbf{R}^n_0$, the map σ_n represents an element of the relative cocyle group $Z^n(\mathbf{R}^n, \mathbf{R}^n_0; \mathbf{Z})$. Since $H^n_S(\mathbf{R}^n, \mathbf{R}^n_0; \mathbf{Z}) = \mathbf{Z}$, the map σ^n is the preferred generator of $\mathbf{Z}$. Repeat the above process instead with the ordered basis $(e_1, \ldots, e_{n-1}, -e_n)$ with an analogous simplex Δ'_n and map σ'_n to get that σ'_n is the negative of the preferred generator of $\mathbf{Z} = H^n_S(\mathbf{R}^n, \mathbf{R}^n_0; \mathbf{Z})$. For example, again with $n = 2$, we have the following:



Note that the map $m : \sigma_n \Delta_n \to \sigma'_n \Delta_n$ is the -1 multiplication map on $H^n_S(\mathbf{R}^n, \mathbf{R}^n_0; \mathbf{Z})$ (described more in Exercise 7 in §2.2 of [Hat02]). Since the definition of orientation of a vector space, given by Definition 3.1.1, says that two orientations of $\mathbf{R}^n$ are the same iff the change of basis matrix between the two has positive determinant, and the map from $(e_1, \ldots, e_n)$ to $(e_1, \ldots, e_{n-1}, -e_n)$ has negative determinant, each generator of the mentioned cohomology group corresponds to an orientation of $\mathbf{R}^n$. ■

Definition B.2. With reference to the construction above, the *preferred generator* of $H^n_S(\mathbf{R}^n, \mathbf{R}^n_0; \mathbf{Z})$ is the cohomology class $u = [\sigma_n \Delta_n] \in H^n_S(\mathbf{R}^n, \mathbf{R}^n_0; \mathbf{Z})$.

All of the above may be applied to a vector space V, and so also to a fiber of a vector bundle.

Corollary B.3. (adapted from Theorem 9.1 in [MS74]) For E an oriented rank n bundle,

$$H^i_S(E, E_0; \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } i = n, \\ 0 & \text{if } i < n. \end{cases}$$

Moreover, the restriction $u|_{(F,F_0)} \in H^n_S(F, F_0; \mathbf{Z})$ of the preferred generator $u \in H^n(F, F_0; \mathbf{Z})$ is the preferred generator of the fiber F, for all fibers F.

C The Chern–Weil theorem

Lemma C.1. (appears as Lemma 4.4.4 in [Huy05]) Let $\gamma_j \in \mathcal{A}^{i_j}(\mathrm{End}(E))$ for $j = 1, \dots, k$, so $P(\gamma_1, \dots, \gamma_k) \in \mathcal{A}^{i_1+\cdots+i_k}(M)$. Then, for any connection ∇ on E,

$$d(P(\gamma_1, \dots, \gamma_k)) = \sum_{j=1}^{k} (-1)^{i_1+\cdots+i_{j-1}} P(\gamma_1, \dots, \gamma_{j-1}, d^\nabla \gamma_j, \gamma_{j+1}, \dots, \gamma_k).$$

Proof: Fix a local trivialization on U. Then $\gamma_j \in \mathcal{A}^{i_j}(\mathbf{K}^{r\times r})$, or equivalently γ_j is a matrix of i_j-forms on U. In this trivalization, as in the discussion before Theorem 4.3.12, the operator d^∇ on $\mathcal{A}^*(\mathrm{End}(E))$ is $d + [A, \cdot]$, for $A \in \mathcal{A}^1(E)$ such that $\nabla = d + A$ locally on E. That is, $d^\nabla \gamma_j = d\gamma_j + [A, \gamma_j]$ for the bracket defined on page 26. Without loss of generality, $\gamma_j = \omega_j \otimes T_j$ for $\omega_j \in \mathcal{A}^{i_j}(M)$ and $T_j \in \Gamma(\mathrm{End}(E))$. First compute

$$\begin{aligned} P(\gamma_1, \dots, \gamma_k) &= P(\omega_1 \otimes T_1, \dots, \omega_k \otimes T_k) \\ &= \omega_1 \wedge \cdots \wedge \omega_k P(T_1, \dots, T_k) \\ &= \omega_1 \wedge \cdots \wedge \omega_k P((T_1)^{a_1}_{b_1} e^{b_1} \otimes e_{a_1}, \dots, (T_k)^{a_k}_{b_k} e^{b_k} \otimes e_{a_k}) \\ &= (\omega_1 (T_1)^{a_1}_{b_1}) \wedge \cdots \wedge (\omega_k (T_k)^{a_k}_{b_k}) \underbrace{P(e^{b_1} \otimes e_{a_1}, \dots, e^{b_k} \otimes e_{a_k})}_{\text{constant function}}, \end{aligned}$$

