mat-blag
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this document contains the posts at www.mat-blag.blogspot.com, with an index

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devoted to the art of moving butts

Part I
Foundational topics

1 Algebra

1.1 Unit and counit adjunction

2016-02-24

Keywords: *unit, counit, adjoint*

Let $\mathcal{F} : C \leftrightarrows D : \mathcal{G}$ be adjoint functors. That is, let $\mathcal{F}$ be left-adjoint to $\mathcal{G}$, and let $\mathcal{G}$ be right-adjoint to $\mathcal{F}$, so that $\mathrm{Hom}_D(\mathcal{F}(X), Y) \cong \mathrm{Hom}_C(X, \mathcal{G}(Y))$ for any $X \in \mathrm{Obj}(C)$ and $Y \in \mathrm{Obj}(D)$.

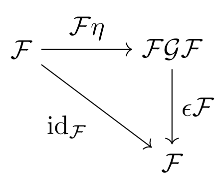
Definition 1.1.1. This isomorphism gives natural maps η_X and ϵ_Y as below:

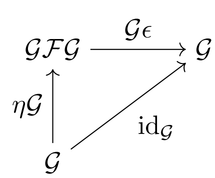
$$\begin{aligned} \mathrm{Hom}_D(\mathcal{F}(X), \mathcal{F}(X)) &\cong \mathrm{Hom}_C(X, \mathcal{G}(\mathcal{F}(X)) \\ \mathrm{id}_{\mathcal{F}(X)} &\mapsto \left(X \xrightarrow{\eta_X} (\mathcal{G} \circ \mathcal{F})(X)\right) \end{aligned} \qquad \begin{aligned} \mathrm{Hom}_C(\mathcal{G}(Y), \mathcal{G}(Y)) &\cong \mathrm{Hom}_D(\mathcal{F}(\mathcal{G}(Y)), Y) \\ \mathrm{id}_{\mathcal{G}(Y)} &\mapsto \left((\mathcal{F} \circ \mathcal{G})(Y) \xrightarrow{\epsilon_Y} Y\right) \end{aligned}$$

These may be viewed as natural transformations called the *unit* η and the *counit* ϵ:

$$\eta : 1_C \to \mathcal{G} \circ \mathcal{F} \qquad \epsilon : \mathcal{F} \circ \mathcal{G} \to 1_D$$

They satisfy the triangle identities, that is, the following diagrams commute.



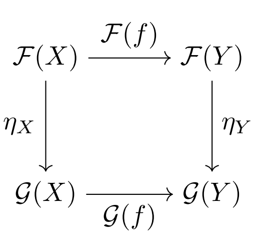


1.2 Limits and colimits

2016-03-09

Keywords: *limit, colimit, natural transformation, constant category*

Definition 1.2.1. Given categories A, B and functors $\mathcal{F}, \mathcal{G} : A \to B$, a *natural transformation* $\eta : \mathcal{F} \to \mathcal{G}$ is a collection of elements $\eta_X \in \mathrm{Hom}_B(\mathcal{F}(X), \mathcal{G}(X))$ for all $X \in \mathrm{Obj}(A)$ such that the diagram

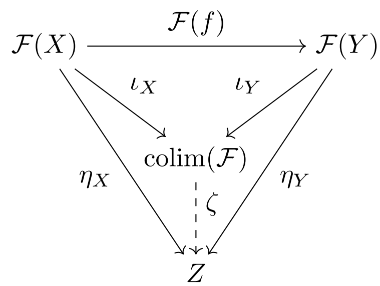


commutes, whenever $f \in \mathrm{Hom}_A(X, Y)$.

Definition 1.2.2. For $X \in \mathrm{Obj}(A)$, define the *constant category* $\underline{X}$ to be the category with $\mathrm{Obj}(\underline{X}) = \{X\}$ and $\mathrm{Hom}_{\underline{X}}(X, X) = \{\mathrm{id}_X\}$. For any other category B, this may also be viewed as a natural transformation $\underline{X} : B \to A$ with $\underline{X}(Y) = X$ and $\underline{X}(f) = \mathrm{id}_X$ for any object Y and any morphism f of B.

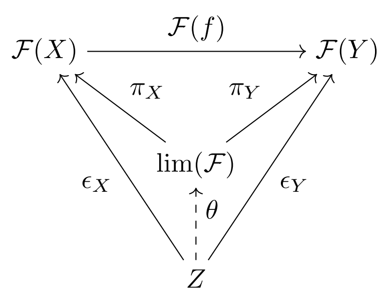
Definition 1.2.3. Let A be a small category and $\mathcal{F} : A \to B$ a functor. The *colimit* $\mathrm{colim}(\mathcal{F})$ of $\mathcal{F}$ is an object $\mathrm{colim}(\mathcal{F}) \in \mathrm{Obj}(B)$ and a natural transformation $\iota : \mathcal{F} \to \underline{\mathrm{colim}(\mathcal{F})}$ that is initial among all such natural transformations. We write $\iota_X : \mathcal{F}(X) \to \mathrm{colim}(\mathcal{F})$ and have $\iota(f) = \mathrm{id}_{\mathrm{colim}(\mathcal{F})}$ for any morphism f of A.

In other words, whenever $Z \in \mathrm{Obj}(B)$ and $\eta : \mathcal{F} \to \underline{Z}$ is a natural transformation, there is a unique map $\zeta : \mathrm{colim}(\mathcal{F}) \to Z$ such that the following diagram commutes:



Definition 1.2.4. Let A be a small category and $\mathcal{F} : A \to B$ a functor. The *limit* $\lim(\mathcal{F})$ of $\mathcal{F}$ is an object $\lim(\mathcal{F}) \in \mathrm{Obj}(B)$ and a natural transformation $\pi : \underline{\lim(\mathcal{F})} \to \mathcal{F}$ that is final among all such natural transformations. We write $\pi_X : \lim(\mathcal{F}) \to \mathcal{F}(X)$ and have $\pi(f) = \mathrm{id}_{\lim(\mathcal{F})}$ for any morphism f of A.

In other words, whenever $Z \in \mathrm{Obj}(B)$ and $\epsilon : \underline{Z} \to \mathcal{F}$ is a natural transformation, there is a unique map $\theta : Z \to \lim(\mathcal{F})$ such that the following diagram commutes:



Examples of colimits are initial objects, coproducts, cokernels, pushouts, direct limits. Examples of limits are final objects, products, kernels, pullbacks, inverse limits.

Remark 1.2.5. Often we take the limit or colimit of an indexed set X_i. In the context described, this means X_i are objects of B, and $A = \mathbf{N}$, the natural numbers, with $\mathcal{F}(i) = X_i$.

Remark 1.2.6. Hom commutes with limits and tensor commutes with colimits. That is:

$$\mathrm{Hom}(A, \lim(B_i)) = \lim\left(\mathrm{Hom}(A, B_i)\right) \qquad (\mathrm{colim}(A_i)) \otimes B = \mathrm{colim}(A_i \otimes B)$$

References: May (A Concise course in Algebraic Topology, Chapter 2.6), Aluffi (Algebra: Chapter 0, Chapter VIII.1)

1.3 Examples of limits and colimits

2016-03-18

Keywords: *limit, colimit, product, coproduct, kernel, cokernel, equalizer, coequalizer, pullback, pushout*

Let C be a category and $X, Y, Z \in \mathrm{Obj}(C)$. Choose I to be a category with $\mathcal{F} : I \to C$ a functor as described below. Then we may consider the limit and colimit of $\mathcal{F}$, noting that they may not always exist, as there may be no suitable natural transformation i or π.

Category I	*Image $\mathcal{F}(I)$*	*Limit*	*Colimit*
$\circlearrowleft\bullet \qquad \bullet\circlearrowright$	$\circlearrowleft X \qquad Y \circlearrowright$	Product of X and Y	Coproduct of X and Y
$\circlearrowleft\bullet \longrightarrow \bullet\circlearrowright$	$\circlearrowleft X \xrightarrow{f} Y \circlearrowright$	Kernel of f	Cokernel of f
$\circlearrowleft\bullet \rightrightarrows \bullet\circlearrowright$	$\circlearrowleft X \underset{g}{\overset{f}{\rightrightarrows}} Y \circlearrowright$	Equalizer of f and g	Coequalizer of f and g
$\circlearrowleft\bullet \longrightarrow \overset{\circlearrowleft}{\bullet} \longleftarrow \bullet\circlearrowright$	$\circlearrowleft Y \xrightarrow{f} \overset{\circlearrowleft}{X} \xleftarrow{g} Z \circlearrowright$	Pullback of f and g	-
$\circlearrowleft\bullet \longleftarrow \overset{\circlearrowleft}{\bullet} \longrightarrow \bullet\circlearrowright$	$\circlearrowleft Y \xleftarrow{f} \overset{\circlearrowleft}{X} \xrightarrow{g} Z \circlearrowright$	-	Pushout of f and g

The limit and colimit of the category I with two points and two arrows going between the points in opposite directions, namely

$$\circlearrowleft\bullet \rightleftarrows \bullet\circlearrowright \,,$$

are not interesting to consider. That is because as a category, it must satisfy compositions, so $f \circ g = \mathrm{id}$, which is a restrictive condition on f and g. We may define a new map $h : X \to X$ with $h = f \circ g$, but then more maps, such as $h \circ f$ and so on need to be defined, which complicate the situation.

References: Borceux (Handbook of Categorical Algebra I, Chapter 2)

1.4 Exactness and derived functors

2016-03-20

Keywords: *functor, exact functor, derived functor, projective, injective, free, resolution, tensor, hom, tor, ext*

Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence of objects in a category A. Let $\mathcal{F} : A \to B$ be a covariant functor.

Definition 1.4.1. The functor $\mathcal{F}$ is *right-exact* if $\mathcal{F}(X) \to \mathcal{F}(Y) \to \mathcal{F}(Z) \to 0$ is an exact sequence. It is *left-exact* if $0 \to \mathcal{F}(X) \to \mathcal{F}(Y) \to \mathcal{F}(Z)$ is an exact sequence. It is *exact* if it is both left- and right-exact.

Example 1.4.2. These are some examples of left- and right-exact functors:
· $\mathrm{Hom}_A(X, -)$ is covariant left-exact
· $\mathrm{Hom}_A(-, X)$ is contravariant left-exact
· $- \otimes_R X$ is covariant right-exact, for X a left R-module

Recall that $X \otimes_R Y$ is naturally isomorphic to $Y \otimes_R X$.

Definition 1.4.3. An object $X \in \mathrm{Obj}(A)$ is *projective* if $\mathrm{Hom}_A(X, -)$ is an exact functor. Similarly, X is *injective* if $\mathrm{Hom}_A(-, X)$ is an exact functor.

