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(updated later)

MATH 549 Exercises

1. (August 24) Find an atlas on the extended complex plane $\mathbf{C} \cup \{\infty\}$.

   Solution: Consider the sets
   $$\begin{array}{rcl} U_0 & = & \mathbf{C}, \\ U_1 & = & \mathbf{C} \setminus \{0\} \cup \{\infty\}, \end{array}$$
   and the two maps
   $$\begin{array}{rcl} \varphi_0 \ : \ \mathbf{C} & \to & \mathbf{R}^2, \\ x+iy & \mapsto & (x,y), \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \varphi_0 \ : \ \mathbf{C} \setminus \{0\} \cup \{\infty\} & \to & \mathbf{R}^2, \\ x+iy & \mapsto & (1/x, 1/y). \end{array}$$
   It is immediate that U_0, U_1 cover the space and that the images of φ_0, φ_1 are open, since they are surjective onto all of $\mathbf{R}^2$. The intersections are easily seen to be
   $$\varphi_0(U_0 \cap U_1) = \varphi_1(U_0 \cap U_1) = \mathbf{R}^2 \setminus \{0\},$$
   so
   $$\begin{aligned} \varphi_0 \circ \varphi_1^{-1}(x,y) &= \varphi_0\left(\frac{1}{x} + \frac{i}{y}\right) = \left(\frac{1}{x}, \frac{1}{y}\right), \\ \varphi_1 \circ \varphi_0^{-1}(x,y) &= \varphi_1(x+iy) = \left(\frac{1}{x}, \frac{1}{y}\right). \end{aligned}$$
   Hence the composition is a $\mathbf{C}^\infty$ map with C^∞ inverse. ■

2. (August 24) Find an atlas on the real projective space $\mathbf{RP}^n = \{$1-dimensional subspaces of $\mathbf{R}^n\}$.

   Solution: Recall that any point in $\mathbf{RP}^n$ is represented by an $(n+1)$-tuple $[x_0 : \cdots : x_n]$, where $x_i \in \mathbf{R}$, the coordinates are never all simultaneously zero, and points are equivalent under non-zero scalar multiplication. So consider the sets
   $$\begin{array}{rcl} U_0 & = & \{[1 : x_1 : \cdots : x_n] \ : \ x_i \in \mathbf{R}_{\neq 0}\}, \\ U_1 & = & \{[x_0 : 1 : x_2 \cdots : x_n] \ : \ x_i \in \mathbf{R}_{\neq 0}\}, \\ \vdots & & \\ U_n & = & \{[x_0 : x_1 : \cdots : x_{n-1} : 1] \ : \ x_i \in \mathbf{R}_{\neq 0}\}, \end{array}$$
   which clearly cover all of $\mathbf{RP}^n$. For our maps, consider
   $$\begin{array}{rcl} \varphi_i \ : \ U_i & \to & \mathbf{R}^n, \\ [x_1 : \cdots : x_{i-1} : 1 : x_{i+1} : \cdots : x_n] & \mapsto & (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n). \end{array}$$
   As these maps are surjective, $\varphi_i(U_i)$ is open. ■

3. (August 28) Show that the stereographic projection $\pi : S^2 \setminus \{N\} \to \mathbf{R}^2$ is a diffeomorphism, for N the "north pole" of the sphere S^2.

Solution: The north pole is chosen to be $(0,0,1)$, and the stereographic projection is given by

$$\begin{array}{rcl} \pi \ : \ S^2 \setminus \{N\} & \to & \mathbf{R}^2, \\ (x,y,z) & \mapsto & \frac{(x,y)}{1-z}. \end{array}$$

Here the unit sphere is centered at the origin of $\mathbf{R}^3$, and we are considering $\mathbf{R}^2$ to be the xy-plane in $\mathbf{R}^3$. Since $z \neq 1$, dividing by $1-z$ is a smooth operation, so π is smooth. The inverse of π is

$$\begin{array}{rcl} \varphi \ : \ \mathbf{R}^2 & \to & S^2 \setminus \{N\}, \\ (X,Y) & \mapsto & \left(\frac{2X}{X^2+Y^2+1}, \frac{2Y}{X^2+Y^2+1}, \frac{X^2+Y^2-1}{X^2+Y^2+1}\right), \end{array}$$

which may be seen to be the inverse as

$$\begin{aligned} \varphi(\pi((x,y,z)) &= \varphi\left(\frac{x}{1-z}, \frac{y}{1-z}\right) \\ &= \left(\frac{\frac{2x}{1-z}}{\frac{x^2}{(1-z)^2}+\frac{y^2}{(1-z)^2}+1}, \frac{\frac{2y}{1-z}}{\frac{x^2}{(1-z)^2}+\frac{y^2}{(1-z)^2}+1}, \frac{\frac{x^2}{(1-z)^2}+\frac{y^2}{(1-z)^2}-1}{\frac{x^2}{(1-z)^2}+\frac{y^2}{(1-z)^2}+1}\right) \\ &= \left(\frac{2x(1-z)}{x^2+y^2+(1-z)^2}, \frac{2y(1-z)}{x^2+y^2+(1-z)^2}, \frac{x^2+y^2-(1-z)^2}{x^2+y^2+(1-z)^2}\right) \\ &= \left(\frac{2x(1-z)}{2-2z}, \frac{2y(1-z)}{2-2z}, \frac{2z-2z^2}{2-2z}\right) \\ &= (x,y,z). \end{aligned}$$

This inverse is also smooth, since $X^2+Y^2+1 \neq 0$, as $X^2, Y^2 \geqslant 0$ and $1 > 0$. Hence the stereographic projection is smooth with a smooth inverse, so we have a diffeomorphism. ■

4. (August 28) Show that $O(n)$, the space of orthogonal $n \times n$ matrices, and $SO(n)$, the space of orthogonal matrices with determinant 1, are both manifolds.

Solution: Recall that $A \in O(n)$ iff $AA^T = I$. Consider the map $F : M_n \to \mathrm{Sym}(M_n)$, given by $A \mapsto AA^T$, where M_n is the set of real-valued $n \times n$ matrices. The derivative of F at A is map DF_A given by

$$\begin{aligned} 0 &= \lim_{\|H\|\to 0} \left[\frac{\|F(A+H) - F(A) - DF_A(H)\|}{\|H\|}\right] \\ &= \lim_{\|H\|\to 0} \left[\frac{\|AA^T + AH^T + HA^T + HH^T - AA^T - DF_A(H)\|}{\|H\|}\right] \\ &= \lim_{\|H\|\to 0} \left[\frac{\|AH^T + HA^T + HH^T - DF_A(H)\|}{\|H\|}\right]. \end{aligned}$$

It follows that $DF_A(H) = HA^T + AH^T$. Consider the case $H = KA$ for some matrix K. Then $D_A(H) = KAA^T + AA^TK^T$, so if $A \in F^{-1}(I)$, then $DF_A(H) = K + K^T$. Suppose we start with a matrix S. Then $DF_A(KA) = K + K^T = S$, so $K = S/2$. Hence $DF_A(H)$ is surjective, and applying the theorem from class, $F^{-1}(I)$ is a manifold.

For $SO(n)$, which is the matrices $A \in O(n)$ with determinant 1, consider the determinant $\det : O(n) \to \mathbf{R}$. It is a smooth function, and the image of det is $\{-1, 1\}$. This means that $O(n)$ has at least two connected components, and no componenet contains matrices with both determinant 1 and -1. Therefore the connected componenets of $O(n)$ that map to $+1$ under det (there happens to be just one, but we do not prove this) are $SO(n)$. Since a connected component of a manifold is a manifold in its own right (by refinements of charts), $SO(n)$ is a manifold. ■

5. (August 31) Show that a smooth map of manifolds is continuous, using the topology of the manifolds.

6. (August 31) Show that $SO(3)$ is diffeomorphic to $\mathbf{RP}^3$.

   Solution: To see this, view $SO(3)$ as the space of rotations in $\mathbf{R}^3$ and $\mathbf{R}P^3$ as $S^3/(x \sim -x)$, the 3-sphere modulo the antipodal relation. Further, view the 3-sphere as the 3-dimensional solid ball with radius π with boundary identified, that is,

   $$S^3 \cong B^3/\partial B^3.$$

   We now construct an identification between the two spaces. An arbitrary element of $SO(3)$ looks like

   $$(\underbrace{(x,y,z)}_{\in S^2},\ \underbrace{\theta}_{\in[-\pi,\pi)}\ ) \in SO(3),$$

   with $((x,y,z),\theta) \sim ((-x,-y,-z),-\theta)$. An arbitrary element of S^3 looks like

   $$(\underbrace{(x,y,z)}_{\in \partial B^2 = S^2},\ \underbrace{\theta}_{\in[-\pi,\pi)}\ ) \in S^3,$$

   where (x,y,z) represents a direction in $\mathbf{R}^3$, and θ is the length of the radius in B^3 (which we are viewing concurrently as having radius 1 (for $SO(3)$) and radius π (for S^3)). When we apply the antipodal map $((x,y,z),\theta) \sim ((-x,-y,-z),-\theta)$ in S^3 (to match the one in $SO(3)$ above), we get $\mathbf{R}P^3$, as desired. The map is a diffeomorphism, since it is the identity as presented. ■

7. (September 2) Show that $C^\infty(M)$, the space of smooth maps $M \to \mathbf{R}$, is a vector space.

   Solution: To show that it is a vector space, we need to show it is closed under addition and scalar multiplication. So let $f,g \in C^\infty(M)$, for which

   $$(f+g)(x) = f(x) + g(x) \in \mathbf{R},$$

   so $f+g \in C^\infty(M)$. Similarly, for any scalar $c \in \mathbf{R}$, we have

   $$(cf)(x) = c \cdot f(x) \in \mathbf{R},$$

   so $cf \in C^\infty(M)$. Hence $C^\infty(M)$ is a vector space. ■

8. (September 2) Describe an n-dimensional analogue of the smooth bump function presented in class.

   Solution: Consider the function f in 1 variable, given by

   $$f(t) = \begin{cases} e^{-1/t} & \text{if } t > 0, \\ 0 & \text{if } t \leqslant 0. \end{cases}$$

This is a C^∞ function. Define

$$g(t) = \frac{f(t)}{f(t)+f(1-t)} \qquad \text{with} \qquad \begin{array}{ll} g(t)=0 & \text{if} \quad t \leqslant 0, \\ g(t)=1 & \text{if} \quad t \geqslant 1. \end{array}$$

Next, define

$$h(t) = g(t+2)g(2-t) \qquad \text{with} \qquad \begin{array}{ll} h(t)=0 & \text{if} \quad |t| \geqslant 2, \\ h(t)=1 & \text{if} \quad |t| \leqslant 1. \end{array}$$

Note this function is also C^∞. Moreover, we can make an n-dimensional analogue, by $k(x_1,\dots,x_n) = h(x_1)h(x_2)\cdots h(x_n)$. In this setup, the function will be 1 if $\|x\| \leqslant 1$, and taking $k(R^{-1}x)$ is identically 1 in a ball of radius R, and is 0 outside a ball of radius $2R$. More specifically, define $h_i(x_i)$ for $1 \leqslant i \leqslant n$ with analogous f and t as above, and note that

$$h(R^{-1}x_i) = g\left(\frac{x_i}{R}+2\right) g\left(2-\frac{x_i}{R}\right) \qquad \text{with} \qquad \begin{array}{ll} h(R^{-1}x_i)=0 & \text{if} \quad |t| \geqslant 2R, \\ h(R^{-1}x_i)=1 & \text{if} \quad |t| \leqslant R. \end{array}$$

So indeed, for $x = (x_1,\dots,x_n) \in \mathbf{R}^n$, whenever $\|x\| \leqslant R$, we have $k(R^{-1}x) = 1$ and whenever $\|x\| \geqslant 2R$, we have $k(R^{-1}x) = 0$. ■

9. (September 4) Let $M \ni a$ be a n-dimensional manifold in coordinates $x_1,\dots,x_n$. Show that $(dx_1)_a,\dots,(dx_n)_a$ span T_a^*M.

Solution: Recall $T_a^*M := C^\infty(M)/Z_a(M)$, where $Z_a(M)$ is the subspace of $C^\infty(M)$ consisting of the smooth maps whose derivative vanishes at a. The dx_i are in T_a^*M, since each dx_i represents the linear function x_i on $\mathbf{R}^n$. This also shows that the dx_i span T_a^*M. To see that they are linearly independent, suppose that

$$0 = \sum_{i=1}^n \lambda_i (dx_i)_a$$

for some $\lambda_i \in \mathbf{R}$. Then

$$0 = \sum_{i=1}^n \lambda_i (dx_i)_a = \sum_{i=1}^n d(\lambda_i x_i) = d\left(\sum_{i=1}^n \lambda_i x_i\right),$$

meaning that $\sum \lambda_i x_i = c$ for some scalar c. Since the x_i are linearly independent coordinates in $\mathbf{R}^n$, the coefficients of x_i have to match up on the left and right sides. Hence $\lambda_i = 0$ for all i and $c = 0$. Therefore the dx_i are linearly independent, and so form a basis of T_a^*M. ■

10. (September 9) Find a basis for T_pS^3, the tangent space of S^3 at a point p.

Solution: Let $p = (p_1,p_2,p_3,p_4) \in S^3$, so a vector $x = (x_1,x_2,x_3,x_4) \in \mathbf{R}^4$ is tangent to S^3 at p (that is, lies in T_pS^3) if and only if $p \cdot x = 0$, for $\cdot$ the dot product. Note that

$$x \cdot p = (x_1,x_2,x_3,x_4)\cdot(p_1,p_2,p_3,p_4) = x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4,$$

and assuming that $p_4 \neq 0$ (if $p_4 = 0$, change the basis vectors so that $p_4 \neq 0$, as there is always one coordinate of p that is non-zero). Then in T_pS^3 we have

$$x_4 = \frac{-x_1p_1 - x_2p_2 - x_3p_3}{p_4},$$

and so T_pS^3 is completely described by the points

$$\left(x_1, x_2, x_3, \frac{-x_1p_1 - x_2p_2 - x_3p_3}{p_4}\right).$$

It follows immediately that a basis for T_pS^3 in $\mathbf{R}^4$ is

$$\left(1, 0, 0, \frac{-p_1}{p_4}\right) \quad , \quad \left(0, 1, 0, \frac{-p_2}{p_4}\right) \quad , \quad \left(0, 0, 1, \frac{-p_3}{p_4}\right).$$

■

11. (September 11) Prove the following statement: Let $F : M \to N$ be a smooth map and $c \in N$ such that for all $a \in F^{-1}(c)$, the derivative DF_a is surjective. Then $F^{-1}(c)$ is a smooth manifold of dimension $\dim(M) - \dim(N)$.