for $T_j = (T_j)^{a_j}_{b_j} e^{b_j} \otimes e_{a_j}$. Then, for $\gamma_j = (\gamma_j)^a_b e^b \otimes e_a$, that is, $(\gamma_j)^a_b = \omega_j (T_j)^a_b$,

$$\begin{aligned} d(P(\gamma_1, \dots, \gamma_k)) &= \sum_{j=1}^{k} (-1)^{i_1+\cdots+i_{j-1}} (\gamma_1)^{a_1}_{b_1} \wedge \cdots \wedge d((\gamma_j)^{a_j}_{b_j}) \wedge \cdots \wedge (\gamma_k)^{a_k}_{b_k} P(e^{b_1} \otimes e_{a_1}, \dots, e^{b_k} \otimes e_{a_k}) \\ &= \sum_{j=1}^{k} (-1)^{i_1+\cdots i_{j-1}} P(\gamma_1, \dots, \gamma_{j-1}, d\gamma_j, \gamma_{j+1}, \dots, \gamma_k). \end{aligned}$$

■

Lemma C.2. [Generalization of infinitesimal invariants]
Let $C_1, \dots, C_k \in \mathcal{A}^{2m}(\mathrm{End}(E))$ for some $m \in \mathbf{Z}$, where m might change for each C_i. Let $B \in \mathcal{A}^1(\mathrm{End}(E))$ and $P : gl(r, \mathbf{K}) \to \mathbf{K}$ be a k-linear symmetric invariant map. Then

$$\sum_{j=1}^{k} (-1)^{i_1+\cdots+i_{j-1}} P(C_1, \dots, C_{j-1}, [B, C_j], C_{j+1}, \dots, C_k) = 0.$$

Proof: By linearity assume without loss of generality that all forms are decomposable. We start with $\omega \in \mathcal{A}^1(M)$, $S \in \Gamma(\mathrm{End}(E))$ and $B = \omega \otimes S$. Also, $\omega_j \in \mathcal{A}^{2m}(M)$ for some m, $T_j \in \Gamma(\mathrm{End}(E))$, and $C_j = \omega_j \otimes T_j$. Then, using the bracket operation introduced in the proof of Theorem 4.3.12, $[B, C_j] = \omega \wedge \omega_j [S, T_j]$ and

$$\begin{aligned} P(C_1, \dots, C_{j-1}, [B, C_j], C_{j+1}, \dots, C_k) &= \omega_1 \wedge \cdots \wedge \omega_{j-1} \wedge (\omega \wedge \omega_j) \wedge \omega_{j+1} \wedge \cdots \wedge \omega_k \\ &\qquad P(T_1, \dots, T_{j-1}, [S, T_j], T_{j+1}, \dots, T_k) \\ &= \omega \wedge (\omega_1 \wedge \cdots \wedge \omega_k) P(T_1, \dots, T_{j-1}, [S, T_j], T_{j+1}, \dots, T_k). \end{aligned}$$

Sum over all j from 1 to k and apply the fact from the previous lemma that in a local trivialization, $d^\nabla \gamma_j = d\gamma_j + [A, \gamma_j]$ to get

$$\begin{aligned} d(P(\gamma_1, \dots, \gamma_k)) &= \sum_{j=1}^{k} (-1)^{i_1+\cdots i_{j-1}} P(\gamma_1, \dots, \gamma_{j-1}, d^\nabla \gamma_j, \gamma_{j+1}, \dots, \gamma_k) \\ &\qquad - \sum_{j=1}^{k} (-1)^{i_1+\cdots+i_{j-1}} P(\gamma_1, \dots, \gamma_{j-1}, [A, \gamma_j], \gamma_{j+1}, \dots, \gamma_k) \\ &= 0. \end{aligned}$$

■

D Fine sheaves

Here we show a generalization from §4.4.3. Let X be a compact topological space and $\mathcal{F}$ a sheaf such that $\mathcal{F}(U)$ is a $C^\infty(U)$-module for all U open in X. Such a sheaf $\mathcal{F}$, in particular, is a fine sheaf, for which $\check{H}^k(X;\mathcal{F}) = 0$ for all $k > 0$. For C^k the space of k-cochains in $\check{H}^k(X;\mathcal{F})$, we show that $Z^k = \ker(\delta : C^k \to C^{k+1}) = \operatorname{im}(\delta : C^{k-1} \to C^k) = B^k$. Since $B^k \subset Z^k$ follows from the definition of the map δ, we show that $Z^k \subset B^k$. Since every cover of a manifold admits a countable subcover, we restrict our attention to countable covers.