Recall that a *projective resolution* of an object X is a sequence of projective objects $\cdots \to P_2 \to P_1 \to P_0$ that may or may not terminate on the left. The homology of the sequence in degree 0 is X, and trivial in other degrees. Similarly, an *injective resolution* of X is a sequence of injective objects $I_0 \to I_1 \to I_2 \to \cdots$ that may or may not terminate on the right. The cohomology is also concentrated in degree 0, and is X there. A *free resolution* is a projective resolution where all the objects are free (whatever that means in the context).

These types of resolutions may not exist. A category "has enough injectives (projectives)" means we can always construct injective (projective) resolutions.

Definition 1.4.4. Let $\mathcal{F} : A \to B$ be a covariant right-exact functor and $\mathcal{G} : A \to B$ a covariant left-exact functor. Let $X \in \mathrm{Obj}(A)$ with $P_\bullet$ a projective resolution of X and $I_\bullet$ an injective resolution of X. The *ith left-derived functor* of $\mathcal{F}$ is $L_i\mathcal{F}(X) = H_i(\mathcal{F}(P_\bullet))$. The *ith right-derived functor* of $\mathcal{G}$ is $R^i\mathcal{G}(X) = H^i(\mathcal{G}(I_\bullet))$.

These objects of B are well-defined up to natural isomorphism. Note that $\mathcal{F}^{op} : A^{op} \to B^{op}$ is a contravariant right-exact functor. Moreover, if $\mathcal{F}$ was contravariant right-exact and $\mathcal{G}$ was contravariant left-exact, then $L_i\mathcal{F}(X) = H_i(\mathcal{F}(I_\bullet))$ and $R^i\mathcal{G}(X) = H^i(\mathcal{G}(P_\bullet))$.

Example 1.4.5. Let R be a ring with X and Y both R-bimodules. Then

$$\begin{aligned} \mathrm{Tor}_i^R(Y,X) &= L_i(- \otimes_R X)(Y) \\ &= L_i(Y \otimes_R -)(X), \end{aligned} \qquad \begin{aligned} \mathrm{Ext}_R^i(X,Y) &= R^i(\mathrm{Hom}_R(X,-))(Y) \\ &= R^i(\mathrm{Hom}_R(-,Y))(X). \end{aligned}$$

Recall that $\mathrm{Tor}_i^R(Y,X)$ is canonically isomorphic to $\mathrm{Tor}_i^R(X,Y)$, but it is not true for Ext. Also note that $\mathrm{Hom}_R(X,-)$ is covariant and $\mathrm{Hom}_R(-,Y)$ is contravariant, while $- \otimes_R X$ and $Y \otimes_R -$ are both covariant functors.

References: Weibel (An introduction to homological algebra, Chapter 2)

2 Geometry

2.1 The real and complex Jacobian

2016-09-03

Keywords: *Jacobian, determinant, complex, holomorphic*

Let $f : \mathbf{C}^n \to \mathbf{C}^n$ be a holomorphic function. We will show that the the real Jacobian is the square of the complex Jacobian. Write $f = (f_1, \ldots, f_n)$ with $f_i = u_i + \sqrt{-1}\, v_i$, where the u_i are functions of the $z_j = x_j + \sqrt{-1}\, y_j$. By the Cauchy–Riemann equations

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial v_i}{\partial y_j} \qquad \text{and} \qquad \frac{\partial u_i}{\partial y_j} = -\frac{\partial v_i}{\partial x_j}$$

and expanding, we have that

$$\begin{aligned}\frac{\partial f_i}{\partial z_j} &= \frac{1}{2}\left(\frac{\partial f_i}{\partial x_j} - \sqrt{-1}\,\frac{\partial f_i}{\partial y_j}\right)\\ &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \sqrt{-1}\,\frac{\partial v_i}{\partial x_j} - \sqrt{-1}\,\left(\frac{\partial u_i}{\partial y_j} + \sqrt{-1}\,\frac{\partial v_i}{\partial y_j}\right)\right)\\ &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial v_i}{\partial y_j} + \sqrt{-1}\,\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial u_i}{\partial y_j}\right)\right)\\ &= \frac{\partial u_i}{\partial x_j} + \sqrt{-1}\,\frac{\partial v_i}{\partial x_j}.\end{aligned}$$

The *complex Jacobian* of f is $J_{\mathbf{C}}f$ (or its determinant), with entries

$$(J_{\mathbf{C}}f)_{i,j} = \frac{\partial f_i}{\partial z_j},$$

and the *real Jacobian* of f is $J_{\mathbf{R}}f$ (or its determinant), with entries

$$\begin{aligned}\begin{bmatrix}(J_{\mathbf{R}}f)_{2i-1,2j-1} & (J_{\mathbf{R}}f)_{2i-1,2j}\\ (J_{\mathbf{R}}f)_{2i,2j-1} & (J_{\mathbf{R}}f)_{2i,2j}\end{bmatrix} &= \begin{bmatrix}\frac{\partial u_i}{\partial x_j} & \frac{\partial u_i}{\partial y_j}\\ \frac{\partial v_i}{\partial x_j} & \frac{\partial v_i}{\partial y_j}\end{bmatrix}\\ &\xrightarrow{R_{2i-1}+\sqrt{-1}\;R_{2i}\to R_{2i-1}} \begin{bmatrix}\frac{\partial f_i}{\partial z_j} & \sqrt{-1}\,\frac{\partial f}{\partial z}\\ \frac{\partial v_i}{\partial x_j} & \frac{\partial v_i}{\partial y_j}\end{bmatrix}\\ &\xrightarrow{C_{2j}-\sqrt{-1}\;C_{2j-i}\to C_{2j}} \begin{bmatrix}\frac{\partial f_i}{\partial z_j} & 0\\ \frac{\partial v_i}{\partial x_j} & \overline{\frac{\partial f_i}{\partial z_j}}\end{bmatrix},\end{aligned}$$

where the row and column operations have been performed for all rows $2i$ and all columns $2j$. Moving all the odd-indexed columns to the left and all odd-indexed rows to the top, we get that

$$J_{\mathbf{R}}f \simeq \begin{bmatrix}A & 0\\ * & B\end{bmatrix} \qquad \text{with} \qquad A_{i,j} = \frac{\partial f_i}{\partial z_j}, \quad B_{i,j} = \overline{\frac{\partial f_i}{\partial z_j}}.$$

Since the number of operations to switch the columns is the same as the number of operations to switch the rows, the sign of the determinant of $J_{\mathbf{R}}f$ will not change. That is,

$$\det(J_{\mathbf{R}}f) = \det(A)\det(B) = \det(J_{\mathbf{C}}f)\overline{\det(J_{\mathbf{C}}f)} = |\det(J_{\mathbf{C}}f)|^2.$$

2.2 Classical Lie groups

2016-09-05

Keywords: *Lie group, Lie algebra, symplectic*

Definition 2.2.1. A *Lie group* G is both a group and a manifold, with a smooth map $G \times G \to G$, given by $(g,h) \mapsto gh^{-1}$. The *Lie algebra* $\mathfrak{g}$ of G is the tangent space T_eG of G at the identity.

We distinguish between *real* and *complex* Lie groups by saying that the base manifold is either real or complex analytic, respectively.

Example 2.2.2. Here are some examples of classical Lie groups and their dimension:

$$\begin{array}{rcrcl} \text{general linear group} & n^2 & GL(n) & = & \{n \times n \text{ matrices with non-zero determinant}\} \\ \text{special linear group} & n^2-1 & SL(n) & = & \{M \in GL(n) \ : \ \det(M) = 1\} \\ \text{orthogonal group} & n(n-1)/2 & O(n) & = & \{M \in GL(n) \ : \ MM^t = I\} \\ \text{special orthogonal group} & n(n-1)/2 & SO(n) & = & \{M \in O(n) \ : \ \det(M) = 1\} \\ \text{unitary group} & n^2 & U(n) & = & \{M \in GL(n, \mathbf{C}) \ : \ MM^* = I\} \\ \text{special unitary group} & n^2-1 & SU(n) & = & \{M \in U(n) \ : \ \det(M) = 1\} \\ \text{symplectic group} & n(2n+1) & Sp(n) & = & \{n \times n \text{ matrices} : \ \omega(Mx, My) = \omega(x,y)\} \end{array}$$

For the symplectic group, the skew-symmetric bilinear form ω is defined as

$$\omega(x,y) = \sum_{i=1}^n x_i y_{i+n} - y_i x_{i+n} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} x \cdot y,$$

where $\cdot$ is the regular dot product (a symmetric bilinear form). Also note that the unitary group is a real Lie group - real because there is no holomorphic map $G \times G \to G$ as would be necessary, so we view the entries of a matrix in $U(n)$ in terms of its real and imaginary parts. Hence the dimension indicated above is real dimension.

References: Kirillov Jr (An introduction to Lie groups and Lie algebras, Chapter 2)

2.3 Degree and orientation

2016-09-28

Keywords: *degree, orientation, relative, excision, homology, cohomology, orientation, Stokes*

Topology

Recall that a *topological manifold* is a Hausdroff space in which every point has a neighborhood homeomorphic to $\mathbf{R}^n$ for some n. An *orientation* on M is a choice of basis of $\mathbf{R}^n$ in each neighborhood such that every path in M keeps the same orientation in each neighborhood. Every manifold $M \ni x$ appears in a long exact sequence (via relative homology) with three terms

$$H_n(M - \{x\}) \xrightarrow{f} H_n(M) \xrightarrow{g} H_n(M, M - \{x\}).$$

The first term is 0, because removing a point from an n-dimensional space leaves only its $(n-1)$-skeleton, which is at most $(n-1)$-dimensional. For U a neighborhood of x in M, the last term (via excision) is

$$H_n(M - U^c, M - \{x\} - U^c) = H_n(U, U - \{x\}) \cong H_n(\mathbf{R}^n, \mathbf{R}^n - \{x\}) \cong H_n(\mathbf{R}^n, S^{n-1}),$$

which in turn fits into a long exact sequence whose interesting part is

$$H_n(\mathbf{R}^n) \to H_n(\mathbf{R}^n, S^{n-1}) \to H_{n-1}(S^{n-1}) \to H_{n-1}(\mathbf{R}^n),$$

and since the first and last terms are zero, $H_n(M, M - \{x\}) = \mathbf{Z}$. Since f is zero, g into $\mathbf{Z}$ must be injective, meaning that $H_n(M) = \mathbf{Z}$ or 0.

Theorem 2.3.1. Let M be a connected compact (without boundary) n-manifold. Then

1. if M is orientable, g is an isomorphism for all $x \in M$, and

2. if M is not orientable, $g = 0$.