Solution: We know the statement is true when $M = \mathbf{R}^m$ and $N = \mathbf{R}^n$. In this case, let M be m-dimensional and N be n-dimensional. So let (U, φ) be a chart on N such that $c \in U$. Let (V, ψ) be a chart on M such that $a \in F^{-1}(c)$ also is in V.

Apply the known theorem to the map $\widetilde{\mathbf{F}} = vp \circ F \circ \psi^{-1} : \mathbf{R}^m \to \mathbf{R}^n$. The derivative of this map is surjective - indeed, surjectivity of such a map means the surjectivity of the homomorphism between tangent spaces. Since this is guaranteed for F and the chart maps φ, ψ have it guaranteed to begin with, we are fine.

So we have that $\widetilde{F}^{-1}(\varphi(c)) \subset \psi(V) \subset \mathbf{R}^m$ is a manifold of dimension $m - n$. Since φ and ψ are invertible homomorphism, we have that $F^{-1}(c) \subset M$ is a manifold of dimension $m - n$. ■

12. (September 11) Let $f : M \to N$ be a diffeomorphism of manifolds. Show that for each $x \in M$, $(df)_x$ is an isomorphism of tangent spaces.

Solution: Recall an isomorphism is an invertible homomorphism. Since f is a diffeomorphism, it has a differentiable inverse $g : N \to M$ such that $g \circ f = \mathrm{id}_M$ and $f \circ g = \mathrm{id}_N$. We claim that $(dg)_{f(x)}$ is the inverse of $(df)_x$. Indeed, apply the chain rule to $g \circ f$ and $f \circ g$ to find that

$$\begin{aligned} \mathrm{id}_{T_xM} &= (d(g \circ f))_x = (dg)_{f(x)} \circ (df)_x, \\ \mathrm{id}_{T_yN} &= (d(f \circ g))_y = (df)_{g(y)} \circ (dg)_y. \end{aligned}$$

Hence dg is the inverse of df, and by the homomorphism properties, this is an isomorphism. ■

13. (September 11) Let X be a manifold with $U \subset X$ open. Show that $T_aU = T_aX$ for all $a \in U$.

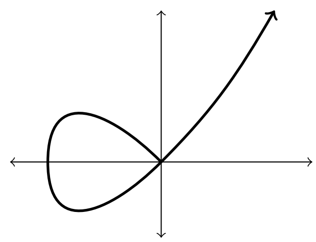
Solution: We use the description of T_aM as the set of derivations at a (that is, maps $v : C^\infty(M) \to \mathbf{R}$ satisfying $v(fg) = g(a)v(f) + f(a)v(g)$). The approach is to show the map $i_* : T_aU \to T_aX$, induced from the inclusion $i : U \to X$, is injective and surjective. The pushforward acts as $i_*(v)(f) = v(f|_U)$ for any $f \in C^\infty(X)$ (and hence $f|_U \in C^\infty(U)$, since restrictions of smooth maps are smooth).

For injectivity, take $v \in T_aU$ and $i_*(v) \in T_aX$, supposing that $i_*(v) = 0$, so $i_*(v)(f) = 0$ for all $f \in C^\infty(X)$. Then $v(f|_U) = 0$, and since f was arbitrary (and may be chosen so that $f|_U = g$, for any $g \in C^\infty(U)$), we have that $v = 0$.

For surjectivity, take $w \in T_aX$, and define $v \in T_aU$ by $v(f) = w(\widetilde{f})$, for $\widetilde{f} \in C^\infty(M)$ any function with $f = \widetilde{f}|_U$ (this is well-defined, since the derivation of functions that agree on an open set are the same). Then $i_*(v)(f) = v(f|_U) = w(\widetilde{f|_U}) = w(f)$, so $i_*(v) = w$. ■

14. (September 14) Consider the map $i : (-1, \infty) \to \mathbf{R}^2$ given by $t \mapsto (t^2 - 1, t(t^2 - 1))$. Show that this map does not give a submanifold of $\mathbf{R}^2$.

Solution: The image of this space M under i looks like in the diagram below.



The subspace topology is $\{U : U = V \cap M \text{ for some } V \subset \mathbf{R}^2 \text{ open}\}$. In the topology of M, we clearly have open intervals $(1-\delta, 1+\delta)$ for all $\delta > 0$. However, there is no open set $V \subset \mathbf{R}^2$ such that $V \cap M = (1-\delta, 1+\delta)$. Hence the topology of M is not the same as the induced subspace topology from $\mathbf{R}^2$, so M is not a submanifold of $\mathbf{R}^2$ with this i. ■

15. (September 14) Let $M \ni x, N \ni y$ be two manifolds. Show that $T_{(x,y)}M \times N \cong T_xM \times T_yN$.

Solution: Consider the maps

$$\begin{array}{rcrcl} \pi_1 & : & M \times N & \to & M, \\ & & (a,b) & \mapsto & a, \end{array} \qquad \begin{array}{rcrcl} \pi_2 & : & M \times N & \to & N, \\ & & (a,b) & \mapsto & b, \end{array} \qquad \begin{array}{rcrcl} i_y & : & M & \to & M \times N, \\ & & a & \mapsto & (a,y), \end{array} \qquad \begin{array}{rcrcl} j_x & : & N & \to & M \times N, \\ & & b & \mapsto & (x,b). \end{array}$$

We will use these to construct maps between the spaces. Each of the maps above have induced maps on tangent spaces, the pushforwards, so we get new maps

$$\begin{array}{rcrcl} \alpha & : & T_{(x,y)}M \times N & \to & T_xM \times T_yN, \\ & & v & \mapsto & (\pi_{1*}(v), \pi_{2*}(v)), \end{array} \qquad \begin{array}{rcrcl} \beta & : & T_xM \times T_yN & \to & T_{(x,y)}M \times N, \\ & & (v,w) & \mapsto & i_{y*}(v) + j_{x*}(w). \end{array}$$

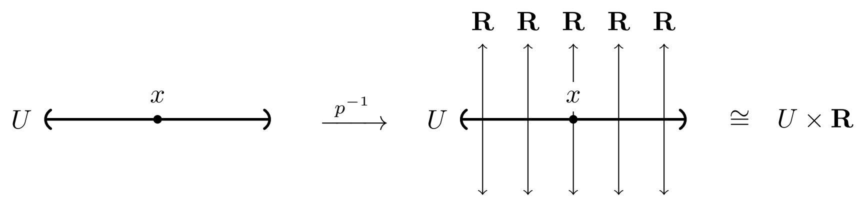
These maps are well defined, smooth, and

$$\begin{aligned} (\alpha \circ \beta)(v,w)(f,g) &= \alpha(i_{y*}(v) + j_{x*}(w))(f,g) \\ &= (\pi_{1*}(i_{y*}(v) + j_{x*}(w)), \pi_{2*}(i_{y*}(v) + j_{x*}(w)))\,(f,g) \\ &= (\pi_{1*}(i_{y*}(v))(f) + \pi_{1*}(j_{x*}(w))(f), \pi_{2*}(i_{y*}(v))(g) + \pi_{2*}(j_{x*}(w))(g)) \\ &= ((\pi_1 \circ i_y)_*(v)(f) + (\pi_1 \circ j_x)_*(w)(f), (\pi_2 \circ i_y)_*(v)(g) + (\pi_2 \circ j_x)(w)(g)) \\ &= (v(f \circ \pi_1 \circ i_y) + w(f \circ \pi_1 \circ j_x), v(g \circ \pi_2 \circ i_y) + w(g \circ \pi_2 \circ j_x)) \\ &= (v(f) + 0, 0 + w(g)) \\ &= (v,w)(f,g). \end{aligned}$$

Hence β is injective and α is surjective. Using either of these facts, since domain and range have the same dimension and both α and β are linear (as they are defined in terms of derivatives), they both are isomorphisms. ■

16. (September 18) Show that the 1-sphere S^1 has trivial tangent bundle.

Solution: First we describe the tangent bundle structure, which is $p : TS^1 \to S^1$, with $p^{-1}(x) = \mathbf{R}$ for all $x \in S^1$. For any such x, choose a neighborhood U, just an open interval on the sphere, and apply p^{-1} to get something diffeomorphic to $U \times \mathbf{R}$. Visually,



It is clear that $g_{UV} = 1$ for all U, V. Recall the product space

$$\begin{tikzcd}
& S^1 \times \mathbf{R} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
S^1 & & \mathbf{R}
\end{tikzcd}$$

with the relevant projection maps. Consider the map

$$\begin{array}{rcl} \Theta \ : \ TS^1 & \to & S^1 \times \mathbf{R}, \\ w & \mapsto & (p(w), w), \end{array}$$

which makes sense, as $w \in TS^1 = \bigoplus_{a \in M} T_aM$ is in $T_aS^1 \cong \mathbf{R}$ for some $a \in M$. Then

$$(\pi_1 \circ \Theta)(w) = \pi_1(p(w), r(w)) = p(w),$$

exactly as desired. The map Θ is a diffeomorphism, so we are done. ■

17. (September 18) Prove the following statement: A manifold M^n has trivial tangent bundle iff there are n vector fields $X_1, \dots, X_n$ on M such that at each $a \in M$, the elements $(X_1)_a, \dots, (X_n)_a$ form a basis for T_aM.

Solution: Suppose that M^n has trivial tangent bundle. That means $M \times \mathbf{R}^n \cong TM$ via some isomorphism φ. Define vector fields

$$\begin{array}{rcl} X_i \ : \ M & \to & TM, \\ a & \mapsto & \varphi(a, e_i), \end{array}$$

for e_i the ith standard basis vector of $\mathbf{R}^n$. These vector fields are indeed vector fields, and they are all smooth. Moreover, by construction the $(X_i)_a$ are linearly independent, and so they form a basis for T_aM, for all $a \in M$.

Now suppose that there are vector fields $X_1, \dots, X_n$ such that $(X_1)_a, \dots, (X_n)_a$ form a basis for T_aM, for all $a \in M$. We will show that TM and $M \times \mathbf{R}^n$ are diffeomorphic. Begin by taking $a \in M$ and (U, φ) a chart for a. Define maps

$$\begin{array}{rcl} \psi_U \ : \ U \times \mathbf{R}^n & \to & \coprod_{p \in U} T_pM, \\ (q, y_1, \dots, y_n) & \mapsto & (q, \sum_i y_i (X_i)_q) \end{array} \quad , \qquad \begin{array}{rcl} \Psi_U \ : \ \coprod_{p \in U} T_pM & \to & \varphi(U) \times \mathbf{R}^n, \\ (q, z) & \mapsto & (\varphi \times \mathrm{id}) \circ \psi_U^{-1}(q, z). \end{array}$$

The map ψ_U is a bijection between the given spaces by assumption, and Ψ_U is a chart map on TM. Now we turn the focus from local to global. Define a map

$$\begin{array}{rcl} F \ : \ M \times \mathbf{R}^n & \to & TM, \\ (p, y) & \mapsto & (p, \sum_i y_i (X_i)_p), \end{array}$$

which we claim is the desired diffeomorphism. To show this is true, we will demonstrate F and F^{-1} are smooth. Take two special charts

$$\begin{array}{l} (U \times \mathbf{R}^n, \varphi \times \mathrm{id}) \text{ for } M \times \mathbf{R}^n, \\ \left(\coprod_{p \in U} T_p U, \Psi_U\right) \text{ for } TM, \end{array}$$

and observe that

$$\Psi_U \circ F \circ (\varphi \times \mathrm{id})^{-1}(\varphi(a), y) = \Psi_U \circ F(a, y) = \Psi_U \left(a, \sum_i y_i (X_i)_a\right) = (\varphi(a), y).$$

Hence $\Psi_U \circ F \circ (\varphi \times \mathrm{id})^{-1} = \mathrm{id}_M$. To show that other charts work, instead choose an arbitrary chart $(\coprod_{p \in V} T_p M, \Psi_V)$ for TM. The picture of the calculations looks like below:

$$\Psi_V \circ F \circ (\varphi \times \mathrm{id})^{-1} = \Psi_V \circ \Psi_U^{-1} \circ \Psi_U \circ F \circ (\varphi \times \mathrm{id})^{-1} = \Psi_V \circ \Psi_U^{-1}$$

$$\begin{tikzcd} M \times \mathbf{R}^n \arrow[r, "F"] & TM \arrow[d, "\Psi_V"'] \\ \mathbf{R}^{2n} \arrow[u, "(\varphi \times \mathrm{id})^{-1}"'] \arrow[r, bend left=120] & \mathbf{R}^n \end{tikzcd}$$

Since the transition maps are smooth, $\Psi_V \circ \Psi_U^{-1}$ is smooth, so F is indeed a diffeomorphism. Hence $M \times \mathbf{R}^n$ and TM are diffeomorphic, meaning that M has trivial tangent bundle. ■

18. (September 18) Prove the following statement: Any linear transformation which satisfies the Leibniz property is a vector field.

Solution: Recall that the tangent space to M at p may be viewed as the space of derivations, that is, the set of linear maps $v : C^\infty(M) \to \mathbf{R}$ such that for all $f, g \in C^\infty(M)$, we have $v(fg) = f(p)v(g) + g(p)v(f)$. Also recall that a vector field is a map $X : M \to TM$ such that $\pi \circ X = \mathrm{id}_M$, where $\pi : TM \to M$ is the natural projection map.

First we need to show that, for $p \in M$, $X_p \in TM$. This is immediate, as the assumption that X satisfies the Leibniz rule is equivalent to the condition of being in TM, even more, to being in $T_p M$.