Let $\mathcal{U} = \{U_0, U_1, \dots\}$ be a cover of X. Let $\eta_\alpha : X \to [0,1]$ be a partition of unity subordinate to $\mathcal{U}$, so $\eta_0(f) + \eta_1(f) + \cdots = 1$ on all of X and $\operatorname{supp}(\eta_{\alpha_i}) \subset U_{\alpha_i}$, for all α_i. Consider new functions $d^i :\ C^i \to C^{i-1}$ for $i = k, k+1$ defined by

$$\left(d^k \check{g}\right)\left(U_{\alpha_0}, \dots, U_{\alpha_{k-1}}\right) = \sum_i \underbrace{\left(\eta_i \check{g}\right)\left(U_i, U_{\alpha_0}, \dots, U_{\alpha_{k-1}}\right)}_{\substack{\text{non-zero only on} \\ U_0 \cap U_{\alpha_0} \cap \cdots \cap U_{\alpha_{k-1}}}}$$

and

$$\left(d^{k+1} \check{h}\right)\left(U_{\alpha_0}, \dots, U_{\alpha_k}\right) = \sum_i \underbrace{\left(\eta_i \check{h}\right)\left(U_i, U_{\alpha_0}, \dots, U_{\alpha_k}\right)}_{\substack{\text{non-zero only on} \\ U_0 \cap U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}}},$$

where $\alpha_i \in \{0, 1, \dots\}$ for all i, and $\alpha_i \neq \alpha_j$ unless $i = j$. Applying the above definitions and the coboundary operator from (4.4.2) gives the desired result. When the map δ is applied, we omit the set to which it is restricted. So

$$\begin{aligned}
\left(d^{k+1} \circ \delta^k\right)(\check{g})\left(U_{\alpha_0}, \dots, U_{\alpha_k}\right) &= \sum_i \left(\eta_i \delta^k \check{g}\right)\left(U_i, U_{\alpha_0}, \dots, U_{\alpha_k}\right) \\
&= \sum_i \left(\left(\eta_i \check{g}\right)\left(U_{\alpha_0}, \dots, U_{\alpha_k}\right) - \sum_j (-1)^j \left(\eta_i \check{g}\right)\left(U_i, U_{\alpha_0}, \dots, \widehat{U_{\alpha_j}}, \dots, U_{\alpha_k}\right) \right)
\end{aligned}$$

and

$$\begin{aligned}
\left(\delta^{k-1} \circ d^k\right)(\check{g})\left(U_{\alpha_0}, \dots, U_{\alpha_k}\right) &= \sum_j (-1)^j \left(d^k \check{g}\right)\left(U_{\alpha_0}, \dots, \widehat{U_{\alpha_j}}, \dots, U_{\alpha_k}\right) \\
&= \sum_j \sum_i (-1)^j \left(\eta_i \check{g}\right)\left(U_i, U_{\alpha_0}, \dots, \widehat{U_{\alpha_j}}, \dots, U_{\alpha_k}\right).
\end{aligned}$$

Since the nested summand of the first expression is the negative of the second expression, adding the two expressions together gives

$$\begin{aligned}
\left(d^{k+1} \circ \delta^k + \delta^{k-1} \circ d^k\right)(\check{g})\left(U_{\alpha_0}, \dots, U_{\alpha_k}\right) &= \sum_i \left(\eta_i \check{g}\right)\left(U_{\alpha_0}, \dots, U_{\alpha_k}\right) \\
&= \left(\sum_i \eta_i\right)(\check{g})\left(U_{\alpha_0}, \dots, U_{\alpha_k}\right) \\
&= (\check{g})\left(U_{\alpha_0}, \dots, U_{\alpha_k}\right),
\end{aligned}$$

so $d^{k+1} \circ \delta^k + \delta^{k-1} \circ d^k$ is the identity operator on C^k. Let $\check{g} \in Z^k$, so $\delta^k \check{g} = 0$. Hence $\delta^{k-1}(d^k \check{g}) = \check{g}$, so $\check{g}$ is also a k-coboundary, that is, $\check{g} \in B^k$. Hence $Z^k \subset B^k$, completing the calculation.

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