Definition 2.3.2. Let $f : M \to N$ be a map of connected, oriented n-manifolds. Since $H_n(M) = H_n(N)$ is infinite cyclic, the induced homomorphism $f_* : H_n(M) \to H_n(N)$ must be of the form $x \mapsto dx$. The number d is called the *degree* of f.

In the special case when we are computing the degree for a map $f : S^n \to S^n$, by excision we get

$$\deg(f) = \sum_{x_i \in f^{-1}(y)} \deg\left(H_n(U_i, U_i - x_i) \xrightarrow{f_*} H_n(V, V - y)\right),$$

for any $y \in Y$, some neighborhood V of y, and preimages U_i of V. This is called the *local degree* of f.

Geometry

Let M be a smooth n-manifold. Recall Ω^r_M is the space of r-forms on M and $d^r : \Omega^r_M \to \Omega^{r+1}_M$ is the differential map. Also recall the *de Rham cohomology groups* $H^r(M) = \ker(d^r)/\mathrm{im}(d^{r-1})$.

Definition 2.3.3. An n-manifold M is *orientable* if it has a nowhere-zero n-form $\omega \in \Omega^n_M$. A choice of ω is called an *orientation* of M.

We also have a map $H^n(M) \to \mathbf{R}$, given by $\alpha \mapsto \int_M \alpha$, where the integral is normalized by the volume of M, so that integrating 1 across M gives back 1. It is immediate that this doesn't make sense when M is not compact, but when M is compact and orientable, we get that $H^n(M) \neq 0$. Indeed, if $\eta \in \Omega^{n-1}_M$ with $d\eta = \omega$, by Stokes' theorem we have

$$\int_M \omega = \int_M d\eta = \int_{\partial M} \eta = \int_\emptyset \eta = 0,$$

as M has no boundary (since it is a manifold). But ω is nowhere-zero, meaning the first expression on the left cannot be zero. Hence ω is not exact and is a non-trivial element of $H^n(M)$.

Theorem 2.3.4. Let M be a smooth, compact, orientable manifold of dimension n. Then $H^n(M)$ is one-dimensional.

Proof: The above discussion demonstrates that $\dim(H^n(M)) \geqslant 1$. We can get an upper bound on the dimension by noting that the space of n-forms on M, given by $\Omega^n_M = \bigwedge^n (TM)^*$, has elements described by $dx_{i_1} \wedge \cdots \wedge dx_{i_n}$, with $\{i_1, \ldots, i_n\} \subset \{1, \ldots, n\}$. By rearranging the order of the dx_{i_j}, every element looks like $\alpha dx_1 \wedge \cdots \wedge dx_n$ for some real number α. Hence $\dim(\Omega^n_M) \leqslant 1$, so $\dim(H^n(M))$ is either 0 or 1. Therefore $\dim(H^n(M)) = 1$. ∎

Definition 2.3.5. Let $f : M \to N$ be a map of smooth, compact, oriented manifolds of dimension n. Since $H^n(M)$ and $H^n(N)$ are 1-dimensional, the induced map $f^* : H^n(N) \to H^n(M)$ must be of the form $x \mapsto dx$. The number d is called the *degree* of f. Equivalently, for any $\omega \in \Omega^n_N$,

$$\int_M f^*\omega = d \int_N \omega$$

References: Hatcher (Algebraic Topology, Chapters 2, 3.3), Lee (Introduction to Smooth Manifolds, Chapter 17)

2.4 The tangent space and differentials

2016-09-29

Keywords: *manifold, tangent space, differential, pushforward, derivation, cotangent, tangent, derivative*

Let M, N be smooth n-manifolds. Here we discuss different definitions of the tangent space and differentials, or pushforwards, of smooth maps $f : M \to N$.

Derivations (Lee)

Definition 2.4.1. A *derivation* of M at $p \in M$ is a linear map $v : C^\infty(M) \to \mathbf{R}$ such that for all $f, g \in C^\infty(M)$,

$$v(fg) = f(p)v(g) + g(p)v(f).$$

The *tangent space* T_pM to M at p is the set of all derivations of M at p.

Given a smooth map $F : M \to N$ and $p \in M$, define the *differential* $dF_p : T_pM \to T_{f(p)}N$, which, for $v \in T_pM$ and $f \in C^\infty(N)$ acts as

$$dF_p(v)(f) = v(f \circ F) \in \mathbf{R}.$$

Dual of cotangent (Hitchin)

Definition 2.4.2. Let $Z_p \subset C^\infty(M)$ be the functions whose derivative vanishes at $p \in M$. The *cotangent space* T_p^*M to M at P is the quotient space $C^\infty(M)/Z_p$. The *tangent space* to M at P is the dual of the cotangent space $T_pM = (T_p^*M)^* = \mathrm{Hom}(T_p^*M, \mathbf{R})$.

Given a smooth map $F : M \to N$ and $p \in M$, define the *differential*

$$\begin{array}{rcl} dF_p \ : \ T_pM & \to & T_{F(p)}N, \\ (f : C^\infty(M)/Z_p \to \mathbf{R}) & \mapsto & \left(\begin{array}{rcl} g \ : \ C^\infty(N)/Z_{F(p)} & \to & \mathbf{R}, \\ h & \mapsto & f(h \circ F). \end{array} \right) \end{array}$$

This definition makes clear the relation to the first approach. Since $h \notin Z_{F(p)}$, the derivative of h does not vanish at $F(p)$. Hence the derivative of $h \circ F$ at p, which is the derivative of h at $F(p)$ multiplied by the derivative of F at p, does not *a priori* vanish at p.

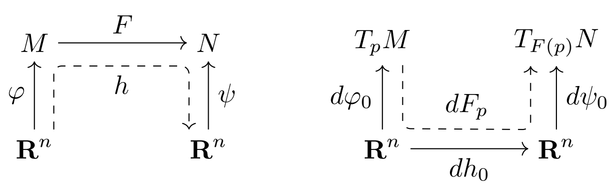
Derivative of chart map (Guillemin and Pollack)

Definition 2.4.3. Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be a smooth map. Then the *derivative* of f at $x \in \mathbf{R}^n$ in the direction $y \in \mathbf{R}^n$ is defined as

$$df_x(y) = \lim_{h \to 0} \left[\frac{f(x + yh) - f(x)}{h} \right].$$

Given $x \in M$ and charts $\varphi : \mathbf{R}^n \to M \subset \mathbf{R}^m$, the *tangent space* to M at p is the image $T_pM = d\varphi_0(\mathbf{R}^n)$, where we assume $\varphi(0) = p$.

Given a smooth map $F : M \to N$ and charts $\varphi : \mathbf{R}^n \to M$, $\psi : \mathbf{R}^n \to N$, with $\varphi(0) = p$ and $\psi(0) = F(p)$, define the *differential* $dF_p : T_pM \to T_{F(p)}N$ via the diagrams below.



Here $h = \psi^{-1} \circ F \circ \varphi$, so dh_0 is well-defined. Hence $dF_p = d\psi_0 \circ dh_0 \circ d\varphi_0^{-1}$ is also well-defined.

Sometimes the differential is referred to as the *pushforward*, in which case it is denoted by $(F_*)_p$.

References: Lee (Introduction to Smooth Manifolds, Chapter 3), Hitchin (Differentiable manifolds, Chapter 3.2), Guillemin and Pollack (Differential topology, Chapter 1.2)

2.5 Vector fields

2016-10-10

Keywords: *vector field, integral curve, flow, Lie derivative, Lie bracket, interior product, differential forms*

Here we will have an overview of vector fields and all things related to them. Let M be an n-dimensional manifold, and $\pi : M \to TM$ its tangent bundle.

Definition 2.5.1. A *vector field* is a map $X : M \to TM$ such that $\pi \circ X = \mathrm{id}_M$.

A vector field may also be viewed as a section of the tangent bundle, and smooth vector fields as the space of smooth sections $\Gamma(TM)$. Given a chart (U, φ) of M near p, we have the pushforward $\varphi_* : T_pM \to T_{\varphi(p)}(\mathbf{R}^n) = \mathbf{R}^n$, where we may assume $\varphi(p) = 0$. Given the standard basis $\{e_i\}$ of $\mathbf{R}^n$, we get a basis of T_pM given by

$$\left\{ \left.\frac{\partial}{\partial x_i}\right|_p = (\varphi_*)^{-1}(e_i) \right\}_{i=1}^n .$$

Recall that TM may be viewed as the space of derivations, or maps $C^\infty(M) \to \mathbf{R}$ satisfying the Leibniz rule. Then for $p \in M$, we have $X(p) : C^\infty(M) \to \mathbf{R}$, so we have $X(p)(f) = X_p(f) \in \mathbf{R}$ for all $f \in C^\infty(M)$. Hence $X_p \in T_pM$, and $X(f) \in C^\infty(M)$. Briefly,

$$\begin{array}{rcccl} f & : & M & \to & \mathbf{R}, \\ X & : & M & \to & TM, \end{array} \qquad \begin{array}{rcccl} Xf & : & M & \to & \mathbf{R}, \\ fX & : & M & \to & TM. \end{array}$$

Definition 2.5.2. Given a vector field $X \in \Gamma(TM)$, an *integral curve* of X is a smooth curve $\gamma : \mathbf{R} \to M$ such that $\gamma'(t) = X_{\gamma(t)}$ for all $t \in \mathbf{R}$.

The domain of γ need not be all of $\mathbf{R}$, though any integral curve may be extended to a *maximal* integral curve, for which the domain can not be made larger. A collection of integral curves for a particular vector field is a *flow*.

Definition 2.5.3. A *flow*, or a *one paramater group of diffeomorphisms*, is a smooth map $\psi : \mathbf{R} \times M \to M$ such that

1. $\psi(t, \cdot)$ is a diffeomorphism of M, for all t,
2. $\psi(0, \cdot) = \mathrm{id}_M$,
3. $\psi(s + t, \cdot) = \psi(s, \cdot) \circ \psi(t, \cdot)$.

For convenience, we write $\psi_t(p) = \psi(t, p)$, Note that fixing $p \in M$, the map $\psi(\cdot, p)$ is a integral curve. Moreover, flows and vector fields are related uniquely by

$$\left.\frac{df}{dt}\psi_t(p)\right|_{t=0} = X_p(f).$$

Indeed, if we have a flow ψ and an element $f \in \mathrm{Hom}(T_p^*M, \mathbf{R})$, this gives us a vector field $X \in \Gamma(TM)$. Conversely, if we have a vector field X, by the existence and uniqueness of solutions to first order ordinary differential equations (with boundary conditions), we can find a ψ that satisfies this equality.