Next we need to show $\pi(X_p) = p$, but this is immediate, as $\pi(T_p M) = p$, and $X_p \in T_p M$. ■

19. (September 18) Let X, Y, Z be vector fields on a manifold M. Show the following properties hold, in coordinates:

(a) $[X, Y + Z] = [X, Y] + [X, Z]$

(b) $[X, Y] = -[Y, X]$

(c) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

(d) $\lambda[X, Y] = [X, \lambda Y]$ for any scalar λ

Solution: This first identity involves some long algebra.

$$\begin{aligned}
[X, Y+Z] &= \left[a_i \frac{\partial}{\partial x_i}, b_j \frac{\partial}{\partial x_j} + c_k \frac{\partial}{\partial x_k}\right] \\
&= a_i \frac{\partial}{\partial x_i}\left(b_j \frac{\partial}{\partial x_j} + c_k \frac{\partial}{\partial x_k}\right) - \left(b_j \frac{\partial}{\partial x_j} + c_k \frac{\partial}{\partial x_k}\right) a_i \frac{\partial}{\partial x_i} \\
&= a_i b_j \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + a_i c_k \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k} - a_i b_j \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} - a_i c_k \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_i} \\
&= \left(a_i b_j \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} - a_i b_j \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}\right) + \left(a_i c_k \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k} - a_i c_k \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_i}\right) \\
&= \left[a_i \frac{\partial}{\partial x_i}, b_j \frac{\partial}{\partial x_j}\right] + \left[a_i \frac{\partial}{\partial x_i}, c_k \frac{\partial}{\partial x_k}\right] \\
&= [X, Y] + [X, Z]
\end{aligned}$$

The second identity just needs some rearranging.

$$\begin{aligned}
[X, Y] &= \left[a_i \frac{\partial}{\partial x_i}, b_j \frac{\partial}{\partial x_j}\right] \\
&= a_i b_j \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} - a_i b_j \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} \\
&= -\left(a_i b_j \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} - a_i b_j \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\right) \\
&= -\left[b_j \frac{\partial}{\partial x_j}, a_i \frac{\partial}{\partial x_i},\right] \\
&= -[Y, X]
\end{aligned}$$

The third identity is an exercise in masochism. We begin by expanding the first term in the identity.

$$\begin{aligned}
[X, [Y, Z]] &= \left[a_i \frac{\partial}{\partial x_i}, \left[b_j \frac{\partial}{\partial x_j}, c_k \frac{\partial}{\partial x_k}\right]\right] \\
&= \left[a_i \frac{\partial}{\partial x_i}, b_j c_k \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k} - b_j c_k \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}\right] \\
&= a_i b_j c_k \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k} - a_i b_j c_k \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_i} - a_i b_j c_k \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j} + a_i b_j c_k \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}
\end{aligned}$$

Denote the first term above by the ordered triple (i, j, k), noting that the order of the smooth coefficient functions does not matter. Generalizing, the sum of the terms in the Jacobi identity contains the sum of the terms in the following table:

(i, j, k)	$-(j, k, i)$	$-(i, k, j)$	(k, j, i)
(j, k, i)	$-(k, i, j)$	$-(j, i, k)$	(i, k, j)
(k, i, j)	$-(i, j, k)$	$-(k, j, i)$	(j, i, k)

The terms in the first column are the negatives of the terms in the second column, and the terms in the third column are the negatives of the terms in the fourth column. Hence adding them all together gives 0, yielding the desired identity.

The last identity is straightforward.

$$\begin{aligned}\lambda[X,Y] &= \lambda\left[a_i\frac{\partial}{\partial x_i}, b_j\frac{\partial}{\partial x_j}\right]\\ &= \lambda a_i b_j \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} - \lambda a_i b_j \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}\\ &= a_i(\lambda b_j)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} - a_i(b_j\lambda)\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}\\ &= \left[a_i\frac{\partial}{\partial x_i}, \lambda b_j\frac{\partial}{\partial x_j}\right]\\ &= [X,\lambda Y]\end{aligned}$$

This completes the proof. ∎

20. (September 21) Let A be a skew-symmetric $m\times m$ matrix, and set $\gamma(t)=\exp(tA)=\sum_{n=0}^{\infty} t^nA^n/n!$.

   (a) Show that γ defines a smooth curve in $SO(m)$.
   (b) Find $\gamma'(0)$, the tangent vector defined by γ at 0.
   (c) Find $T_ISO(m)$.
   (d) Find $T_gSO(m)$, for arbitrary $g\in SO(m)$.

(Contributed by Nathan Lopez)
Solution:
(a) The sum converges uniformly and each partial sum $\sum_{n=0}^{k} t^nA^n/n!$ is smooth, so $\exp(tA)$ is smooth. To show that $\gamma(t)\in SO(m)$, we need to show $\gamma(t)^T=\gamma(t)^{-1}$ and $\det(\gamma(t))=1$. For the first, note

$$\begin{aligned}\gamma(t)^T &= \exp(tA)^T\\ &= \left(\sum_{n=0}^{\infty}\frac{t^nA^n}{n!}\right)^T\\ &= \left(\lim_{k\to\infty}\left[\sum_{n=0}^{k}\frac{t^nA^n}{n!}\right]\right)^T && \text{(uniform convergence)}\\ &= \lim_{k\to\infty}\left[\left(\sum_{n=0}^{k}\frac{t^nA^n}{n!}\right)^T\right] && \text{(continuity of lim and } T)\\ &= \lim_{k\to\infty}\left[\sum_{n=0}^{k}\frac{t^n(A^n)^T}{n!}\right] && \text{(properties of } T)\\ &= \lim_{k\to\infty}\left[\sum_{n=0}^{k}\frac{t^n(A^T)^n}{n!}\right] && \text{(properties of } T)\\ &= \sum_{n=0}^{\infty}\frac{t^n(A^T)^n}{n!}\\ &= \exp(tA^T)\\ &= \exp(-tA). && (A \text{ is skew-symmetric})\end{aligned}$$

A tedious algebra argumwent shows that if two matrices X, Y commute (that is, $XY = YX$), then $\exp(X)\exp(Y) = \exp(X+Y)$. Since $(tA)(-tA) = -t^2A^2 = (-tA)(tA)$, we have that

$$\gamma(t)\gamma(t)^T = \exp(tA)\exp(-tA) = \exp(tA - tA) = \exp(0) = I \quad\Longrightarrow\quad \gamma(t)^T = \gamma(t)^{-1}.$$

Finally, Jacobi's identity says that $\det(\exp(X)) = e^{\operatorname{tr}(A)}$, and we know that the trace of a skew-symmetric matrix is 0, so $\det(\exp(tA)) = e^0 = 1$, and therefore $\gamma \in SO(m)$.

(b) Since the sum converges uniformly, we may compute the derivative term by term. That is,

$$\frac{d}{dt}\gamma(t) = \frac{d}{dt}\lim_{k\to\infty}\left[\sum_{n=0}^{k}\frac{t^nA^n}{n!}\right] = \lim_{k\to\infty}\left[\frac{d}{dt}\sum_{n=0}^{k}\frac{t^nA^n}{n!}\right] = \lim_{k\to\infty}\left[\sum_{n=1}^{k}\frac{t^{n-1}A^n}{(n-1)!}\right] = A + tA^2 + \frac{t^2A^3}{2} + \cdots,$$

so $\gamma'(0) = A$.

(c) First note that part (b) gives us a tangent vector in $T_ISO(m)$, since $\gamma(0) = I$. That is, any skew-symmetric matrix is in this tangent space. Next, since

$$\dim(m\times m \text{ skew-symmetric matrices}) = \frac{n(n-1)}{2} = \dim(SO(m)) = \dim(T_ISO(m)),$$

a basis of skew-symmetric matrices is a basis of $T_ISO(m)$. Hence $T_ISO(m)$ is simply the space of $m \times m$ skew-symmetric matrices.

(d) Now let $g \in SO(m)$ be arbitrary. To find $T_gSO(m)$, define a new path $\widetilde{\gamma}(t) = g\exp(tA)$, for which the exact same calculations as above may be repeated. The changes are that $\gamma'(0) = gA$, meaning that, for $SS(m)$ the space of $m\times m$ skew-symmetric matrices, we get $T_gSO(m) = gSS(m)$. ■

21. (October 12) Show that a smooth vector field on a manifold M that vanishes outside a compact set $K \subset M$ generates a 1-parameter group of diffeomorphisms on M.

Solution: Take $p \in K$, for which there exists an open neighborhood U_p of p and $\epsilon_p > 0$ such that $\varphi^p : (-\epsilon_p, \epsilon_p) \to M$ is a maximal integral curve of X going through p. Since K is compact, there is a finite set $p_1, \dots, p_k$ such that $U_{p_1} \cup \cdots \cup U_{p_k} = K$. Let $\epsilon = \min_i\{\epsilon_{p_i}\}$, so that $\varphi^{p_i} : (-\epsilon, \epsilon) \to M$ is a maximal integral curve through p_i.

For $p \in M \setminus K$, the maximal integral curve through p is constant, so is clearly defined on $(-\epsilon, \epsilon)$ for any $\epsilon > 0$. Hence every point of M has a maximal integral curve going through it, defined on $(-\epsilon, \epsilon)$. By some finagling (see "Uniform time lemma", p.216 in Lee), it follows directly that there is a maximal integral curve defined on all of $\mathbf{R}$ and all of M. This is equivalent to saying that there is a 1-parameter group of diffeomorphisms on all of M. ■

22. (October 14) Consider $S^2 \subset \mathbf{R}^3$ in coordinates (x, y, z), and let $X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ and $Y = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$ be vector fields on S^2. Calculate $[X, Y]$.

Solution: This is just some calculations with the product rule.

$$\begin{aligned}
[X,Y] &= \left[y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right] \\
&= \left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right) - \left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right)\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) \\
&= y\frac{\partial z}{\partial x}\frac{\partial}{\partial y} + yz\frac{\partial^2}{\partial x\partial y} - y\frac{\partial y}{\partial x}\frac{\partial}{\partial z} - y^2\frac{\partial^2}{\partial x\partial z} - x\frac{\partial z}{\partial y}\frac{\partial}{\partial y} - xz\frac{\partial^2}{\partial y^2} + x\frac{\partial y}{\partial y}\frac{\partial}{\partial z} + xy\frac{\partial^2}{\partial y\partial z} \\
&\qquad - z\frac{\partial y}{\partial y}\frac{\partial}{\partial x} - zy\frac{\partial^2}{\partial y\partial x} + z\frac{\partial x}{\partial y}\frac{\partial}{\partial y} + zx\frac{\partial^2}{\partial y^2} + y\frac{\partial y}{\partial z}\frac{\partial}{\partial x} + y^2\frac{\partial^2}{\partial z\partial x} - y\frac{\partial x}{\partial z}\frac{\partial}{\partial y} - yx\frac{\partial^2}{\partial z\partial y} \\
&= yz\frac{\partial^2}{\partial x\partial y} - y^2\frac{\partial^2}{\partial x\partial z} + x\frac{\partial}{\partial z} + xy\frac{\partial^2}{\partial y\partial z} - z\frac{\partial}{\partial x} - zy\frac{\partial^2}{\partial y\partial x} + y^2\frac{\partial^2}{\partial z\partial x} - yx\frac{\partial^2}{\partial z\partial y} \\
&= x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}.
\end{aligned}$$

The second and third equalities were just expanding, the fourth was reducing inverse terms, and the last equality was reducing by Fubini's theorem. ■

23. (October 16) Let X, Y be vector fields on a smooth manifold M. Give the definition of the Lie bracket $[X, Y]$ as a differential operator on smooth functions. Also show that $L_{[X,Y]} = [L_X, L_Y]$, for $L_X Y = [X, Y]$ the *Lie derivative.*

(Contributed by Dan Solomon)
Solution: Let f be a smooth function. Fix some lical coordinates $x_1, \dots, x_n$ on M, and define the Lie bracket of two vector fields $X = a_i \frac{\partial}{\partial x_i}$ and $Y = b_j \frac{\partial}{\partial x_j}$ on f to be

$$\begin{aligned}
[X,Y]f &= (XY - YX)f \\
&= X(Yf) - Y(Xf) \\
&= a_i\frac{\partial}{\partial x_i}\left(b_j\frac{\partial f}{\partial x_j}\right) - b_j\frac{\partial}{\partial x_j}\left(a_i\frac{\partial f}{\partial x_i}\right) \\
&= a_i\frac{\partial b_j}{\partial x_i}\frac{\partial f}{\partial x_j} + a_i b_j\frac{\partial^2 f}{\partial x_i\partial x_j} - b_j\frac{\partial a_i}{\partial x_j}\frac{\partial f}{\partial x_i} - a_i b_j\frac{\partial^2 f}{\partial x_j\partial x_i} && \text{(product rule)} \\
&= a_i\frac{\partial b_j}{\partial x_i}\frac{\partial f}{\partial x_j} - b_j\frac{\partial a_i}{\partial x_j}\frac{\partial f}{\partial x_i} && \text{(order of differentiation)} \\
&= a_i\frac{\partial b_j}{\partial x_i}\frac{\partial f}{\partial x_j} - b_i\frac{\partial a_j}{\partial x_i}\frac{\partial f}{\partial x_j} && \text{(renaming of indices)} \\
&= \left(a_i\frac{\partial b_j}{\partial x_i} - b_i\frac{\partial a_j}{\partial x_i}\right)\frac{\partial f}{\partial x_j}.
\end{aligned}$$

This gives a clear definition of how the Lie bracket acts on smooth functions. To check that the given identity holds, Let Z be another vector field, for which

$$\begin{aligned}
L_{[X,Y]}Z &= L_{XY-YX}Z \\
&= [XY - YX, Z] \\
&= XYZ - YXZ - ZXY + ZYX
\end{aligned}$$

and

$$\begin{aligned}[L_X, L_Y](Z) &= L_X(L_Y Z) - L_Y(L_X Z) \\ &= [X,[Y,Z]] - [Y,[X,Z]] \\ &= [X, YZ - ZY] - [Y, XZ - ZX] \\ &= XYZ - XZY - YZX + ZYX - YXZ + YZX + XZY - ZXY \\ &= XYZ + ZYX - YXZ - ZXY,\end{aligned}$$

which are both the same. ■

24. (October 16) Let $v_1, \dots, v_n$ be a basis of an n-dimensional vector space V. Show that the elements $v_{i_1} \wedge \cdots \wedge v_{i_p}$, for $1 \leqslant i_1 < \cdots < i_p \leqslant n$, form a basis for $\bigwedge^p V$.