Definition 2.5.4. Let $X, Y \in \Gamma(TM)$ and ψ be the associated flow of X. The *Lie derivative* of Y in the direction of X, or *Lie bracket* of X and Y, is an element of $\Gamma(TM)$ given by

$$\begin{aligned} (\mathcal{L}_X Y)_p(f) &= \left.\frac{df}{dt}\right|_{t=0} \left((\psi_t)_*^{-1}(Y_{\psi_t(p)}(f)) \right) \\ &= [X, Y]_p(f) \\ &= X_p(Y(f)) - Y_p(X(f)) \end{aligned}$$

The Lie derivative has some properties, among them $\mathcal{L}_X(fY) = X(fY) + f(\mathcal{L}_X Y)$ for any $f \in C^\infty(M)$. If we let Y be the map $M \to TM$ given by

$$\begin{array}{rcl} Y : M & \to & \mathrm{Hom}(T^*M, \mathbf{R}), \\ p & \mapsto & \left(\begin{array}{rcl} f_p : C^\infty(M) & \to & \mathbf{R}, \\ g & \mapsto & g(p), \end{array} \right), \end{array}$$

then $Yf = f$, so $\mathcal{L}_X Y = X - X = 0$, and we have $\mathcal{L}_X f = Xf$.

Remark 2.5.5. Vector fields are 1-forms, or elements of $\mathcal{A}^0_M(TM) = \Gamma(TM \otimes \bigwedge^0 T^*M) = \Gamma(TM)$. We may generalize the definition above to consider the Lie derivative $\mathcal{L}_X\omega$ of a differential k-form ω . Note that a differential k-form takes in k vector fields and gives back a smooth function $M \to \mathbf{R}$. With this in mind, we may define new operations on vector fields:

$$\begin{aligned}
(\mathcal{L}_X\omega)(Y_1,\dots,Y_k) &= \mathcal{L}_X(\omega(Y_1,\dots,Y_k)) - \sum_{i=1}^{k}\omega(Y_1,\dots,\mathcal{L}_XY_i,\dots,Y_k)\\
(d\omega)(Y_1,\dots,Y_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i-1}Y_i(\omega(Y_1,..,\widehat{Y_i},..,Y_{k+1})) + \sum_{j>i=1}^{k+1}(-1)^{i+j}\omega([Y_i,Y_j],Y_1,..,\widehat{Y_i},..,\widehat{Y_j},..,Y_{k+1})\\
(i_X\omega)(Y_1,\dots,Y_{k-1}) &= \omega(X,Y_1,\dots,Y_{k-1})
\end{aligned}$$

The last is the *interior product*. All three are related by *Cartan's formula* $\mathcal{L}_X\omega = d(i_X\omega) + i_X(d\omega)$:

$$\begin{aligned}
(\mathcal{L}_{Y_1}\omega)(Y_2,\dots,Y_{k+1}) &= Y_1(\omega(Y_2,\dots,Y_{k+1})) - \sum_{i=2}^{k+1}\omega(Y_2,\dots,[Y_1,Y_i],\dots,Y_k)\\
&= Y_1(\omega(Y_2,\dots,Y_{k+1})) - \sum_{i=2}^{k+1}(-1)^i\omega([Y_1,Y_i],Y_2,\dots,\widehat{Y_i},\dots,Y_k)\\
(d(i_{Y_1}\omega))(Y_2,\dots,Y_{k+1}) &= \sum_{i=2}^{k+1}(-1)^iY_i(\omega(Y_1,..,\widehat{Y_i},..,Y_{k+1})) - \sum_{j>i=2}^{k+1}(-1)^{i+j}\omega([Y_i,Y_j],Y_1,..,\widehat{Y_i},..,\widehat{Y_j},..,Y_{k+1})\\
(i_{Y_1}(d\omega))(Y_2,\dots,Y_{k+1}) &= (d\omega)(Y_1,\dots,Y_{k+1})\\
&= \sum_{i=1}^{k+1}(-1)^{i-1}Y_i(\omega(Y_1,..,\widehat{Y_i},..,Y_{k+1})) + \sum_{j>i=1}^{k+1}(-1)^{i+j}\omega([Y_i,Y_j],Y_1,..,\widehat{Y_i},..,\widehat{Y_j},..,Y_{k+1})
\end{aligned}$$

Remark 2.5.6. The action of a k-differential form on a k-vector field is given by

$$(dx_1\wedge\cdots\wedge dx_k)\left(\frac{\partial}{\partial y_1},\dots,\frac{\partial}{\partial y_p}\right) = \det\begin{bmatrix} dx_1\frac{\partial}{\partial y_1} & dx_1\frac{\partial}{\partial y_2} & \cdots & dx_1\frac{\partial}{\partial y_p}\\ dx_2\frac{\partial}{\partial y_1} & dx_2\frac{\partial}{\partial y_2} & \cdots & dx_2\frac{\partial}{\partial y_p}\\ \vdots & \vdots & \ddots & \vdots\\ dx_p\frac{\partial}{\partial y_1} & dx_p\frac{\partial}{\partial y_2} & \cdots & dx_p\frac{\partial}{\partial y_p}\end{bmatrix} \quad = \quad \det\left(dx_i\frac{\partial}{\partial y_j}\right).$$

This may be generalized to get a map $\wedge^kT^*M \oplus \Gamma(TM)^{\oplus\ell} \to \bigwedge^{k-\ell}T^*M$, for $\ell \leqslant k$. For example, given a basis x, y of our space M,

$$(dx\wedge dy)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) = dx\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)dy - dy\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)dx = x\ dy - y\ dx.$$

When $\ell = 1$, this is just the interior product.

References: Lee (Introduction to smooth manifolds, Chapter 8), Hitchin (Differentiable manifolds, Chapter 3)

2.6 Explicit pushforwards and pullbacks

2016-11-01

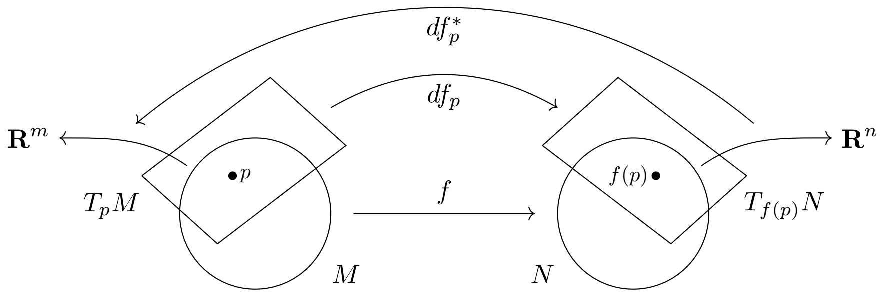
Keywords: *tangent space, cotangent space, differential, pushforward, pullback*

Here we consider a map $f : M \to N$ between manifolds of dimension m and n, respectively, and the maps that it induces. Let $p \in M$ with $x_1,\dots,x_m$ a local chart for $U \ni p$ and $y_1,\dots,y_n$ a local chart for $V \ni f(p)$. Induced from

f are the *differential* (or *pushforward*) df and the *pullback* df^*, which are duals of each other:

$$\begin{aligned} df_p &: T_pM \to T_{f(p)}N & \qquad df_p^* &: T^*_{f(p)}N \to T^*_pM \\ df &: TM \to TN & df^* &: T^*N \to T^*M \\ \alpha &\mapsto (\beta \mapsto \alpha(\beta\circ f)) & \omega &\mapsto \omega\circ f \\ & & \textstyle\bigwedge^k T^*N &\to \textstyle\bigwedge^k T^*M \\ & & \omega\, dy_1\wedge\cdots\wedge dy_k &\mapsto (\omega\circ f)\ d(y_1\circ f)\wedge\cdots\wedge d(y_k\circ f) \end{aligned}$$

These maps may be described by the diagram below.



Example 2.6.1. For example, consider the map $f : \mathbf{R}^3 \to \mathbf{R}^3$ given by $f(x, y, z) = (x - y, 3z^2, xz + yz)$, with the image having coordinates (u, v, w). With elements

$$2x\frac{\partial}{\partial x} - 5z\frac{\partial}{\partial y} \in TM, \qquad 2uv + \sqrt{w} - 5 \in C^\infty(N), \qquad \cos(uv) \in T^*N,$$

we have

$$\begin{aligned} df_p\left(2x\frac{\partial}{\partial x} - 5z\frac{\partial}{\partial y}\right)(2uv+\sqrt{w}-5) &= \left(2x\frac{\partial}{\partial x} - 5z\frac{\partial}{\partial y}\right)\left(6(x-y)z^2 + \sqrt{xz+yz} - 5\right)(p), \\ df_p^*(\cos(uv)) &= \cos((x-y)3z^2), \\ \left(\textstyle\bigwedge^2 df_p^*\right)(\cos(uv)du\wedge dw) &= \cos((x-y)3z^2)d(3z^2)\wedge d(xz+yz) \\ &= \cos((x-y)3z^2)\left(-6z^2\ dx\wedge dz - 6z^2\ dy\wedge dz\right). \end{aligned}$$

2.7 Images of manifolds and transversality

2016-11-07

Keywords: *immersion, embedding, transversality, regular value, Sard, preimage theorem*

Let X, Y be manifolds embedded in $\mathbf{R}^n$, and $f : X \to Y$ a map, with $df_x : T_xX \to T_{f(x)}Y$ the induced map on tangent spaces.

Definition 2.7.1. The map f is a

- *homeomorphism* if it is continuous and has a continuous inverse,
- *diffeomorphism* if it is smooth and has a smooth inverse,
- *injection* if $f(a) = f(b)$ implies $a = b$,
- *immersion* if df_x is injective for all $x \in X$,
- *embedding* if it is an immersion and df_x is a homeomorphism onto its image,
- *submersion* if df_x is surjective for all $x \in X$.

Transversality is a mathematical relic whose only practical use is, perhaps, in classical algebraic geometry.

Definition 2.7.2. The manifolds X and Y are *transverse* if $T_pX \oplus T_pY \cong \mathbf{R}^n$ for every $p \in X \cap Y$. The map f and Y are *transverse* if $\mathrm{im}(f)$ and Y are transverse.

Note that being transverse (or transversal) is a symmetric, but not a reflexive, nor a transitive relation. Recall that a *regular value* of f is $y \in Y$ such that $df_x : T_xX \to T_{f(x)}Y$ is surjective for all $x \in f^{-1}(y)$. If y is not in the image of f, then $f^{-1}(y)$ is empty, so y is trivially a regular value. Every value that is not a regular value is a *critical value.*

Theorem 2.7.3. [Preimage theorem]
For every regular value y of f, the subset $f^{-1}(y) \subset X$ is a submanifold of X of dimension $\dim(X) - \dim(Y)$.

Now let M be a submanifold of Y.

Corollary 2.7.4. If f is transverse to M, then $f^{-1}(M)$ is a manifold, with $\text{codim}_Y(M) = \text{codim}_X(f^{-1}(M))$.

Theorem 2.7.5. [Transversality theorem]
Let $\{g_s : X \to Y \mid s \in S\}$ be a smooth family of maps. If $g : X \times S \to Y$ is transverse to M, then for almost every $s \in S$ the map g_s is transverse to M.

If we replace f with df, and ask that it be transverse to M, then $df|_s$ is also transverse to M.

Example 2.7.6. Consider the map $g_s : X \to \mathbf{R}^n$ given by $g_s(X) = i(X) + s = X + s$, where i is the embedding of X into $\mathbf{R}^n$. Since $g(X \times \mathbf{R}^n) = \mathbf{R}^n$ and g varies smoothly in both variables, we have that g is transverse to X. Hence by the transversality theorem, X is transverse to its translates $X + s$ for almost all $s \in \mathbf{R}^n$.

Theorem 2.7.7. [Sard]
For f smooth and $\partial Y = \emptyset$, almost every $y \in Y$ is a regular value of f and $f|_{\partial X}$. Equivalently, the set of critical values of f has measure zero.

Resources: Guillemin and Pollack (Differential topology, Chapters 1, 2), Lee (Introduction to smooth manifolds, Chapter 6)

2.8 Differential 1-forms are closed if and only if they are exact

2016-11-10

Keywords: *differential forms, paths, integration*

The title refers to 1-forms in Euclidean n-space $\mathbf{R}^n$, for $n \geqslant 2$. This theorem is instructive to do in the case $n = 2$, but we present it in general. We will use several facts, most importantly that the integral of a function $f : X \to Y$ over a curve $\gamma : [a, b] \to X$ is given by

$$\int_\gamma f \ dx_1 \wedge \cdots \wedge dx_k = \int_a^b (f \circ \gamma) \ d(x_1 \circ \gamma) \wedge \cdots \wedge d(x_n \circ \gamma),$$

where $x_1, \dots, x_n$ is some local frame on X. We will also use the fundamental theorem of calculus and one of its consequences, namely

$$\int_a^b \frac{\partial f}{\partial t}(t) \ dt = f(b) - f(a).$$

Theorem 2.8.1. A 1-form on $\mathbf{R}^n$ is closed if and only if it is exact, for $n \geqslant 2$.

Proof: Let $\omega = a_1dx_1 + \cdots a_ndx_n \in \Omega^1_{\mathbf{R}^n}$ be a 1-form on $\mathbf{R}^n$. If there exists $\eta \in \Omega^0_{\mathbf{R}^n}$ such that $d\eta = \omega$, then $d\omega = d^2\eta = 0$, so the reverse direction is clear. For the forward direction, since ω is closed, we have

$$0 = d\omega = \sum_{i=1}^n \frac{\partial a_1}{\partial x_i} dx_i \wedge dx_1 + \cdots + \sum_{i=1}^n \frac{\partial a_n}{\partial x_n} dx_i \wedge dx_n \qquad \Longrightarrow \qquad \frac{\partial a_i}{\partial x_j} = \frac{\partial a_j}{\partial x_i} \ \forall \ i \neq j.$$

Now fix some $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathbf{R}^n$, and define $f \in \Omega^0_{\mathbf{R}^n}$ by

$$f(\mathbf{x}_1, \dots, \mathbf{x}_n) = \int_{\gamma(\mathbf{x}_1, \dots, \mathbf{x}_n)} \omega,$$

for γ the composition of the paths

$$\begin{array}{rcl} \gamma_1 : [0,\mathbf{x}_1] & \to & \mathbf{R}^n, \\ t & \mapsto & (t,0,\dots,0), \end{array} \qquad \begin{array}{rcl} \gamma_2 : [0,\mathbf{x}_2] & \to & \mathbf{R}^n, \\ t & \mapsto & (\mathbf{x}_1,t,0,\dots,0), \end{array} \quad \cdots \quad \begin{array}{rcl} \gamma_n : [0,\mathbf{x}_n] & \to & \mathbf{R}^n, \\ t & \mapsto & (\mathbf{x}_1,\dots,\mathbf{x}_{n-1},t). \end{array}$$

By applying the definition of a pullback and the change of variables formula (use $s = \gamma_i(t)$ for every i),

$$\begin{aligned}
\int_{\gamma(\mathbf{x}_1,\dots,\mathbf{x}_n)} \omega &= \sum_{i=1}^{n} \int_{\gamma_i} a_1 dx_1 + \dots + \sum_{i=1}^{n} \int_{\gamma_i} a_n dx_n \\
&= \sum_{i=1}^{n} \int_{\gamma_i} a_1(x_1,\dots,x_n)\ dx_1 + \dots + \sum_{i=1}^{n} \int_{\gamma_i} a_n(x_1,\dots,x_n)\ dx_n \\
&= \sum_{i=1}^{n} \int_0^{\mathbf{x}_i} a_1(\gamma_i(t))\ d(x_1 \circ \gamma_i)(t) + \dots + \sum_{i=1}^{n} \int_0^{\mathbf{x}_i} a_n(\gamma_i(t))\ d(x_n \circ \gamma_i)(t) \\
&= \int_0^{\mathbf{x}_1} a_1(\gamma_1(t))\gamma_1'(t)\ dt + \dots + \int_0^{\mathbf{x}_n} a_n(\gamma_n(t))\gamma_n'(t)\ dt \\
&= \int_{(0,\dots,0)}^{(\mathbf{x}_1,0,\dots,0)} a_1(s)\ ds + \dots + \int_{(\mathbf{x}_1,\dots,\mathbf{x}_{n-1},0)}^{(\mathbf{x}_1,\dots,\mathbf{x}_n)} a_n(s)\ ds \\
&= \int_0^{\mathbf{x}_1} a_1(s,0,\dots,0)\ ds + \dots + \int_0^{\mathbf{x}_n} a_n(\mathbf{x}_1,\dots,\mathbf{x}_{n-1},s)\ ds.
\end{aligned}$$

To take the derivative of this, we consider the partial derivatives first. In the last variable, we have

$$\frac{\partial f}{\partial \mathbf{x}_n} = \frac{\partial}{\partial \mathbf{x}_n} \int_0^{\mathbf{x}_n} a_n(\mathbf{x}_1,\dots,\mathbf{x}_{n-1},s)\ ds = a_n(\mathbf{x}_1,\dots,\mathbf{x}_n) = a_n.$$

In the second-last variable, applying one of the identities from ω being closed, we have

$$\begin{aligned}
\frac{\partial f}{\partial \mathbf{x}_{n-1}} &= \frac{\partial}{\partial \mathbf{x}_{n-1}} \int_0^{\mathbf{x}_{n-1}} a_{n-1}(\mathbf{x}_1,\dots,\mathbf{x}_{n-2},s,0)\ ds + \frac{\partial}{\partial \mathbf{x}_{n-1}} \int_0^{\mathbf{x}_n} a_n(\mathbf{x}_1,\dots,\mathbf{x}_{n-1},s)\ ds \\
&= a_{n-1}(\mathbf{x}_1,\dots,\mathbf{x}_{n-1},0) + \int_0^{\mathbf{x}_n} \frac{\partial a_n}{\partial \mathbf{x}_{n-1}}(\mathbf{x}_1,\dots,\mathbf{x}_{n-1},s)\ ds \\
&= a_{n-1}(\mathbf{x}_1,\dots,\mathbf{x}_{n-1},0) + \int_0^{\mathbf{x}_n} \frac{\partial a_{n-1}}{\partial s}(\mathbf{x}_1,\dots,\mathbf{x}_{n-1},s)\ ds \\
&= a_{n-1}(\mathbf{x}_1,\dots,\mathbf{x}_{n-1},0) + a_{n-1}(\mathbf{x}_1,\dots,\mathbf{x}_n) - a_{n-1}(\mathbf{x}_1,\dots,\mathbf{x}_{n-1},0) \\
&= a_{n-1}(\mathbf{x}_1,\dots,\mathbf{x}_n) \\
&= a_{n-1}.
\end{aligned}$$

This pattern continues. For the other variables we have telescoping sums, and we compute the partial derivative in the first variable as an example:

$$\begin{aligned}
\frac{\partial f}{\partial \mathbf{x}_1} &= \frac{\partial}{\partial \mathbf{x}_1} \int_0^{\mathbf{x}_1} a_1(s,0,\dots,0)\ ds + \sum_{i=2}^{n} \frac{\partial}{\partial \mathbf{x}_1} \int_0^{\mathbf{x}_i} a_i(\mathbf{x}_1,\dots,\mathbf{x}_{i-1},s,0,\dots,0)\ ds \\
&= a_1(\mathbf{x}_1,0,\dots,0) + \sum_{i=2}^{n} \int_0^{\mathbf{x}_i} \frac{\partial a_i}{\partial \mathbf{x}_1}(\mathbf{x}_1,\dots,\mathbf{x}_{i-1},s,0,\dots,0)\ ds \\
&= a_1(\mathbf{x}_1,0,\dots,0) + \sum_{i=2}^{n} \int_0^{\mathbf{x}_i} \frac{\partial a_1}{\partial s}(\mathbf{x}_1,\dots,\mathbf{x}_{i-1},s,0,\dots,0)\ ds \\
&= a_1(\mathbf{x}_1,0,\dots,0) + \sum_{i=2}^{n} \left(a_1(\mathbf{x}_1,\dots,\mathbf{x}_i,0,\dots,0) - a_1(\mathbf{x}_1,\dots,\mathbf{x}_{i-1},0,\dots,0)\right) \\
&= a_1(\mathbf{x}_1,\dots,\mathbf{x}_n) \\
&= a_1.
\end{aligned}$$

Hence we get that

$$df = \frac{\partial f}{\partial x_1}dx_1 + \cdots + \frac{\partial f}{\partial x_n}dx_n = a_1dx_1 + \cdots + a_ndx_n = \omega,$$

so ω is exact. ■

References: Lee (Introduction to smooth manifolds, Chapter 11)

2.9 Loose ends of smooth manifolds

2016-11-18

Keywords: *inverse function theorem, Stokes theorem, classification, manifold, orientation, tangent space*

Here we round up some theorems that have escaped previous roundings-up. Let X, Y be smooth manifolds and $f : X \to Y$ a smooth map.

Theorem 2.9.1. [Inverse function theorem]
If df_p is invertible for some $p \in M$, then there exist $U \ni p$ and $V \ni f(p)$ connected such that $f|_U : U \to V$ is a diffeomorphism.

Corollary 2.9.2. [Stack of records theorem]
If $\dim(X) = \dim(Y)$, then every regular value $y \in Y$ has a neighborhood $V \ni y$ such that $f^{-1}(Y) = U_1 \sqcup \cdots \sqcup U_k$, where $f|_{U_i} : U_i \to V$ is a diffeomorphism.