Solution: It suffices to show an element $w = \alpha w_1 \wedge \cdots \wedge w_p$ may be expressed in terms of the given elements, for a scalar α and $w_i \in V$. Note that for each i, we have

$$w_i = \sum_{j=1}^{n} \alpha_i^j v_j,$$

for some scalars α_j^i and the v_j a basis for V. Then by p-multilinearity, we have

$$w = \alpha \left(\sum_{j_1=1}^{n} \alpha_1^{j_1} v_{j_1} \right) \wedge \cdots \wedge \left(\sum_{j_p=1}^{n} \alpha_p^{j_p} v_{j_p} \right) = \sum_{j_1=1}^{n} \cdots \sum_{j_p=1}^{n} \alpha_1^{j_1} v_{j_1} \wedge \cdots \wedge \alpha_p^{j_p} v_{j_p}.$$

Given $v_{i_1} \wedge \cdots \wedge v_{i_p} \in \bigwedge^p V$, view $(i_1 \ \cdots \ i_p)$ as an element of S_p, the symmetric group on p elements. Then there exists $\sigma \in S_p$ such that $i_{\sigma(1)} < \cdots < i_{\sigma(p)}$, so $v_{i_1} \wedge \cdots \wedge v_{i_p} = \mathrm{sgn}(\sigma) v_{i_{\sigma(1)}} \wedge \cdots \wedge v_{i_{\sigma(p)}}$, for $\mathrm{sgn}(\sigma)$ either $+1$ or -1, depending on the number of transpositions done. Hence we have

$$w = \sum_{j_1=1}^{n} \cdots \sum_{j_p=1}^{n} \left(\prod_{k=1}^{p} \alpha_k^{j_k} \right) \mathrm{sgn}(\sigma_{j_1 \cdots j_p}) v_{\sigma_{j_1 \cdots j_k}(j_1)} \wedge \cdots \wedge v_{\sigma_{j_1 \cdots j_p}(j_p)},$$

where $\sigma_{j_1 \cdots j_p}(j_1) < \cdots < \sigma_{j_1 \cdots j_p}(j_p)$ for all $j_1, \dots, j_p$. We have now written w as a sum of wedges of v_is with increasing indeces. Many of the terms in the sum are 0 because of the quotiented relations, though that does not affect the correctness of the expression above. ■

25. (October 16) For $n \geqslant 1$, show that $SL(n)$ is a smooth manifold, and find its dimension.

(Contributed by Charlotte Greenblatt)
Solution: Recall that $SL(n)$ is the space of $n \times n$ matrices with determinant 1. Since $\det : \mathbf{R}^{n^2} \to \mathbf{R}$ is smooth with $SL(n) = \det^{-1}(1)$, if we can show that the derivative of det is surjective at every point, then it will follow that $SL(n)$ is a smooth manifold of dimension $n^2 - 1$.

Let $x_{11}, x_{12}, x_{13}, \dots, x_{nn}$ be basis vectors for $\mathbf{R}^{n^2}$, and for $A \in SL(n)$, let B_{ij} be the matrix with zeros everywhere except a 1 in the (i,j)-th position. Finally, let M_{ij} be the (i,j)-minor of A (the determinant of A with the ith row and jth column removed). Note that the (k,j)-minor of $A + tB_{ij}$ is the same as the (k,j)-minor of A, since we have removed the jth column. Hence the derivative of det in the x_{ij} direction is

$$
\begin{aligned}
\frac{\partial \det}{\partial x_{ij}}(A) &= \lim_{t\to 0}\left[\frac{\det(A+tB_{ij})-\det(A)}{t}\right] && \text{(definition of derivative)}\\
&= \lim_{t\to 0}\left[\frac{1}{t}\left(\sum_{k=1}^{n}(-1)^{k+j}(A+tB_{ij})_{kj}M_{kj}-1\right)\right] && \text{(minor decomposition)}\\
&= \lim_{t\to 0}\left[\frac{1}{t}\left(\sum_{\substack{k=1\\k\neq i}}^{n}(-1)^{k+j}(A+tB_{ij})_{kj}M_{kj}+(-1)^{i+j}(A+tB_{ij})_{ij}M_{ij}-1\right)\right]\\
&= \lim_{t\to 0}\left[\frac{1}{t}\left(\sum_{\substack{k=1\\k\neq i}}^{n}(-1)^{k+j}(A)_{kj}M_{kj}+(-1)^{i+j}(A)_{ij}M_{ij}+(-1)^{i+j}tM_{ij}-1\right)\right]\\
&= \lim_{t\to 0}\left[\frac{1}{t}\left(\sum_{k=1}^{n}(-1)^{k+j}(A)_{kj}M_{kj}+(-1)^{i+j}tM_{ij}-1\right)\right]\\
&= \lim_{t\to 0}\left[\frac{1}{t}\left(\det(A)+(-1)^{i+j}tM_{ij}-1\right)\right]\\
&= \lim_{t\to 0}\left[\frac{1}{t}\left((-1)^{i+j}tM_{ij}\right)\right] && \text{(since } \det(A)=1)\\
&= (-1)^{i+j}M_{ij}.
\end{aligned}
$$

Since $\mathbf{R}$ is 1-dimensional, only one of the minors has to be non-zero, since that will make the derivative surjective. Since $\det(A) = 1$ for any $A \in SL(n)$, all the minors cannot be zero, since that would mean $\det(A) = 0$. Hence at least one of the minors is non-zero, so the derivative is surjective at every $A \in SL(n)$, and so $SL(n)$ is a smooth manifold of dimension $n^2 - 1$. ■

26. (October 16) Let M, N be smooth manifolds with $M \subset N$ a submanifold. Show that if X is a vector field defined on an open neighborhood of M, then there exists a vector field Y on N such that $Y|_M = X|_M$.

Solution: Let M be a k-dimensional submanifold of N with open neighborhood $\widetilde{M}$ also k-dimensional. Let $X = \sum_{i=1}^{k} a_i \frac{\partial}{\partial x_i} \in \Gamma(T\widetilde{M})$ be a vector field on $\widetilde{M}$ and set $a_{k+1} = \cdots = a_n = 0$, so that we may extend X to all of N. Let (U, φ) be a chart on N such that

$$x \in U \cap \widetilde{M} \quad\iff\quad \varphi(x) = (\underbrace{*, \dots, *}_{k}, \underbrace{0, \dots, 0}_{n-k}).$$

Such a U and φ is possible to find because $\widetilde{M}$ is k-dimensional and by restricting the charts. For the next step, recall that a diffeomorphism $a : B \to C$ of manifolds induces a map $a_* : \Gamma(TB) \to \Gamma(TC)$ on the vector fields, given by

$$(a_* Z)_p(h) = (Z)_{a^{-1}(p)}(h \circ a),$$

for $Z \in \Gamma(TB)$, $p \in C$, and $h \in C^\infty(C)$, so $h \circ a \in C^\infty(B)$. Using a slight variation of this, define a vector field $Z_U \in \Gamma(T\varphi(U))$ given by

$$(Z_U)_q(h) := \left(a_i \frac{\partial}{\partial x_i}\right)_{\varphi^{-1}(q_1,\dots,q_k,0,\dots,0)} (h \circ \varphi),$$

for $h \in C^\infty(\varphi(U))$ and $q = (q_1, \ldots, q_n) \in \varphi(U)$. This vector field is almost $(\varphi_* \widetilde{X})_q(h)$, but not quite, since the point at which the vector field is evaluated is slightly different from $\varphi^{-1}(q)$. Now define a vector field $\widetilde{Z}_U \in \Gamma(TU)$ by

$$(\widetilde{Z}_u)_p(h) := ((\varphi^{-1})_* Z_U)_{\varphi(p)}(h \circ \varphi^{-1}),$$

for $h \in C^\infty(U)$ and $p \in U$. Finally, take $K = \{U_\alpha, \varphi_\alpha\}$ to be a refinement (by restriction) of an atlas covering $\widetilde{M}$ and an atlas covering N. Set $Z_{U_\alpha} = 0$ whenever $U_\alpha \cap M = \emptyset$. Let ψ_α be a partition of unity subordinate to K such that $\sum_{U_\alpha \cap M \neq \emptyset} \psi_\alpha(p) = 1$ whenever $p \in M$, which is possible, since $\widetilde{M}$ is an open neighborhood of M. Define a vector field $Y \in \Gamma(TN)$ by

$$Y_p(h) := \sum_\alpha \psi_\alpha (Z_{U_\alpha})_p(h|_{U_\alpha})$$

for $h \in C^\infty(N)$ and $p \in N$. From the choice of partition of unity, it is immediate that

$$Y|_M = X|_M \qquad \text{but} \qquad Y|_{\widetilde{M}} \neq X,$$

since the partition of unity decreases the effect of X only on $\widetilde{M} \setminus M$. This defines the desired vector field $Y \in \Gamma(TN)$.

If in addition we know that M is closed, we have a bump function $\varphi : N \to \mathbf{R}$ that is 1 on M and 0 outside of the open neighborhood of M. Then $\varphi X \in \Gamma(TN)$.

If M is closed but the vector field X is only defined on M, a simpler approach is also possible. For open sets U_α and maps φ_α covering M (open in N) such that $\varphi_\alpha(U_\alpha) = (x_1, \ldots, x_k, 0, \ldots, 0)$ for M codimension $n-k$ in N, define $X' \in \Gamma(T\varphi_\alpha(U_\alpha))$ by

$$X'(x_1, \ldots, x_n) = (\varphi_\alpha)_*(X(\varphi_\alpha^{-1}(x_1, \ldots, x_k, 0, \ldots, 0))).$$

This gives a vector field $X'' \in \Gamma(TU_\alpha)$ by

$$X''(a_1, \ldots, a_n) = (\varphi_\alpha)_*^{-1}(X'(\varphi_\alpha(a_1, \ldots, a_n))).$$

Given a partition of unity ψ_α subordinate to $\{V_\beta\} \supset \{U_\alpha\}$ on N, define $X''' \in \Gamma(TN)$ by

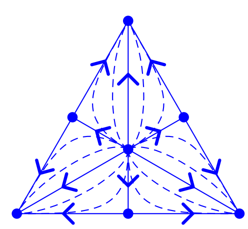
$$X'''(a_1, \ldots, a_n) = \begin{cases} \psi_\alpha X''(a_1, \ldots, a_n), & \text{if } (a_1, \ldots, a_n) \in U_\alpha, \\ 0, & \text{else.} \end{cases}$$

■

27. (October 21) Show that every compact manifold has a vector field with finitely many zeros.

Solution: Every compact manifold may be triangulated, and every n-simplex in the manifold M may be considered in terms of its barycentric subdivision. We will construct a vector field on M that has zeros at all the intersection points of this subdivison (of which there are finitely many). This is clear in the 1-simlex and 2-simplex case:

1-simplex



2-simplex

The vector field continues in the empty spaces, following the pattern on the sides. The only zeroes are at the emphasized points. This generalizes to n-simplices, and so gives a vector fields with finitely many zeros on the whole manifold. ■

28. (October 21) Calculate $F^*\alpha$ for $F : \mathbf{R}^3 \to \mathbf{R}^2$ given by $F(x_1, x_2, x_3) = (x_1x_2, x_2+x_3)$ and $\alpha = xdx\wedge dy$.

Solution: This question is just a long calculation. Recall the rules that $F^*f = f \circ F$ and $F^*(df) = d(f \circ F)$ for f a smooth 0-form. Write $(x, y) = F(x_1, x_2, x_3)$ to get

$$\begin{aligned} F^*\alpha &= F^*(xdx \wedge dy) \\ &= (x \circ F)d(x \circ F) \wedge d(F \circ y) \\ &= (x_1x_2)d(x_1x_2) \wedge d(x_2 + x_3) \\ &= x_1x_2(x_2dx_1 + x_1dx_2) \wedge (dx_2 + dx_3) \\ &= x_1x_2^2dx_1 \wedge dx_2 + x_1x_2^2dx_1 \wedge dx_3 + x_1^2x_2dx_2 \wedge dx_2 + x_1^2x_2dx_2 \wedge dx_3 \\ &= x_1x_2^2dx_1 \wedge dx_2 + x_1x_2^2dx_1 \wedge dx_3 + x_1^2x_2dx_2 \wedge dx_3. \end{aligned}$$

■

29. (October 23) Let M be a smooth n-manifold and ω a k-form on M. Give $d\omega$ in local coordinates and show why it is independent of the basis chosen for M.

Solution: Let $(x^1, \dots, x^n)$ be local coordinates on M. A k-form on M is

$$\omega = \sum_{|I|=k} f_I dx^I,$$

where $I = \{i_1 < \cdots < i_k\} \subset \{1, \dots, n\}$ is a multi-index, and $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ and $f_I \in C^\infty(M)$ for all I. To describe $d\omega$, it suffices to describe $d\omega$ for ω a pure wedge, as the result extends by linearity. So

$$\omega = f dx^{i_1} \wedge \cdots \wedge dx^{i_k} \qquad \Longrightarrow \qquad d\omega = \frac{\partial f}{\partial x^i} dx^i \wedge dx^{i_1} \cdots \wedge dx^{i_k},$$

with an implied sum in $d\omega$ over i (using Einstein notation). Now suppose that $(y^1, \dots, y^n)$ are also local coordinates on M. By the chain rule, we have

$$\frac{\partial}{\partial y^j} = \frac{\partial x^i}{\partial y^j}\frac{\partial}{\partial x^i} \qquad \text{and} \qquad dy^j = \frac{\partial y^j}{\partial x^i} dx^i.$$

Therefore

$$\begin{aligned} d\omega &= \frac{\partial f}{\partial y^j} dy^j \wedge dy^{j_1} \wedge \cdots \wedge dy^{j_k} \\ &= \frac{\partial x^i}{\partial y^j}\frac{\partial f}{\partial x^i}\frac{\partial y^j}{\partial x^i} dx^i \wedge \frac{\partial y^{j_1}}{\partial x^{\iota_1}} dx^{\iota_1} \wedge \cdots \wedge \frac{\partial y^{j_k}}{\partial x^{\iota_k}} dx^{\iota_k} \\ &= g\frac{\partial f}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \end{aligned}$$

where $g \in C^\infty(M)$ is a smooth function in terms of $\frac{\partial y^j}{\partial x^i}$ over some (possibly all) i, j. Hence $d\omega$ is independent, up to scaling by a smooth function, of basis chosen for M. ■

30. (October 26) Let $U \subset \mathbf{R}^n$, $V \subset \mathbf{R}^m$ be open sets with coordinates x_i, y_i, respectively, and $\theta : U \to V$ be a smooth map. Show that, for $\theta_i = y_i \circ \theta$,

$$\theta^*(dy_i) = \frac{\partial \theta_i}{\partial x_j} dx_j.$$

Solution: Since $\theta : U \to V$, we have $\theta^* : T^*V \to T^*U$. The definition of the pullback gives

$$\theta^*(dy_i) = \theta^*(1 \cdot dy_i) = (1 \circ \theta)\ d(y_i \circ \theta) = 1 \cdot d\theta_i = 1 \cdot \sum_{j=1}^{n} \frac{\partial \theta_i}{\partial x_j} dx_j = \frac{\partial \theta_i}{\partial x_j} dx_j,$$

using Einstein notation. ■

31. (October 26) Define the *Hodge star* operator

$$\begin{aligned} * \ : \ \Omega^k(\mathbf{R}^m) &\to \Omega^{m-k}(\mathbf{R}^m), \\ dx_{i_1} \wedge \cdots \wedge dx_{i_k} &\mapsto \operatorname{sgn}(\sigma) dx_{j_1} \wedge \cdots \wedge dx_{j_{m-k}}, \end{aligned}$$

with $1 \leqslant i_1 < \cdots < i_k \leqslant m$ and $1 \leqslant j_1 < \cdots < j_{m-k} \leqslant m$. Also $\{i_1, \ldots, i_k, j_1, \ldots, j_{m-k}\} = \{1, \ldots, m\}$ and σ is the permutation $(i_1 \ \cdots \ i_k \ j_1 \ \cdots \ j_{m-k}) \in S_m$ (the symmetric group on m elements). Let $\omega = a_{12} dx_1 \wedge dx_2 + a_{13} dx_1 \wedge dx_3 + a_{23} dx_2 \wedge dx_3$.