Proof: Since $y \in Y$ is a regular value, df_x is surjective for all $x \in f^{-1}(y)$. Since $\dim(X) = \dim(Y)$ and df_x is linear, df_x is an isomorphism, hence invertible. By the inverse function theorem, there exist $U \ni x$ and $V \ni y$ connected such that $f|_U : U \to V$ is a diffeomorphism. Before we actually apply this, we need to show that $f^{-1}(y)$ is a finite set.

First we note that by the preimage theorem, since y is a regular value, $f^{-1}(y)$ is a submanifold of X of dimension $\dim(X) - \dim(Y) = 0$. Next, if $f^{-1}(y) = \{x_i\}$ were infinite, since X is compact, there would be some limit point $p \in X$ of $\{x_i\}$. But then by continuity,

$$y = \lim_{i\to\infty} [f(x_i)] = f\left(\lim_{i\to\infty} [x_i]\right) = f(p),$$

so $p \in f^{-1}(y)$. But then either p cannot be separated from other elements of $f^{-1}(y)$, meaning $f^{-1}(y)$ is not a manifold, or the sequence $\{x_i\}$ is finite in length. Hence $f^{-1}(y) = \{x_1, \dots, x_k\}$. Let $U_i \ni x_i$ and $V_i \ni y$ be the sets asserted to exist by the inverse function theorem (the U_i may be assumed to be disjoint without loss of generality). Let $V = \bigcap_{i=1}^k V_i$ and $U_i' = f^{-1}(V) \cap U_i$, for which we still have $f|_{U_i'} : U_i' \to V$ a diffeomorphism. ■

Theorem 2.9.3. [Classification of manifolds]
Up to diffeomorphism,

- the only 0-dimensional manifolds are collections of points,
- the only 1-dimensional manifolds are S^1 and $\mathbf{R}$, and
- the only 2-dimensional compact manifolds are $S^2\#(T^2)^{\#n}$ or $S^2\#(\mathbf{RP}^2)^{\#n}$, for any $n \geqslant 0$.

Compact 2-manifolds are homeomorphic iff they are both (non)-orientable and have the same Euler characteristic. Note that

$$\chi\left(S^2\#(T^2)^{\#n}\right) = 2 - 2n, \qquad \chi\left(S^2\#(\mathbf{RP}^2)^{\#n}\right) = 2 - n.$$

These surfaces are called *orientable* (on the left) and *non-orientable* (on the right) *surfaces of genus n*.

Theorem 2.9.4. [Stokes' theorem]
For X oriented and $\omega \in \Omega_X^{n-1}$, $\int_X d\omega = \int_{\partial X} \omega$.

Theorem 2.9.5. The tangent bundle TX is always orientable.

Proof: Let $U, V \subset X$ with $\varphi : U \to \mathbf{R}^n$ and $\psi : V \to \mathbf{R}^n$ trivializing maps, and $\psi \circ \varphi^{-1} : \mathbf{R}^n \to \mathbf{R}^n$ the transition function. To show that TX is always orientable, we need to show the Jacobian of the induced transition function (determinant of the derivative) on TX is always non-negative. On TU and TV, we have trivializing maps $(\varphi, d\varphi)$ and $(\psi, d\psi)$, giving a transition function

$$(\psi \circ \varphi^{-1}, d\psi \circ d\varphi^{-1}) = (\psi \circ \varphi^{-1}, d(\psi \circ \varphi^{-1})).$$

The Jacobian of this is

$$\det(d(\psi \circ \varphi^{-1}, d(\psi \circ \varphi^{-1}))) = \det(d(\psi \circ \varphi^{-1}), d(\psi \circ \varphi^{-1}))) = \det(d(\psi \circ \varphi^{-1})) \cdot \det(d(\psi \circ \varphi^{-1})) \geqslant 0,$$

and since $d(\psi \circ \varphi^{-1}) \neq 0$ (as $\psi \circ \varphi^{-1}$ is a diffeomorphism, its derivative is an isomorphism), the result is always positive. ■

References: Lee (Introduction to smooth manifolds, Chapter 4), Guillemin and Pollack (Differential topology, Chapter 1)

3 Topology

3.1 Complexes and their homology

2016-09-16

Keywords: *simplex, simplicial complex, delta complex, cell, cell complex, CW complex, homology*

Here I'll present complexes from the most restrictive to the most general. Recall the standard *n-simplex* is

$$\Delta^n = \{x \in \mathbf{R}^{n+1} \ : \ \sum x_i = 1, x_i \geqslant 0\}.$$

Definition 3.1.1. Let V be a finite set. A *simplicial complex* X on V is a set of distinct subsets of V such that if $\sigma \in X$, then all the subsets of σ are in X.

Every n-simplex in a simplicial complex is uniquely determined by its vertices, hence no pair of lower dimensional faces of a simplex may be identified with each other.

Definition 3.1.2. Let A, B be two indexing sets. A Δ-*complex* (or *delta complex*) X is

$$X = \bigsqcup_{\alpha \in A} \Delta_\alpha^{n_\alpha} \Big/ \left\{\mathcal{F}_\beta^{k_\beta}\right\}_{\beta \in B}, \qquad \mathcal{F}_\beta^{k_\beta} = \{\Delta_1^{k_\beta}, \dots, \Delta_{m_\beta}^{k_\beta}\},$$

such that if σ appears in the disjoint union, all of its lower dimensional faces also appear. The identification of the k-simplices in $\mathcal{F}^k$ is done in the natural (linear) way, and restricting to identified faces gives the identification of the $\mathcal{F}^{k-1}$ where the faces appear.

To define *simplicial homology* of a simplicial or Δ-complex X, fix an ordering of the set of 0-simplices (which gives an ordering of every $\sigma \in X$), define C_k to be the free abelian group generated by all $\sigma \in X$ of dimension k (defined by $k+1$ 0-simplices), and define a *boundary map*

$$\begin{array}{rcl} \partial_k \ : \ C_k & \to & C_{k-1}, \\ [v_0, \dots, v_k] & \mapsto & \sum_{i=0}^k (-1)^i [v_0, \dots, \widehat{v_i}, \dots, v_k]. \end{array}$$

Then $H_k(X) := \ker(\partial_k)/\mathrm{im}(\partial_{k+1})$.

Recall the standard *n-cell* is $e^n = \{x \in \mathbf{R}^n \ : \ |x| \leqslant 1\}$, also known as the n-disk or n-ball.

Definition 3.1.3. Let X_0 be a finite set. A *cell complex* (or *CW complex*) is a collection $X_0, X_1, \dots$ where

$$X_k := X_{k-1} \bigsqcup_{\alpha \in A_k} e_\alpha^k \Big/ \left\{\partial e_\alpha^k \sim f_{k,\alpha}(\partial e_\alpha^k)\right\}_{\alpha \in A_k},$$

where the $f_{k,\alpha}$ describe how to attach k-cells to the $(k-1)$-skeleton X_{k-1}, for $k \geqslant 1$. X_k may also be described by pushing out $e^k \sqcup_{\partial e^k} X_{k-1}$. Note that $\partial e^k = S^{k-1}$, the $(k-1)$-sphere.

To define *cellular homology*, we need more tools (relative homology and excision) that require a blog post of their own.

References: Hatcher (Algebraic topology, Chapter 2.1)

3.2 Tools of (co)homology

2016-10-13

Keywords: *homology, reduced homology, relative homology, excision, local homology, Mayer–Vietoris, universal coefficient theorem, Kunneth formula, Poincaré duality, Alexander duality*

Let X, Y be topological spaces, G a group, and R a unital commutative ring.

Defining homology groups

Theorem 3.2.1. If (X, A) is a good pair (there exists a neighborhood $U \subset X$ of A such that U deformation retracts onto A), then for $i : A \hookrightarrow X$ the inclusion and $q : X \twoheadrightarrow X/A$ the quotient maps, there exists a long exact sequence of reduced homology groups

$$\cdots \to \widetilde{H}_n(A) \xrightarrow{i_*} \widetilde{H}_n(X) \xrightarrow{q_*} \widetilde{H}_n(X/A) \to \cdots .$$

Theorem 3.2.2. For any pair (X, A), there exists a long exact sequence of homology groups

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X, A) \to \cdots ,$$

where the last is called a *relative homology* group. Hence $H_n(X, A) \cong \widetilde{H}_n(X/A)$ for a good pair (X, A).

Theorem 3.2.3. [Excision]
For any triple of spaces (Z, A, X) with $\text{cl}(Z) \subset \text{int}(A)$, there is an isomorphism $H_n(X - Z, A - Z) \cong H_n(X, A)$.

For any $x \in X$, the *local homology* of X at x is the relative homology groups $H_n(X, X - \{x\})$. By excision, these are isomorphic to $H_n(U, U - \{x\})$ for U any neighborhood of x. If X is nice enough around x (that is, if $U \cong \mathbf{R}^k$), then these groups are isomorphic to $H_n(\mathbf{R}^k, \mathbf{R}^k - \{x\}) \cong H_n(D^k, \partial D^k) = H_n(S^k)$.

Theorem 3.2.4. [Mayer–Vietoris] For $X = A \cup B$, there is a long exact sequence of homology groups

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to \cdots ,$$

and if $A \cap B$ is non-empty, there is an analogous sequence for reduced homology groups.

Extending with coefficients

Recall the Tor and Ext groups, which were, respectively, the left and right derived functors of, respectively, $\otimes$ and Hom (see post "Exactness and derived functors," 2016-03-20). Here we only need Tor_1 and Ext^1, which are given by, for any groups (that is, $\mathbf{Z}$-modules) A, B,

$$\begin{aligned} \text{Tor}(A, B) &= H_1(\text{projres}(A) \otimes B) &= H_1(A \otimes \text{projres}(B)), \\ \text{Ext}(A, B) &= H^1(\text{Hom}(A, \text{injres}(B))) &= H^1(\text{Hom}(\text{projres}(A), B)). \end{aligned}$$

Note that Tor is symmetric in its arguments, while Ext is not. Recall that $\text{Tor}_0(A, B) = A \otimes B$ and $\text{Ext}^0(A, B) = \text{Hom}(A, B)$.