(a) Calculate $*\omega$ for $\omega \in \Omega^2(\mathbf{R}^3)$.

(b) Calculate $*\omega$ for $\omega \in \Omega^2(\mathbf{R}^4)$.

Solution: (a) Using the definition, we get

$$*\omega = a_{12} dx_3 - a_{13} dx_2 + a_{23} dx_1.$$

(b) Similarly, we find

$$*\omega = a_{12} dx_3 \wedge dx_4 - a_{13} dx_2 \wedge dx_4 + a_{23} dx_1 \wedge dx_4.$$

■

32. (October 26) Show that the formula $\mathscr{L}_X \alpha = d(i_X \alpha) + i_X(d\alpha)$ agrees with the definition of $\mathscr{L}_X \alpha$.

Solution: Let $\alpha = f dg$ be a p-form and X a vector field. The result will extend linearly to all p-forms. The right-hand side expands as

$$\begin{aligned} d(i_X \alpha) + i_X(d\alpha) &= d(f X(g)) + i_X(df \wedge dg) \\ &= df \wedge d(X(g)) + f d(X(g)) + X(f) \wedge dg - df \wedge d(X(g)) \\ &= f d(X(g)) + X(f) \wedge dg. \end{aligned}$$

The left-hand side, for φ the 1-parameter group of diffeomorphisms associated to X, is just

$$\mathscr{L}_X \alpha = \frac{\partial}{\partial t} \varphi_t^* \alpha \bigg|_{t=0} = \lim_{t \to 0} \left[\frac{\varphi_t^* \alpha - \varphi_0^* \alpha}{t} \right] = \cdots$$

I'm not sure how to finish this and I feel we have not learned enough in class to finish this. However, if we simply consider the action on vector fields, the result follows from the definitions. Indeed, the

Lie derivative, exterior derivative of a differential form, and interior product may be described as

$$(\mathcal{L}_X\omega)(Y_1,\dots,Y_k) = \mathcal{L}_X(\omega(Y_1,\dots,Y_k)) - \sum_{i=1}^{k}\omega(Y_1,\dots,\mathcal{L}_XY_i,\dots,Y_k),$$

$$(d\omega)(Y_1,\dots,Y_{k+1}) = \sum_{i=1}^{k+1}(-1)^{i-1}Y_i(\omega(Y_1,..,\widehat{Y_i},..,Y_{k+1})) + \sum_{\substack{i=1\\ j>i}}^{k+1}(-1)^{i+j}\omega([Y_i,Y_j],Y_1,..,\widehat{Y_i},..,\widehat{Y_j},..,Y_{k+1}),$$

$$(i_X\omega)(Y_1,\dots,Y_{k-1}) = \omega(X,Y_1,\dots,Y_{k-1}).$$

By expanding out the given terms (this is called *Cartan's formula*), the result follows quickly.

$$\begin{aligned}
(\mathcal{L}_{Y_1}\omega)(Y_2,\dots,Y_{k+1}) &= Y_1(\omega(Y_2,\dots,Y_{k+1})) - \sum_{i=2}^{k+1}\omega(Y_2,\dots,[Y_1,Y_i],\dots,Y_k)\\
&= Y_1(\omega(Y_2,\dots,Y_{k+1})) - \sum_{i=2}^{k+1}(-1)^i\omega([Y_1,Y_i],Y_2,\dots,\widehat{Y_i},\dots,Y_k)
\end{aligned}$$

$$(d(i_{Y_1}\omega))(Y_2,\dots,Y_{k+1}) = \sum_{i=2}^{k+1}(-1)^iY_i(\omega(Y_1,..,\widehat{Y_i},..,Y_{k+1})) - \sum_{\substack{i=2\\ j>i}}^{k+1}(-1)^{i+j}\omega([Y_i,Y_j],Y_1,..,\widehat{Y_i},..,\widehat{Y_j},..,Y_{k+1})$$

$$\begin{aligned}
(i_{Y_1}(d\omega))(Y_2,\dots,Y_{k+1}) &= (d\omega)(Y_1,\dots,Y_{k+1})\\
&= \sum_{i=1}^{k+1}(-1)^{i-1}Y_i(\omega(Y_1,..,\widehat{Y_i},..,Y_{k+1})) + \sum_{\substack{i=1\\ j>i}}^{k+1}(-1)^{i+j}\omega([Y_i,Y_j],Y_1,..,\widehat{Y_i},..,\widehat{Y_j},..,Y_{k+1})
\end{aligned}$$

■

33. (October 28) Let $F : M\times[0,1]\to N$ be a smooth map and $\alpha\in H^p(N)$. Give a description of $F^*\alpha = \beta + dt\wedge\gamma$ in local coordinates.

Solution: Let $y\in N$ and let $\alpha_y = \alpha_y^{i_1\cdots i_p}dy_{i_1}\wedge\cdots\wedge dy_{i_p}\in H^p(N)$. Let M be an n-manifold and N an m-manifold, so we may write $F(x) = (F_1(x),\dots,F_m(x))$. Then for $x\in M$,

$$\begin{aligned}
(F^*\alpha)_x &= \alpha_{F(x)}^{i_1\cdots i_p}F^*dy_{i_1}\wedge\cdots\wedge F^*dy_{i_p}\\
&= \alpha_{F(x)}^{i_1\cdots i_p}\left(\frac{\partial F_{i_1}}{\partial x_{j_1}}dx_{j_1} + \frac{\partial F_{i_1}}{\partial t}dt\right)\wedge\cdots\wedge\left(\frac{\partial F_{i_p}}{\partial x_{j_p}}dx_{j_p} + \frac{\partial F_{i_p}}{\partial t}dt\right)\\
&= \underbrace{\beta_{F(x)}^{k_1\cdots k_p}dx_{k_1}\wedge\cdots dx_{k_p}}_{\in\ H^p(M)} + \underbrace{\gamma_{F(x)}^{\ell_1,\dots,\ell_{p-1}}dx_{\ell_1}\wedge\cdots\wedge dx_{\ell_{p-1}}}_{\in\ H^{p-1}(M)}\wedge dt.
\end{aligned}$$

■

34. (October 30) Let M be a smooth manifold. Show that $H^p(M\times\mathbf{R}^n)\cong H^p(M)$ for any p. This result is known as *Poincare's lemma*.

Solution: Since $\mathbf{R}^n$ is contractible, $M\times\mathbf{R}^n$ is homotopic to M. That is, there exist smooth maps

$$F : M\times\mathbf{R}^n\to M \qquad\text{and}\qquad G : M\to M\times\mathbf{R}^n$$

such that $G \circ F \cong \mathrm{id}_{M\times\mathbf{R}^n}$ and $F \circ G \cong \mathrm{id}_M$, where $\cong$ signifies homotopy equivalence. For every p, they induce group homomorphisms

$$F^* : H^p(M \times \mathbf{R}^n) \to H^p(M) \qquad \text{and} \qquad G^* : H^p(M) \to H^p(M \times \mathbf{R}^n).$$

By a theorem from class, we know that $(G \circ F)^* = (\mathrm{id}_{M\times\mathbf{R}^n})^*$ and $(F \circ G)^* = (\mathrm{id}_M)^*$. This means that for any p,

$$\begin{aligned} F^* \circ G^* &= (G \circ F)^* = (\mathrm{id}_{M\times\mathbf{R}^n})^* = \mathrm{id}_{H^p(M\times\mathbf{R}^n)}, \\ G^* \circ F^* &= (F \circ G)^* = (\mathrm{id}_M)^* = \mathrm{id}_{H^p(M)}. \end{aligned}$$

Since these homomorphisms are inverses of each other, $H^p(M \times \mathbf{R}^n) \cong H^p(M)$ for all p. ■

35. (October 30) Prove that $H^p(S^n) = \mathbf{R}$ if $p = 0, n$ and 0 otherwise. You may assume the result for $n = 1$.

We proceed by induction, assuming the case for $n = 1$. Decompose S^n into two sets

$$U = S^n - S \cong \mathbf{R}^{n-1} \qquad \text{and} \qquad V = S^n - N \cong \mathbf{R}^{n-1},$$

where S is the south pole and N is the north pole. Recall that cohomology is diffeomorphism invariant, so $H^k(U) = H^k(V) = H^k(\mathbf{R}^{n-1}) = \mathbf{R}$ if $k = 0$ and 0 otherwise. Finally, note that $U \cap V \cong S^{n-1} \times \mathbf{R}$ via the stereographic projection, and by the Poincaré lemma (the previous question), $H^k(U \cap V) \cong H^k(S^{n-1} \times \mathbf{R}) \cong H^k(S^{n-1})$.

For all the cases below, we take $\omega \in \Omega^k(S^n)$ to be closed, so $d\omega = 0$. For any k-form η, we write η_Y instead of $\eta|_Y$ when restricting to some set Y.

$\underline{k = 0}$: Since S^n is connected, $H^0(S^n) = \mathbf{R}$.

$\underline{k = 1}$: Note that $H^1(U) = H^1(V) = 0$, so forms are closed iff they are exact. Since $d\omega_U = 0$ and $d\omega_V = 0$, there exist $f \in \Omega^0(U)$ and $g \in \Omega^0(V)$ such that $df = \omega_U$ and $dg = \omega_V$. Hence $d(f_{U\cap V} - g_{U\cap V}) = 0$, so $f_{U\cap V} = g_{U\cap V} + C$ for some constant C. Define

$$h = \begin{cases} f & \text{on } U, \\ g + C & \text{on } V, \end{cases} \quad \in \Omega^0(S^n)$$

and h is well-defined since the two definitions agree on the overlaps. Then $dh = \omega$, so ω is exact. Hence $H^1(S^n) = 0$.

$\underline{1 < k < n}$: Note that $H^k(U) = H^k(V) = 0$, so forms are closed iff they are exact. Since $d\omega_U = 0$ and $d\omega_V = 0$, there exist $\alpha \in \Omega^{k-1}(U)$ and $g \in \Omega^{k-1}(V)$ such that $d\alpha = \omega_U$ and $d\beta = \omega_V$. Hence $d(\alpha_{U\cap V} - \beta_{U\cap V}) = 0$. Since $H^{k-1}(U \cap V) \cong H^{k-1}(S^{n-1})$ by the remark above, and $H^{k-1}(S^{n-1}) = 0$ by induction, closed forms are exact. Hence there exists $\gamma \in \Omega^{k-2}(U \cap V)$ such that

$$d\gamma = \alpha_{U\cap V} - \beta_{U\cap V}.$$

Let $\{\psi_U, \psi_V\}$ be a partition of unity subordinate to the cover $\{U, V\}$ of S^n. Then $(\psi_U)_{U\cap V}\gamma$ extends, by 0 at S, to a $(k-2)$-form on V. Similarly, $(\psi_V)_{U\cap V}\gamma$ extends, by 0 at N, to a $(k-2)$-form on U. Now we may define

$$\begin{aligned} \delta_1 &:= \beta + d(\psi_U)_{U\cap V}\gamma \in \Omega^{k-1}(V), \\ \delta_2 &:= \alpha - d(\psi_V)_{U\cap V}\gamma \in \Omega^{k-1}(U). \end{aligned}$$

We claim that on $U\cap V$, these two forms are actually the same. Indeed, by noting that $\gamma = (\psi_U)_{U\cap V}\gamma + (\psi_V)_{U\cap V}\gamma$, we get that

$$\begin{aligned}\beta_{U\cap V} + d(\psi_U)_{U\cap V}\gamma &= \alpha_{U\cap V} - d\gamma + d(\psi_U)_{U\cap V}\gamma\\ &= \alpha_{U\cap V} - (d(\psi_U)_{U\cap V}\gamma + d(\psi_V)_{U\cap V}\gamma) + d(\psi_U)_{U\cap V}\gamma\\ &= \alpha_{U\cap V} - d(\psi_V)_{U\cap V}\gamma.\end{aligned}$$

Hence we may define

$$\delta = \begin{cases}\delta_1 & \text{on } U,\\ \delta_2 & \text{on } V,\end{cases} \quad \in \Omega^{k-1}(S^n),$$

for which

$$d\delta = \psi_U d\delta + \psi_v d\delta = \psi_U d\delta_2 + \psi_V d\delta_1 = \psi_U d\alpha + \psi_U d\beta = \psi_U \omega_U + \psi_V\omega_V = \omega.$$

Then $d\delta = \omega$, so ω is exact. Hence $H^k(S^n) = 0$.

$\underline{k=n}$: This case is left unfinished. $\square$

36. (November 2) Consider the space of straight lines in $\mathbf{R}^3$.

(a) Describe this space as a manifold.

(b) What is the dimension of this manifold?

(c) Show this manifold is not orientable.

Solution: (a) Call this space of lines X, and construct an atlas on it with three charts, namely

$$\begin{aligned}U_x &= \{\ell \subset \mathbf{R}^3 \ : \ \ell \text{ is not parallel to the } yz\text{-plane}\},\\ U_y &= \{\ell \subset \mathbf{R}^3 \ : \ \ell \text{ is not parallel to the } xz\text{-plane}\},\\ U_z &= \{\ell \subset \mathbf{R}^3 \ : \ \ell \text{ is not parallel to the } xy\text{-plane}\}.\end{aligned}$$

Consider U_x first. Since each element of U_x is determined by where it uniquely intersects the yz-plane and then by a direction vector from that point, it follows immediately that $U_x \cong \mathbf{R}^2 \times \mathbf{RP}^2$, and the same goes for U_y and U_z. To complete the description of X as a manifold, we need to show the transition functions are diffeomorphisms, which is done in part (c) below.

(b) The dimension of this manifold is $2+2=4$, as the charts are 4-dimensional.

(c) To show this manifold is not orientable, we will show that the transition functions do not always have positive determinant (while some do). Begin with an element of U_x, which looks like

$$\ell = \{(0,y,z) + p(1:s:t) \ : \ p \in \mathbf{R}\},$$

where $(0,y,z)$ is where ℓ intersects the yz-plane. Assuming that $\ell \in U_y$ as well (so $s \neq 0$), we note that

$$\begin{aligned}\ell &= \{(0,y,z) + (p-y)(\tfrac{1}{s} : 1 : \tfrac{t}{s}) \ : \ p \in \mathbf{R}\}\\ &= \{(-\tfrac{y}{s}, 0, z - \tfrac{yt}{s}) + p(\tfrac{1}{s} : 1 : \tfrac{t}{s}) \ : \ p \in \mathbf{R}\},\end{aligned}$$

so ℓ intersects the xz-plane at $(-\frac{y}{s}, 0, z - \frac{yt}{s})$. This tells us the transition function φ_{xy} is

$$\begin{aligned}\varphi_{xy} \ : \ \mathbf{R}^2 \times \mathbf{RP}^2 &\to \mathbf{R}^2 \times \mathbf{RP}^2,\\ (y,z,s,t) &\mapsto (-\tfrac{y}{s}, z - \tfrac{yt}{s}, \tfrac{1}{s}, \tfrac{t}{s}),\end{aligned}$$

and its derivative is

$$J(\varphi_{xy}) = \begin{bmatrix} \frac{\partial y}{\partial y} & \frac{\partial y}{\partial z} & \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial y} & \frac{\partial z}{\partial z} & \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \\ \frac{\partial s}{\partial y} & \frac{\partial s}{\partial z} & \frac{\partial s}{\partial s} & \frac{\partial s}{\partial t} \\ \frac{\partial t}{\partial y} & \frac{\partial t}{\partial z} & \frac{\partial t}{\partial s} & \frac{\partial t}{\partial t} \end{bmatrix} = \begin{bmatrix} -\frac{1}{s} & 0 & \frac{y}{s^2} & 0 \\ -\frac{t}{s} & 1 & \frac{yt}{s^2} & -\frac{y}{s} \\ 0 & 0 & -\frac{1}{s^2} & 0 \\ 0 & 0 & -\frac{t}{s^2} & \frac{1}{s} \end{bmatrix} \quad , \quad \det(J(\varphi_{xy})) = \frac{1}{s^4}.$$

Now let $\ell \in U_y$ and assume that $\ell \in U_z$ as well (so $t \neq 0$). Then we may rewrite the points in the line as above to get

$$\begin{aligned} \ell &= \{(x,0,z) + p(r:1:t) \ : \ p \in \mathbf{R}\} \\ &= \{(x,0,z) + (p-z)(\tfrac{r}{t} : \tfrac{1}{t} : 1) \ : \ p \in \mathbf{R}\} \\ &= \{(x - \tfrac{zr}{t}, -\tfrac{z}{t}, 0) + p(\tfrac{r}{t} : \tfrac{1}{t} : 1) \ : \ p \in \mathbf{R}\}, \end{aligned}$$

so ℓ intersects the xy-plane at $(x - \frac{zr}{t}, -\frac{z}{t}, 0)$. This tells us the transition function φ_{yz} is

$$\begin{array}{rccl} \varphi_{yz} & : & \mathbf{R}^2 \times \mathbf{RP}^2 & \to & \mathbf{R}^2 \times \mathbf{RP}^2, \\ & & (x,z,r,t) & \mapsto & (x - \frac{zr}{t}, -\frac{z}{t}, \frac{r}{t}, \frac{1}{t}), \end{array}$$

and its derivative is

$$J(\varphi_{yz}) = \begin{bmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial z} & \frac{\partial x}{\partial r} & \frac{\partial x}{\partial t} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial z} & \frac{\partial z}{\partial r} & \frac{\partial z}{\partial t} \\ \frac{\partial r}{\partial x} & \frac{\partial r}{\partial z} & \frac{\partial r}{\partial r} & \frac{\partial r}{\partial t} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial z} & \frac{\partial t}{\partial r} & \frac{\partial t}{\partial t} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{r}{t} & -\frac{z}{t} & \frac{zr}{t^2} \\ 0 & -\frac{1}{t} & 0 & \frac{z}{t^2} \\ 0 & 0 & \frac{1}{t} & -\frac{r}{t^2} \\ 0 & 0 & 0 & -\frac{1}{t^2} \end{bmatrix} \quad , \quad \det(J(\varphi_{yz})) = \frac{1}{t^4}.$$

Finally, let $\ell \in U_z$ and assume that $\ell \in U_x$ as well (so $r \neq 0$). Then we may rewrite the points in the line as above to get

$$\begin{aligned} \ell &= \{(x,y,0) + p(r:s:1) \ : \ p \in \mathbf{R}\} \\ &= \{(x,y,0) + (p-x)(1 : \tfrac{s}{r} : \tfrac{1}{r}) \ : \ p \in \mathbf{R}\} \\ &= \{(0, y - \tfrac{xs}{r}, -\tfrac{x}{r}) + p(1 : \tfrac{s}{r} : \tfrac{1}{r}) \ : \ p \in \mathbf{R}\}, \end{aligned}$$

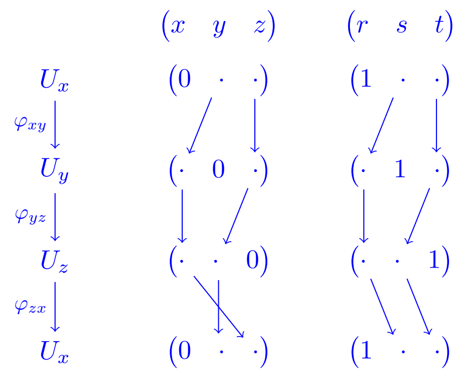
so ℓ intersects the yz-plane at $(0, y - \frac{xs}{r}, -\frac{x}{r})$. This tells us the transition function φ_{zx} is

$$\begin{array}{rccl} \varphi_{zx} & : & \mathbf{R}^2 \times \mathbf{RP}^2 & \to & \mathbf{R}^2 \times \mathbf{RP}^2, \\ & & (x,y,r,s) & \mapsto & (-\frac{x}{r}, y - \frac{xs}{r}, \frac{s}{r}, \frac{1}{r}), \end{array}$$

and its derivative is

$$J(\varphi_{zx}) = \begin{bmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} & \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} & \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \\ \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial r} & \frac{\partial r}{\partial s} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} & \frac{\partial s}{\partial r} & \frac{\partial s}{\partial s} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{s}{r} & \frac{xs}{r^2} & -\frac{x}{r} \\ 0 & -\frac{1}{r} & \frac{x}{r^2} & 0 \\ 0 & 0 & -\frac{s}{r^2} & \frac{1}{r} \\ 0 & 0 & -\frac{1}{r^2} & 0 \end{bmatrix} \quad , \quad \det(J(\varphi_{zx})) = -\frac{1}{r^4}.$$

The reason why the determinant changes sign is the choice of where to send each of the coordinates, since in any given chart, we only have two of them being non-zero. The process is given in the diagram below.



I am not completely sure why we switch the coordinates in the real part of φ_{zx} but not in the projective part, but I think it is because in $\mathbf{RP}^2$ orientation does not matter, but in $\mathbf{R}^2$ it does. Even more, if both would switch, then all determinants would be positive, and the question clearly states "show this is not orientable." This shows that X is not orientable. ■

37. (November 2) Prove that the tangent bundle of a smooth manifold is orientable.

Solution: Let M be a smooth n-manifold with atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ and local coordinates $(x_1, \ldots, x_n)$ on U_α. We claim that TM is a smooth $2n$-manifold with atlas $\{(V_\alpha, \psi_\alpha)\}_{\alpha\in A}$, where

$$V_\alpha = \bigcup_{p\in U_\alpha} T_pM \qquad \text{and} \qquad \begin{array}{rcl} \psi_\alpha : V_\alpha & \to & \mathbf{R}^{2n}, \\ (p, v) & \mapsto & (\varphi_\alpha(p), vx_1, \ldots, vx_n). \end{array}$$

The action of ψ_α may also be given by

$$\psi_\alpha\left(p,\, a_i \frac{\partial}{\partial x_i}\bigg|_p\right) = (\varphi_\alpha(p), a_1, \ldots, a_n).$$

To check that we actually have a manifold, we need the transition functions on the overlaps of the V_α to be diffeomorphisms. So take $\alpha, \beta \in A$ and suppose that

$$V_\alpha \ni \left(p, a_i\frac{\partial}{\partial x_i}\right) = \left(p, b_j\frac{\partial}{\partial y_j}\right) \in V_\beta,$$

for $(y_1, \ldots, y_n)$ local coordinates on V_β. It is immediate that $b_j = a_i\frac{\partial y_j}{\partial x_i}$, so going from V_α to V_β, the transition function $(\mathrm{id}, \frac{\partial y_j}{\partial x_i})$ is a diffeomorphism. Now that we have shown TM is a manifold, we need to show it is orientable. This means that the determinant of all transition maps is positive. In matrix form, the transition function from V_α to V_β is given by

$$g_{\alpha\beta} = \begin{bmatrix} \dfrac{\partial y_j}{\partial x_i} & \dfrac{\partial y_j}{\partial a_i} \\ \dfrac{\partial b_j}{\partial x_i} & \dfrac{\partial b_j}{\partial a_i} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial y_j}{\partial x_i} & 0 \\ \dfrac{\partial b_j}{\partial x_i} & \dfrac{\partial a_k}{\partial a_i}\dfrac{\partial y_j}{\partial x_k} + a_k\dfrac{\partial^2 y_j}{\partial a_i \partial x_i} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial y_j}{\partial x_i} & 0 \\ \dfrac{\partial b_j}{\partial x_i} & \dfrac{\partial y_j}{\partial x_i} \end{bmatrix}.$$

This follows from the product rule and noting that $\frac{\partial y_j}{\partial a_i} = 0$, since the y_j only depend on the x_i, not the a_i, which are in the tangent space already. The element $\frac{\partial b_j}{\partial x_i}$ doesn't matter if all we want is the determinant, as we have an upper-triangular matrix, and so

$$\det(g_{\alpha\beta}) = \left(\frac{\partial y_j}{\partial x_i}\right)^2 > 0.$$

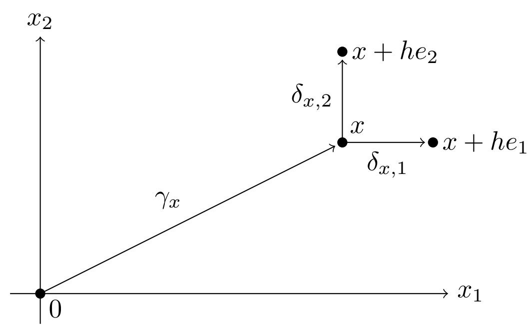
This holds since $\frac{\partial y_j}{\partial x_i}$ is a diffeomorphism, so has non-zero determinant. Therefore the transition functions all have positive determinant, meaning TM is an orientable $2n$-manifold. ■

38. (November 2) Let α be a smooth 1-form on $\mathbf{R}^2$. Show that α is exact if and only if it is closed.

Solution: Describe α as $\alpha = f_1 dx_1 + f_2 dx_2$. First suppose that α is exact, so $\alpha = d\eta$ for some 0-form η. Then $d\alpha = d^2\eta = 0$, so α is closed. Conversely, suppose that α is closed, so $d\alpha = 0$. By Stokes' theorem, for any closed path $\gamma \subset \mathbf{R}^2$ and submanifold $M \subset \mathbf{R}^2$ with $\partial M = \gamma$,

$$\int_\gamma \alpha = \int_M d\omega = \int_M 0 = 0.$$

Let $h > 0$, $\{e_1, e_2\} = \{(1,0),(0,1)\}$ be the standard basis vectors of $\mathbf{R}^2$, and define paths $\gamma_x, \delta_{x,1}, \delta_{x,2}$ as in the diagram below.



Define a 0-form $\eta(x) = \int_{\gamma_x} \omega$, for which we claim that $d\eta = \omega$. By Stokes' theorem above, for $i \in \{1,2\}$,

$$0 = \int_{\gamma_x + \delta_{x,i} - \gamma_{x+he_i}} \omega = \int_{\gamma_x} \omega + \int_{\delta_{x,i}} \omega - \int_{\gamma_{x+he_i}} \omega = \eta(x) - \eta(x+he_i) + \int_{\delta_{x,i}} \omega.$$

Parametrize $\delta_{x,i}$ as $\delta_{x,i} : [0,1] \to \mathbf{R}^2$ given by $\delta_{x,i}(t) = x + the_i$. Rearranging and simplifying the last integral, we get

$$\begin{aligned}
\eta(x+he_i) - \eta(x) &= \int_{\delta_{x,i}} \omega \\
&= \int_0^1 (f_1(\delta_{x,i}(t)) + f_2(\delta_{x,i}(t)))\delta'_{x,i}(t)dt \\
&= \int_0^1 (f_1(x+the_i) + f_2(x+the_i))he_i dt \\
&= h\int_0^1 f_i(x+the_i)dt.
\end{aligned}$$

For $i \in \{1,2\}$, define new functions $g_i : \mathbf{R} \to \mathbf{R}$ by

$$g_i(t) = \int_0^t f_i(x+re_i)dr.$$

Now we finally get to the derivative of η. Observe that

$$\begin{aligned}
\frac{\partial \eta}{\partial x_i} &= \lim_{h\to 0}\left[\frac{\eta(x+he_i)-\eta(x)}{h}\right] && \text{(definition)}\\
&= \lim_{h\to 0}\left[\int_0^1 f_i(x+the_i)dt\right] && \text{(above)}\\
&= \lim_{h\to 0}\left[\frac{1}{h}\int_0^h f_i(x+re_i)dr\right] && \text{(substitution } r=th\text{)}\\
&= \lim_{h\to 0}\left[\frac{g_i(h)-g_i(0)}{h}\right] && \text{(definition)}\\
&= g_i'(0) && \text{(definition)}\\
&= f_i(x+0e_i) && \text{(fundamental theorem of calculus)}\\
&= f_i(x).
\end{aligned}$$

Therefore

$$\alpha = f_1dx_1 + f_2dx_2 = \frac{\partial\eta}{\partial x_1}dx_1 + \frac{\partial\eta}{\partial x_2}dx_2 = d\eta,$$

and so α is an exact 1-form. ■

39. (November 2) Let M be the complement of the origin in $\mathbf{R}^3$. Construct a 2-form on M which is closed but not exact.