Theorem 3.2.5. [Universal coefficient theorem]
There exist isomorphisms

$$\begin{aligned} H_n(X; G) &\cong \text{Hom}(H^n(X), G) \oplus \text{Ext}(H^{n+1}(X), G) &\cong H_n(X) \otimes G \oplus \text{Tor}(H_{n-1}(X), G), \\ H^n(X; G) &\cong \text{Hom}(H_n(X), G) \oplus \text{Ext}(H_{n-1}(X), G) &\cong H^n(X) \otimes G \oplus \text{Tor}(H^{n+1}(X), G). \end{aligned}$$

Here are some common Hom, Tor, and Ext groups:

$$\begin{array}{ccc} \text{Hom}(\mathbf{Z}, G) = G & \text{Tor}(\mathbf{Z}, G) = 0 & \text{Ext}(\mathbf{Z}, G) = 0 \\ \text{Hom}(\mathbf{Z}_m, \mathbf{Z}) = 0 & \text{Tor}(G, \mathbf{Z}) = 0 & \text{Ext}(\mathbf{Z}_m, \mathbf{Z}) = \mathbf{Z}_m \\ \text{Hom}(\mathbf{Z}_m, \mathbf{Z}_n) = \mathbf{Z}_{\gcd(m,n)} & \text{Tor}(\mathbf{Z}_m, \mathbf{Z}_n) = \mathbf{Z}_{\gcd(m,n)} & \text{Ext}(\mathbf{Z}_m, \mathbf{Z}_n) = \mathbf{Z}_{\gcd(m,n)} \\ \text{Hom}(\mathbf{Q}, \mathbf{Z}_n) = 0 & & \text{Ext}(\mathbf{Q}, \mathbf{Z}_n) = 0 \\ \text{Hom}(\mathbf{Q}, \mathbf{Q}) = \mathbf{Q} & & \text{Ext}(G, \mathbf{Q}) = 0 \end{array}$$

Theorem 3.2.6. [Künneth formula]
For X, Y CW-complexes, F a field, and $H^k(Y; G)$ or $H^k(X; G)$ finitely generated for all k, there are isomorphisms, for all k,

$$H_k(X \times Y; F) \cong \bigoplus_{i+j=k} H_i(X; F) \otimes_F H_j(Y; F), \qquad H^k(X \times Y; G) \cong \bigoplus_{i+j=k} H^i(X; G) \otimes_G H^j(Y; G)$$

Dualities

Theorem 3.2.7. [POINCARÉ DUALITY]
For X a closed n-manifold (compact, without boundary) that is R-orientable (consistent choice of R-generator for each local homology group), for $k = 0, \dots, n$ there are isomorphisms

$$H^k(X;R) \cong H_{n-k}(X;R).$$

Note that a simply *orientable* manifold means $\mathbf{Z}$-orientable. A manifold that is not $\mathbf{Z}$-orientable is always $\mathbf{Z}_2$-orientable (in fact all manifolds are $\mathbf{Z}_2$-orientable).

Theorem 3.2.8. [ALEXANDER DUALITY]
For $X \subsetneq S^n$ a non-empty closed locally contractible subset, for $k = 0, \dots, n-1$ there are isomorphisms

$$\widetilde{H}^k(X) \cong \widetilde{H}_{n-k-1}(S^n - X).$$

References: Hatcher (Algebraic topology, Chapters 2, 3), Aguilar, Gitler, and Prieto (Algebraic Topology from a Homotopical Viewpoint, Chapter 7)

3.3 Basic topological constructions

2016-10-25

Keywords: *cone, suspension, wedge, smash, join, homology*

Let X, Y be topological spaces based at x_0, y_0, respectively, and $I = [0,1]$ the unit interval.

$$\begin{array}{rrcl}
\text{cone} & CX & = & X \times I / X \times \{0\} \\
\text{suspension} & \Sigma X & = & X \times I / X \times \{0\}, X \times \{1\} \\
\text{reduced suspension} & \widetilde{\Sigma} X & = & X \times I / X \times \{0\}, X \times \{0\}, \{x_0\} \times I \\
\text{wedge} & X \vee Y & = & X \sqcup Y / \{x_0\} \sim \{y_0\} \\
\text{smash} & X \wedge Y & = & X \times Y / X \times \{y_0\}, \{x_0\} \times Y \\
\text{join} & X * Y & = & X \times Y \times I \Big/ \begin{matrix} X \times \{y\} \times \{0\} & \forall\, y \in Y \\ \{x\} \times Y \times \{1\} & \forall\, x \in X \end{matrix} \\
\text{connected sum} & X \# Y & = & (X \setminus D^n_X) \sqcup (Y \setminus D^n_Y) / \partial D^n_X \sim \partial D^n_Y
\end{array}$$

In the last description, X and Y are assumed to be n-manifolds, with D^n_X a closed n-dimensional disk in X (similarly for Y). The quotient identification may also be made via some non-trivial map. In fact, only the interior of each n-disk is removed from X and Y, so that the quotient makes sense.

Remark 3.3.1. Some of the above constructions may be expressed in terms of others, for example

$$X \wedge Y = X \times Y / X \vee Y, \qquad X * Y = \Sigma(X \wedge Y).$$

The first is clear by viewing $X = X \times \{y_0\}$ and $Y = \{x_0\} \times Y$ as sitting inside $X \times Y$. The second is clear by letting $X \times \{y\} \times \{0\}$ be identified to $\{x_0\} \times \{y\} \times \{0\}$ for every $y \in Y$, and analogously with Y.

Example 3.3.2. Here are some of the constructions above applied to some common spaces.

$$\begin{array}{ccc}
CX \simeq \text{pt} & \Sigma S^n = S^{n+1} & S^n \wedge S^m = S^{n+m} \\
\Sigma X = S^1 \wedge X & S^n * S^m = S^{n+m+1} &
\end{array}$$

Remark 3.3.3. We may also calculate the homology of the new spaces in terms of the old ones.

$$\begin{array}{rcll}
\widetilde{H}_k(CX) & = & 0 & \text{via homotopy} \\
\widetilde{H}_k(\Sigma X) & = & \widetilde{H}_{k-1}(X) & \text{via Mayer–Vietoris} \\
\widetilde{H}_k(X \vee Y) & = & \widetilde{H}_k(X) \oplus \widetilde{H}_k(Y) & \text{via Mayer–Vietoris} \\
\widetilde{H}_k(X \wedge S^\ell) & = & \widetilde{H}_{k-\ell}(X) & \text{via Künneth} \\
\widetilde{H}_k(X \# Y) & = & \widetilde{H}_k(X) \oplus \widetilde{H}_k(Y) & \text{via Mayer–Vietoris and relative homology}
\end{array}$$

The last equality holds for $k < n-1$, for M and N both n-manifolds, and for $k = n-1$ when at least one of them is orientable.

References: Hatcher (Algebraic Topology, Chapters 0, 2)

3.4 Tools of homotopy

2016-11-04

Keywords: *connectedness, homotopy, good pair, homotopy extension property, fundamental group, free group, Borsuk–Ulam, ham sandwich, van Kampen*

Let X, Y be topological spaces and A a subspace of X. Recall that a *path* in X is a continuous map $\gamma : I \to X$, and it is *closed* (or a *loop*), if $\gamma(0) = \gamma(1)$. When X is pointed at x_0, we often require $\gamma(0) = x_0$, and call such paths (and similarly loops) *based*.

Definitions

Definition 3.4.1.
- X is *connected* if it is not the union of two disjoint nonempty open sets.
- X is *path connected* if any two points in X have a path connecting them, or equivalently, if $\pi_0(X) = 0$.
- X is *simply connected* if every loop is contractible, or equivalently, if $\pi_1(X) = 0$.
- X is *semi-locally simply connected* if every point has a neighborhood whose inclusion into X is π_1-trivial.

Path connectedness and simply connectedness have *local* variants. That is, for P either of those properties, a space is *locally* P if for every point x and every neighborhood $U \ni x$, there is a subset $V \subset U$ on which P is satisfied.

Remark 3.4.2. In general, X is *n-connected* whenever $\pi_r(X) = 0$ for all $r \leqslant n$. Note that 0-connected is path connected and 1-connected is simply connected and connected. Also observe that the suspension of path connected space is simply connected.

Definition 3.4.3.
- A *retraction* (or *retract*) from X to A is a map $r : X \to A$ such that $r|_A = \mathrm{id}_A$.
- A *deformation retraction* (or *deformation retract*) from X to A is a family of maps $f_t : X \to X$ continuous in t, X such that $f_0 = \mathrm{id}_X$, $f_1(X) = A$, and $f_t|_A = \mathrm{id}_A$ for all t.
- A *homotopy* from X to Y is a family of maps $f_t : X \to Y$ continuous in t, X.
- A *homotopy equivalence* from X to Y is a map $f : X \to Y$ and a map $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

Definition 3.4.4. A pair (X, A), where $A \subset X$ is a closed subspace, is a *good pair*, or has the *homotopy extension property (HEP)*, if any of the following equivalent properties hold:

1. there exists a neighborhood $U \subset X$ of A such that U deformation retracts onto A,
2. $X \times \{0\} \cup A \times I$ is a retract of $X \times I$, or
3. the inclusion $i : A \hookrightarrow X$ is a cofibration.

In some texts such a pair (X, A) is called a *neighborhood deformation retract pair*, and HEP is reserved for any map $A \to X$, not necessarily the inclusion, that is a cofibration. For more on cofibrations, see a previous blog post (2016-07-31, "(Co)fibrations, suspensions, and loop spaces").

Definition 3.4.5. There is a functor $\pi_1 : \mathrm{Top}_* \to \mathrm{Grp}$ called the *fundamental group*, that takes a pointed topological space X to the space of all pointed loops on X, modulo path homotopy.

This may be generalized to π_n, which takes X to the space of all pointed embeddings of S^n.

Definition 3.4.6. Let G, H be groups. The *free product* of G and H is the group

$$G * H = \{a_1 \cdots a_n \ : \ n \in \mathbf{Z}_{\geqslant 0}, a_i \in G \text{ or } H, a_i \in G(H) \implies a_{i+1} \in H(G)\},$$

with group operation concatenation, and identity element the empty string $\emptyset$. We also assume $e_G e_H = e_H e_G = e_G = e_H = \emptyset$, for e_G (e_H) the identity element of G (H).

The above construction may be generalized to a collection of groups $G_1 * \cdots * G_m$, where the index may be uncountable. If every $G_\alpha = \mathbf{Z}$ (equivalently, has one generator), then $*_{\alpha \in A} G_\alpha$ is called the *free group on $|A|$ generators*.

Theorems

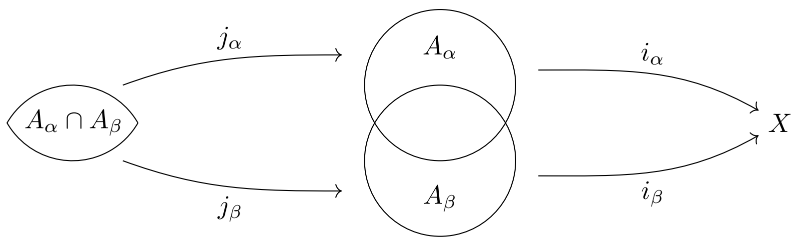
Theorem 3.4.7. [Borsuk–Ulam]
Every continuous map $S^n \to \mathbf{R}^n$ takes a pair of antipodal points to the same value.