Solution: Let $(x,y,z)\in\mathbf{R}^3\setminus\{0\}$ with radius r, given by $r^2=x^2+y^2+z^2$, and consider the 3-form

$$\omega = \frac{x}{r^3}dy\wedge dz - \frac{y}{r^3}dx\wedge dz + \frac{z}{r^3}dx\wedge dy.$$

We claim ω is closed but not exact. To see it is closed, first note that

$$dr = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2xdx+2ydy+2zdz) = r^{-1}(xdx+ydy+zdz).$$

Now calculate

$$\begin{aligned}
d\left(xr^{-3}dy\wedge dz\right) &= (r^{-3}dx - 3r^{-4}xdr)\wedge dy\wedge dz = (r^{-3}-3r^{-5}x^2)dx\wedge dy\wedge dz\\
d\left(yr^{-3}dx\wedge dz\right) &= (r^{-3}dy - 3r^{-4}ydr)\wedge dx\wedge dz = -(r^{-3}-3r^{-5}y^2)dx\wedge dy\wedge dz\\
d\left(zr^{-3}dx\wedge dy\right) &= (r^{-3}dz - 3r^{-4}zdr)\wedge dx\wedge dy = (r^{-3}-3r^{-5}z^2)dx\wedge dy\wedge dz.
\end{aligned}$$

Combining them gives

$$d\omega = (3r^{-3}-3r^{-5}(x^2+y^2+z^2))dx\wedge dy\wedge dz = (3r^{-3}-3r^{-5}r^2)dx\wedge dy\wedge dz = 0,$$

and so ω is closed. To see ω is not exact, apply Stokes' theorem. If ω were to be exact, then $\omega = d\alpha$ for some 1-form α. Consider the solid unit ball M in $\mathbf{R}^3\setminus\{0\}$ and its boundary $\partial M = S^2$. Then we would have

$$\int_{S^2}\omega = \int_{\partial M}\omega = \int_M d\omega = \int_M dd\alpha = \int_M 0 = 0.$$

However, we will show that $\int_{S^2}\omega \neq 0$. Parametrize the sphere by

$$\begin{aligned}
S^2 \ :\ [0,\pi]\times[0,2\pi] &\to \mathbf{R}^3,\\
(s,t) &\mapsto (\sin(s)\cos(t), \sin(s)\sin(t), \cos(s)),
\end{aligned}$$

for which we get

$$\begin{aligned}
dx &= \cos(s)\cos(t)ds - \sin(s)\sin(t)dt,\\
dy &= \cos(s)\sin(t)ds + \sin(s)\cos(t)dt,\\
dz &= -\sin(s)ds.
\end{aligned}$$

Hence on S^2, where $r^3 = 1$,

$$\begin{aligned}
\frac{x}{r^3}dy \wedge dz &= \sin(s)\cos(t)(\cos(s)\sin(t)ds + \sin(s)\cos(t)dt) \wedge (-\sin(s)ds) = \sin^3(s)\cos^2(t)ds \wedge dt,\\
\frac{y}{r^3}dx \wedge dz &= \sin(s)\sin(t)(\cos(s)\cos(t)ds - \sin(s)\sin(t)dt) \wedge (-\sin(s)ds) = -\sin^3(s)\sin^2(t)ds \wedge dt,\\
\frac{z}{r^3}dx \wedge dy &= \cos(s)(\cos(s)\cos(t)ds - \sin(s)\sin(t)dt) \wedge (\cos(s)\sin(t)ds + \sin(s)\cos(t)dt)\\
&= (\sin(s)\cos^2(s)\cos^2(t) + \sin(s)\cos^2(s)\sin^2(t))ds \wedge dt\\
&= \sin(s)\cos^2(s)ds \wedge dt.
\end{aligned}$$

Now we integrate these separately to get

$$\begin{aligned}
\int_{S^2} \frac{x}{r^3}dy\ dz &= \int_0^{2\pi} \cos^2(t) \int_0^{\pi} \sin^3(s)ds\ dt\\
&= \int_0^{2\pi} \cos^2(t) \left(\frac{\cos(3s)}{12} - \frac{3\cos(s)}{4}\right)\Bigg|_{s=0}^{s=\pi} dt\\
&= \frac{4}{3}\int_0^{2\pi} \cos^2(t)dt\\
&= \frac{4}{3}\left(\frac{2t + \sin(2t)}{4}\right)\Bigg|_{t=0}^{t=2\pi}\\
&= \frac{4\pi}{3}
\end{aligned}$$

and

$$\begin{aligned}
\int_{S^2} \frac{y}{r^3}dx\ dz &= -\int_0^{2\pi} \sin^2(t) \int_0^{\pi} \sin^3(s)ds\ dt\\
&= -\int_0^{2\pi} \sin^2(t) \left(\frac{\cos(3s)}{12} - \frac{3\cos(s)}{4}\right)\Bigg|_{s=0}^{s=\pi} dt\\
&= -\frac{4}{3}\int_0^{2\pi} \sin^2(t)dt\\
&= -\frac{4}{3}\left(\frac{2t - \sin(2t)}{4}\right)\Bigg|_{t=0}^{t=2\pi}\\
&= -\frac{4\pi}{3}
\end{aligned}$$

and

$$\begin{aligned}
\int_{S^2} \frac{z}{r^3}dx\ dy &= \int_0^{2\pi}\int_0^{\pi} \sin(s)\cos^2(s)ds\ dt\\
&= 2\pi\left(\frac{-\cos^3(s)}{3}\right)\Bigg|_{s=0}^{s=\pi}\\
&= \frac{4\pi}{3}.
\end{aligned}$$

Hence $\int_{S^2} \omega = 4\pi/3 + 4\pi/3 + 4\pi/3 = 4\pi \neq 0$, and so ω is not exact. ■

40. (November 4) Construct a smooth map $f : S^2 \to \mathbf{RP}^2$ and show, by contradiction, that $\mathbf{RP}^2$ is not orientable (by pulling back an orientation form on $\mathbf{RP}^2$ to an orientation form on S^2).

Solution: Consider the map that takes a point $x \in S^2$ to its equivalence class $[x] = \{x, -x\} \in \mathbf{RP}^2$. There is an induced map on top cohomology groups, given by $f^* : H^2_{dR}(\mathbf{RP}^2) \to H^2_{dR}(S^2)$. However, since $H^2(\mathbf{RP}^2; \mathbf{Z}) = 0$, and by the de Rham theorem, singular and de Rham cohomology groups agree, it follows that $H^2_{dR}(\mathbf{RP}^2) = 0$. Hence no non-zero cohomology classes exist in $H^2_{dR}(\mathbf{RP}^2)$, so there is nothing to pull back to S^2, and $\mathbf{RP}^2$ is not orientable. ■

41. (November 6) Let M, N be smooth manifolds of dimension n, and $f : M \to N$ a smooth bijective immersion. Show that f is a diffeomorphism.

Solution: An immersion $f : M \to N$ between manifolds has injective differential, and since the manifolds are of the same dimension, the differential is also surjective. Hence the differential is invertible. By the inverse function theorem, f is a local diffeomorphism. Since f is bijective, f is a diffeomorphism. ■

42. (November 6) Let M be a connected manifold without boundary. Show that if S, T are finite sets in M of the same size, then there is a diffeomorphism $f : M \to M$ sending S to T (that is, $f(S) = T$).

Solution: Let $S = \{s_1, \ldots, s_m\}$ and $T = \{t_1, \ldots, t_m\}$, and first consider the case when M is 1-dimensional. Assume that $s_1 < s_2 < \cdots < s_m$ and $t_1 < t_2 < \cdots < t_m$. Let $f_1 : M \to M$ be a map with support on a neighborhood of $[s_1, t_1]$ not containing any other points of T that takes s_1 to t_1. Let $f_2 : M \to M$ be a map with support on a neighborhood of $[f_1(s_2), t_2]$ not containing any other points of T that takes $f_1(s_2)$ to t_2. Let $f_3 : M \to M$ be a map with support on a neighborhood of $[f_2(f_1(s_3)), t_3]$ not containing any other points of T that takes $f_2(f_1(s_3))$ to t_3. Keep going in this manner until all the points are tale care of. Then $F = f_m \circ f_{m-1} \circ \cdots \circ f_1$ takes S to T.

Next consider the case $M = \mathbf{R}^n$ for $n \geqslant 2$. Let $\gamma_i : [0, 1] \to \mathbf{R}^n$ be a path in $\mathbf{R}^n$ with

$$\begin{aligned} \gamma_i(0) &= s_i, & \qquad \gamma_i(x) &\neq s_j, t_j \ \forall\ j,\ \forall\ x \in (0,1), \\ \gamma_i(1) &= t_i, & \qquad & \gamma_i \text{ is not self-intersecting.} \end{aligned}$$

We will construct a "tunnel" around γ_i that does not touch any of the other points, so that we have maps that take s_i to t_i without disturbing any of the other points. Let

$$\epsilon_i = \min_{j \neq i} \{d(\gamma_i, s_j), d(\gamma_i, t_j)\} \qquad \text{and} \qquad V_i = \bigcup_{x \in [0,1]} B(\gamma_i(x), \epsilon_i/2).$$

Here V_i is an open neighborhood of γ_i that only contains s_i, t_i of all of S, T. Since we have local compactness, there exist $x_1, \ldots, x_\ell$ such that

$$\widetilde{V}_i = \bigcup_{k=1}^{\ell} B(\gamma_i(x_k), \epsilon_i/2)$$

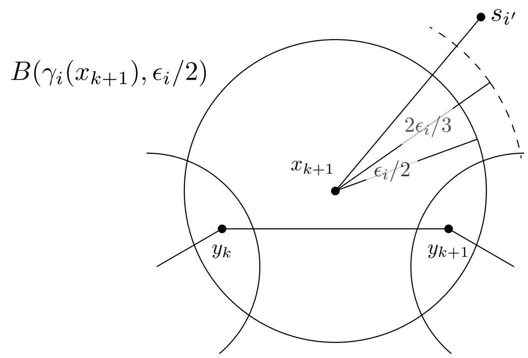
is still an open neighborhood of γ_i. Fix $y_1 = s_i$, $y_{\ell+1} = t_i$ and

$$y_k \in B(\gamma(x_k), \epsilon_i/2) \cap B(\gamma_i(x_{k+1}, \epsilon_i/2)$$

so $y_k = \gamma_i(x)$ for some x (that is, y_k is on the path γ_i). Define maps f_k and bump functions φ_k by

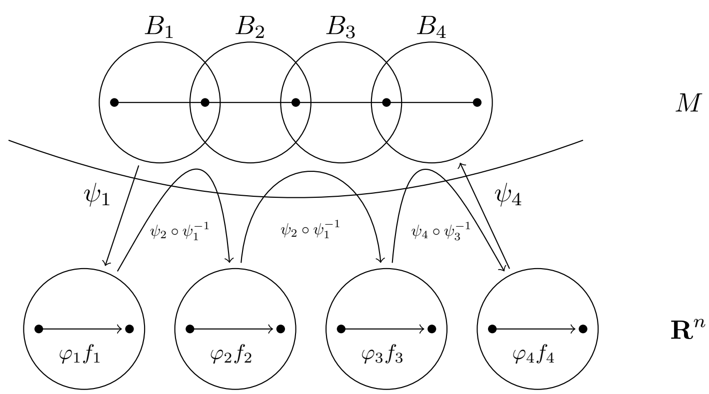
$$\begin{array}{rcl} f_k : \mathbf{R}^n & \to & \mathbf{R}^n, \\ a & \mapsto & a - y_k + y_{k+1}, \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \varphi_k : \mathbf{R}^n & \to & \mathbf{R}, \\ a & \mapsto & 1 \text{ if } a \in B(\gamma_i(x_k), \epsilon_i/2), \\ a & \mapsto & 0 \text{ if } a \notin B(\gamma_i(x_k), 2\epsilon_i/3). \end{array}$$

Then $F_k = \varphi_k f_k$ is a smooth function taking y_k to y_{k+1} and not disturbing any of the other y_k's. The picture looks like in the diagram below.



Let $G_i = F_\ell \circ F_{\ell-1} \circ \cdots \circ F_1$, which is a smooth map on $\mathbf{R}^n$ with $G_i(s_i) = t_i$ and $G_i(s_j) = s_j$ and $G_i(t_j) = t_j$ for all $j \neq i$. Then $G = G_m \circ G_{m-1} \circ \cdots \circ G_1$ takes s_i to t_i for all i.

Now consider some compact manifold M. Let $\gamma_i : [0,1] \to M$ be a path in M with the same conditions as above. Proceed exactly as above until the construction of the maps F_k. Assume that $\psi_k : B_i = B(\gamma_i(x_{k+1}), \epsilon_1/2) \to \mathbf{R}^n$ are charts. Define $\widetilde{F}_k = \psi_k^{-1} \circ (\varphi_k f_k) \circ \psi_k$, which is a smooth map taking $y_k \in M$ to $y_{k+1} \in M$. Let $\widetilde{G}_i = \widetilde{F}_\ell \circ \widetilde{F}_{\ell-1} \circ \cdots \circ \widetilde{F}_1$, which takes s_i to t_i without disturbing any of the other s_j's and t_j's. The situation looks like in the diagram below.



Hence $\widetilde{G} = \widetilde{G}_m \circ \widetilde{G}_{m-1} \circ \cdots \circ \widetilde{G}_1$ takes s_i to t_i for all i, and is a smooth map of M. ■

43. (November 6) Let M be a compact smooth orientable n-manifold. Show that there exists a smooth map $f : M \to S^n$ of non-zero degree.

Solution: We present a solution that works for a non-orientable non-compact manifold as well. Let $p \in M$ and $U \ni p$ a neighborhood of p, and $\varphi : U \to \mathbf{R}^n$ a chart. Let $\epsilon > 0$ such that $B(\varphi(p), \epsilon) \subset \varphi(U)$. For S^n, let $V = \mathbf{S}^n \setminus \{\text{south pole}\}$ and $\psi : V \to \mathbf{R}^n$ the stereographic projection. Define

$$\begin{array}{rcl} g : \mathbf{R}^n & \to & \mathbf{R}^n, \\ x & \mapsto & x - \varphi(p), \end{array} \qquad \text{and} \qquad \begin{array}{rcl} h : \mathbf{R}^n & \to & \mathbf{R}^n, \\ x & \mapsto & x \cdot \frac{1}{\epsilon - |x|}. \end{array}$$

Then $g(B(\varphi(p), \epsilon)) = B(0, \epsilon)$, and $h(B(0, \epsilon)) = B(0, \infty) = \psi(V)$. Let $\widetilde{U} := \varphi^{-1}(B(\varphi(p), \epsilon))$ and define a map

$$\begin{array}{rcl} f : M & \to & S^n, \\ x \in \widetilde{U} & \mapsto & (\psi^{-1} \circ h \circ g \circ \varphi)(x), \\ x \notin \widetilde{U} & \mapsto & \{\text{south pole}\}. \end{array}$$

This is a smooth map, because all the components are smooth or the zero map (which is also smooth). To find the degree of the map, recall that

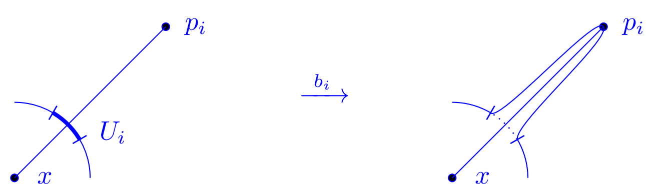
$$\begin{array}{rcl} f^* : H^n(S^n) & \to & H^n(M), \\ [\omega_S] & \mapsto & \deg(f)[\omega_M], \end{array}$$

where $[\omega_S]$ is the orientation class of S^n and $[\omega_M]$ is the orientation class of M. Further, Recall $H^n M = \bigwedge^n T^* M$, and the map f on $\widetilde{U}$ is a diffeomorphism, which is an isomorphism on the cohomologies. Since $f(\widetilde{U})$ is all of S^n minus one point, and on $\widetilde{U}$ the map f^* is an isomorphism (so $\deg(f) = 1$), it follows that $\deg(f) = 1$ everywhere. Hence we have a smooth map $M \to S^n$ of degree $1 \neq 0$. ■

44. (November 9) Let $P \subset \mathbf{R}^3$ be a finite set. Show that there is a smooth embedding $f : S^2 \to \mathbf{R}^3$ such that $P \subset f(S^2)$.

Solution: Let $P = \{p_1, \dots, p_k\} \subset \mathbf{R}^3$ be the given finite set. Let $\ell_{p_i x} \subset \mathbf{R}^3$ be the line segment connecting p_i to some $x \in \mathbf{R}^3$. We claim that there is some $x \in \mathbf{R}^3$ such that $\ell_{p_i x} \cap \ell_{p_j x} = \{x\}$ for all $i \neq j$. This is immediate as the set $L = \{x \in L_{p_i p_j} \ : \ \forall\ 1 \leqslant i, j \leqslant k\}$ is a proper subset of $\mathbf{R}^3$ (even more, a Lebesgue-measure zero subset of $\mathbf{R}^3$), where $L_{p_i p_j}$ is the unique line intersecting p_i and p_j.

Choose $x \in \mathbf{R}^3 \setminus L$, let r be the distance between x and P, and let $S = S(x, 0 < r' < r)$ be the 2-sphere of radius r' centered at x. Let $x_i = S \cap \ell_{p_i x}$ and $U_i \subset S$ some closed neighborhood of x_i such that $U_i \cap U_j = \emptyset$ iff $i \neq j$, for all $1 \leqslant i \leqslant k$. Then for all i, there is a smooth bump function b_i on S with support only on U_i that takes x_i to p_i, as in the diagram below.



Let $i(S^2) = S$, so i is a smooth embedding that takes the standard sphere S^2 to a sphere of radius d' centered at x in $\mathbf{R}^3$. Then b_i is a diffeomorphism for all i, and for $f = b_k \circ b_{k-1} \circ \cdots \circ b_2 \circ b_1 \circ i$ a smooth embedding as well, we have that $P \subset f(S^2)$. ■

45. (November 9) Let C be a closed curve in $\mathbf{R}^2$, given by the zero locus of $f(x, y)$, and $\omega = x\,dy$ a 1-form on $\mathbf{R}^2$. Show that the integral of ω over f is equal to the area enclosed by the curve.

Solution: Recall Green's theorem, which says that for a simple closed curve C in $\mathbf{R}^2$ and D the region enclosed by C, if g, h are C^1 in x, y, then

$$\int_C (g\ dx + h\ dy) = \iint_D \left(\frac{\partial h}{\partial x} - \frac{\partial g}{\partial y} \right) dx\ dy.$$

In this case we have $g = 0$ and $h = x$, which are C^1 in both variables. Hence

$$\int_C \omega = \int_C x\ dy = \iint_D \frac{\partial x}{\partial x} dx\ dy = \iint_D dx\ dy = \text{area}(D).$$

Equivalently, we can use Stokes' theorem (of which Green's is a special case), by letting Ω be the area enclosed by C and $\partial\Omega = C$. Then

$$\int_C \omega = \int_{\partial\Omega} \omega = \int_\Omega d\omega = \int_\Omega d(x\ dy) = \int_\Omega dx\ dy = (\text{area of } \Omega).$$

■

46. (November 9) Describe an equivalent statement to the exercise above, but for surfaces in $\mathbf{R}^3$.

Solution: A simple closed curve C in 2-space becomes a simple closed surface S in 3-space (which still may be described as the zero locus of some $f(x, y, z)$). The "area enclosed" by C now is the 3-dimensional manifold Σ with boundary $\partial\Sigma = S$. To see how ω generalizes, consider the generalization of Green's theorem, which is the divergence theorem (both of which are special cases of Stokes' theorem). It says that, given S a simple closed surface in $\mathbf{R}^3$ and Σ the region enclosed by S, if g, h, k are C^1 in x, y, z, then

$$\int_S (g\ dx + h\ dy + k\ dz) = \iiint_\Sigma \left(\frac{\partial g}{\partial x} + \frac{\partial h}{\partial y} + \frac{\partial k}{\partial z} \right) dx\ dy\ dz.$$

In this case we can use $\omega = xdx, ydy$, or zdz. We could also use $\omega = \frac{1}{3}(xdx + ydy + zdz)$. In all of those cases, we would have g, h, k being C^1 in all the variables, allowing us to say

$$\int_S \omega = \iiint_\Sigma dx\ dy\ dz = \text{volume}(\Sigma).$$

Equivalently, we may ask: "Let C be a closed surface in $\mathbf{R}^3$, given by the zero locus of $f(x, y, z)$, and $\omega = x\ dy\ dz$ a 2-form on $\mathbf{R}^3$. Show that the integral of ω over f is equal to the volume enclosed by the surface." The answer would be the same as above:

$$\int_C \omega = \int_{\partial\Omega} \omega = \int_\Omega d\omega = \int_\Omega d(x\ dy\ dz) = \int_\Omega dx\ dy\ dz = (\text{volume of } \Omega).$$

■

47. (November 9) Let $M \ni x$ be an n-manifold without boundary and $B(x) \subseteq M$ a closed neighborhood of x diffeomorphic to the unit n-ball. Prove that $M - \{x\}$ is diffeomorphic to $M - B(x)$.

Solution: Let B_n be the closed unit ball centered at x, and B_n^ϵ the closed ball of radus $1 + \epsilon$ centered at x. Without loss of generality, assume that $B(x) \subsetneq B_n$ and $B_n^\epsilon \subsetneq M$. If these do not hold, change the radii of the defined balls. Let $f : B(x) \to B_n$ be the diffeomorphism given, and let $b : M \to M$ be a bump function given by

$$b(y) = \begin{cases} f(y) & y \in B(x), \\ y & y \notin B_n^\epsilon. \end{cases}$$

We may assume that $b(x) = x$, so the above is also a map $M - \{x\} \to M - \{x\}$ that takes $B(x)$ to B_n. Next consider the following map, which we claim is a diffeomorphism between $M - \{x\}$ and $M - B_n$:

$$\begin{aligned} g \ : \ M - \{x\} &\to M - B_n, \\ y \in B_n^\epsilon &\mapsto \tfrac{1+\epsilon}{\|y\|} y, \\ y \notin B_n^\epsilon &\mapsto y. \end{aligned}$$

This map is smooth, its inverse is smooth, and both it and its inverse are bijective, so it is a diffeomorphism (all of these things are clear, because the map is just multiplication). Now consider the map

$$\begin{aligned} h \ : \ M - \{x\} &\to M - B(x), \\ y &\mapsto b^{-1}(g(x)). \end{aligned}$$

Since g and b were diffeomorphisms, so is h. Finally, since b takes $B(x)$ to B_n, its inverse b^{-1} takes $M - B_n$ to $M - B(x)$, exactly as desired. Therefore $M - \{x\}$ is diffeomorphic to $M - B(x)$.

A more direct approach is to use Whitney's embedding theorem to embed M in $\mathbf{R}^N$, for a N large enough. Any continuous map defined on a compact subset of $\mathbf{R}^N$ extends to all of it (this is the Tietze extension theorem), so we apply this to the given diffeomorphism f, assuming the unit ball is a subset of $B(x)$ (otherwise shrink the ball). In fact, we only need to extend f to some open neighborhood U of $B(x)$, then apply a partition of unity to define it on M. This gives a map $\alpha : M - B(x) \to M -$ (unit ball), and by stretching an ϵ-shell of the unit ball to an $(\epsilon + 1)$-shell of the point, we get a diffeomorphism $M - B(x) \to M - \{x\}$. ■

48. (November 11) Show that $S^1 \times S^1$ is not diffeomorphic to S^2.

(Contributed by Nathan Lopez)
Solution: First note that $S^1 \times S^2$ is the torus and S^2 is the sphere. The torus is not simply connected, since it has non-trivial elements in its fundamental group, and the sphere is simply connected. If there were to exist a diffeomorphism $f : S^1 \times S^1 \to S^2$, then f would induce an isomorphism on homology groups. However, $H^1(S^1 \times S^1; \mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$ and $H^1(S^2; \mathbf{Z}) = 0$, which are clearly not isomorphic. Hence no such diffeomorphism exists. ■

49. (November 13) Let M be a manifold with boundary ∂M. Show that an orientation M defines an orientation on ∂M.

Solution: Let $\omega \in \Omega^n(M)$ be an orientation of M. Then we know $[0] \neq [\omega] \in H^n(M)$, so there does not exist $\eta \in \Omega^{n-1}(M)$ such that $[d\eta] = [\omega]$. Let $x_1, \dots, x_n$ be local coordinates on M such that the image of ∂M lies in $x_n = 0$. Write

$$\omega = f dx_1 \wedge \cdots \wedge dx_n \qquad \text{for} \qquad f \in C^\infty(M), f > 0.$$

We may choose f to be positive (this is the positive orientation of M). Note that we may consider $f|_{\partial M} \in C^\infty(\partial M)$ as well, and since $f > 0$ on M, we have $f|_{\partial M} > 0$. Consider

$$\widetilde{\omega} = f|_{\partial M} dx_1 \wedge \cdots \wedge dx_{n-1} \in \Omega^{n-1}(\partial M),$$

which is indeed in $\Omega^{n-1}(\partial M)$, by our choice of chart. To show $\widetilde{\omega}$ is an orientation on ∂M, we need to show $[0] \neq \widetilde{\omega}$ in $H^{n-1}(\partial \Omega)$. For contradiction, suppose that there exists $\widetilde{\eta} \in \Omega^{n-2}(\partial M)$ such that $d\widetilde{\eta} = \widetilde{\omega}$. Then wedging with dx_n we get

$$(d\widetilde{\eta}) \wedge dx_n = \widetilde{\omega} \wedge dx_n = f' dx_1 \wedge \cdots \wedge dx_n,$$

where $f' \in C^\infty(M)$ is some strictly positive extension of $f|_{\partial M}$ to all of M, so $[\widetilde{\omega} \wedge dx_n] = [\omega]$. Then

$$\widetilde{\omega} \wedge dx_n = (d\widetilde{\eta}) \wedge dx_n + \underbrace{(-1)^{n-2}\widetilde{\eta} \wedge d(dx_n)}_{=0} = d(\widetilde{\eta} \wedge dx_n),$$

by the Leibniz rule. However, $\widetilde{\eta} \wedge dx_n \in \Omega^{n-1}(M)$, giving an η for which $[d\eta] = [\omega]$, a contradiction. Hence no such $\widetilde{\eta}$ exists, and $[0] \neq [\widetilde{\omega}] \in H^{n-1}(\partial M)$. This shows that an orientation ω on M defines an orientation $\widetilde{\omega}$ on ∂M. ■

50. (November 13) Let M be a compact orientable manifold with boundary ∂M. Recall that a *retract* of M onto a subset $N \subset M$ is a continuous map $r : M \to N$ such that $r(n) = n$ for all $n \in N$. Show that there is no smooth retract $M \to \partial M$.

Solution: Since M is orientable, there exists a non-vanishing orientation form $\omega \in \Omega^n(M)$. By a previous homework question, this induces a non-vanishing orientation form $\widetilde{\omega} \in \Omega^{n-1}(\partial M)$. This means that $\int_{\partial M} \widetilde{\omega} > 0$, where we have chosen the positive orientation.

Suppose there exists a smooth retract $f : M \to \partial M$. Since f is smooth, there is an induced map $f^* : \Omega^{n-1}(\partial M) \to \Omega^{n-1}(M)$. Since f is a retract, $f^* = \mathrm{id}$ on $\Omega^{n-1}(M)$. That is, $\widetilde{\omega} \in \Omega^{n-1}(\partial M)$ is also $f^*\widetilde{\omega} \in \Omega^{n-1}(M)$. Then

$$\begin{aligned}
0 &< \int_{\partial M} \widetilde{\omega} && \text{(hypothesis)} \\
&= \int_{\partial M} f^*\widetilde{\omega} && \text{(assumption)} \\
&= \int_M d(f^*\widetilde{\omega}) && \text{(Stokes' theorem)} \\
&= \int_M f^*(d\widetilde{\omega}). && \text{(pullbacks and } d \text{ commute)}
\end{aligned}$$

Since M is orientable, $H^n(M)$ is 1-dimensional. If $[d\widetilde{\omega}] \in H^n(M)$ is not the zero class $[0]$, it must be a multiple of the orientation class $[\omega]$. But then $[\omega] = [d\widetilde{\omega}]$, contradicting the fact that ω is closed but not exact. Hence $d\widetilde{\omega} = 0$, giving us a contradiction. Hence no such f exists, and there is no retract $M \to \partial M$. ■