Theorem 3.4.8. [Ham Sandwich theorem]
Let $U_1, \dots, U_n$ be bounded open sets in $\mathbf{R}^n$. There exists a hyperplane in $\mathbf{R}^n$ that divides each of the open sets U_i into two sets of equal volume.

Volume is taken to be Lebsegue measure. The Ham sandwich theorem is an application of Borsuk–Ulam (see Terry Tao's blog post for more).

Theorem 3.4.9. If X and Y are path-connected, then $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.

Now suppose that $X = \bigcup_\alpha A_\alpha$ is based at x_0 with $x_0 \in A_\alpha$ for all α. There are natural inclusions $i_\alpha : A_\alpha \to X$ as well as $j_\alpha : A_\alpha \cap A_\beta \to A_\alpha$ and $j_\beta : A_\alpha \cap A_\beta \to A_\beta$.



Both i_α and j_α induce maps on the fundamental group, each (and all) of the $i_{\alpha *} : \pi_1(A_\alpha) \to \pi_1(X)$ extending to a map $\Phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$.

Theorem 3.4.10. [van Kampen]
1. If $A_\alpha \cap A_\beta$ is path-connected, then Φ is a surjection.
2. If $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected, then $\ker(\Phi) = \langle j_{\alpha *}(g)(j_{\beta *}(g))^{-1} \mid g \in \pi_1(A_\alpha \cap A_\beta, x_0)\rangle$.

As a consequence, if triple intersections are path connected, then $\pi_1(X) \cong *_\alpha A_\alpha / \ker(\Phi)$. Moreover, if all double intersections are contractible, then $\ker(\Phi) = 0$ and $\pi_1(X) \cong *_\alpha A_\alpha$.

Proposition 3.4.11. If $\pi_1(X) = 0$ and $\widetilde{H}_n(X) = 0$ for all n, then X is contractible.

References: Hatcher (Algebraic topology, Chapter 1), Tao (blog post "The Kakeya conjecture and the Ham Sandwich theorem")

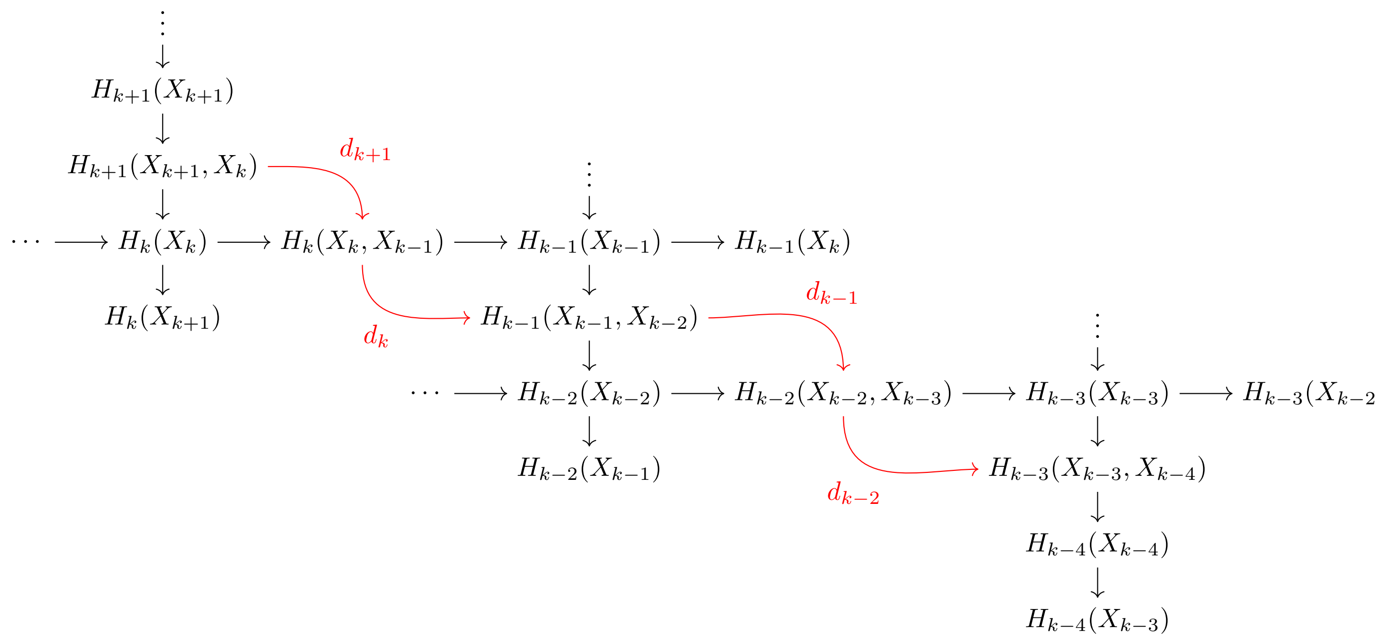
3.5 More (co)homological constructions

2016-11-08
Keywords: *CW complex, homology, cellular homology, chain, cochain, cup product, cap product, cohomology*

Recall a previous post (2016-09-16, "Complexes and their homology") that focused on constructing topological spaces in different ways and recovering the homology. Here we complete that task, introducing cellular homology. Recall a *cell complex* (or *CW complex*) X was a sequence of skeleta X_k for $k = 0, \dots, \dim(X)$ consisting of k-cells e_i^k and their attaching maps to the $(k-1)$-skeleton.

Cellular homology

Definition 3.5.1. The long exact sequence in relative homology for the pair X_k, X_{k-1} shares terms with the long exact sequence for the pair X_{k+1}, X_k, as well as X_{k-1}, X_{k-2}. By letting d_k be the composition of maps in different long exact sequences, for $k > 1$, that make the diagram

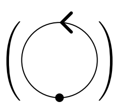
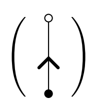


commute, we get a complex of equivalence classes of chains

$$\cdots \to H_{k+1}(X_{k+1}, X_k) \xrightarrow{d_{k+1}} H_k(X_k, X_{k-1}) \xrightarrow{d_k} H_{k-1}(X_{k-1}, X_{k-2}) \to \cdots \to H_1(X_1, X_0) \xrightarrow{d_1} H_0(X_0) \xrightarrow{d_0} 0,$$

whose homology $H_k^{CW}(X) = \ker(d_k)/\mathrm{im}(d_{k-1})$ is called the *cellular homology* of X. The map d_1 is the connecting map in the long exact sequence of the pair X_1, X_0, and $d_0 = 0$.

This seems quite a roundabout way of defining homology groups, but it turns out to be very useful. Note that for $k = 1$, the map d_1 is the same as for a simplicial complex, hence

d_1  $= \bullet - \bullet = 0,$ $\qquad$ d_1  $= \circ - \bullet.$

Theorem 3.5.2. In the context above,

1. for $k \geqslant 0$, $H_k^{CW}(X) \cong H_k(X)$;
2. for $k \geqslant 1$, $H_k(X_k, X_{k-1}) = \mathbf{Z}^\ell$, where ℓ is the number of k-cells in X; and
3. for $k \geqslant 2$, $d_k(e_i^k) = \sum_j \deg(\underbrace{\partial e_i^k}_{S^{k-1}} \xrightarrow{f_{k,i}} X_{k-1} \xrightarrow{\pi} \underbrace{X_{k-1}/X_{k-1} - e_j^{k-1}}_{S^{k-1}}) e_j^{k-1}$.

Example 3.5.3. Real projective space $\mathbf{RP}^n$ has a cell decomposition with one cell in each dimension, and 2-to-1 attaching maps $\partial(e_k) = 2X_{k-1}$ for $k > 1$. This gives us a construction

$$X_0 = e_0, \qquad X_1 = e_1 \bigsqcup_{\partial(e_1)=e_0} X_0, \qquad X_2 = e_2 \bigsqcup_{\partial(e_2)=2e_1} X_1, \qquad X_3 = e_3 \bigsqcup_{\partial(e_3)=2e_2} X_2, \ldots$$

It is immediate that $d_0 = d_1 = 0$, and for higher degrees, we have

$$d_k(e^k) = \deg(S^{k-1} \to \mathbf{RP}^{k-1} \to S^{k-1}) e^{k-1}.$$

Since this is a map between spheres, we may apply local degree calculations. The first part is the 2-to-1 cover, where every point in $\mathbf{RP}^{k-1}$ is covered by two points from S^{k-1}, one in each hemisphere. One covers it via the identity, the other via the antipodal map. As long as we choose a point not in $\mathbf{RP}^{k-2} \subset \mathbf{RP}^{k-1}$, the second step doesn't affect these degree calculations. The antipodal map $S^{k-1} \to S^{k-1}$ has degree $(-1)^k$, hence for a the antipodal map, the composition has degree

$$\deg(S^{k-1} \to \mathbf{RP}^{k-1} \to S^{k-1}) = \deg(\mathrm{id}_{S^{k-1}}) + \deg(a_{S^{k-1}}) = 1 + (-1)^k = \begin{cases} 2 & k \text{ even}, \\ 0 & k \text{ odd}. \end{cases}$$

Products in (co)homology

Recall that an *n-chain* on X is a map $\sigma : \Delta^n \to X$, where $\Delta^n = [v_0, \ldots, v_n]$ is an n-simplex. These form the group C_n of n-chains. An *n-cochain* is an element of $C^n = \mathrm{Hom}(C_n, \mathbf{Z})$, though the coefficient group does not need to be $\mathbf{Z}$ necessarily.

Definition 3.5.4. The diagonal map $X \to X \times X$ induces a map on cohomology $H^*(X \times X) \to H^*(X)$, and by Künneth, this gives a map $H^*(X) \otimes H^*(X) \to H^*(X)$, and is called the *cup product*.

For $a \in H^p(X)$ and $b \in H^q(X)$, representatives of the class a are in $\mathrm{Hom}(C_p, \mathbf{Z})$ and representatives of the class b are in $\mathrm{Hom}(C_q, \mathbf{Z})$, though we will conflate the notation for the class with that of a representative. Hence for a $(p+q)$-chain σ the cup product of a and b acts as

$$(a \smile b)\sigma = a\left(\sigma|_{[v_0,\ldots,v_p]}\right) \cdot b\left(\sigma|_{[v_p,\ldots,v_{p+q}]}\right).$$

Definition 3.5.5. The *cap product* combines p-cochains with q-chains to give $(q-p)$-chains, by

$$\begin{array}{rcl} \frown : H^p(X) \times H_q(X) & \to & H_{q-p}(X), \\ (a, \sigma) & \mapsto & a\left(\sigma|_{[v_0,\ldots,v_p]}\right) \cdot \sigma|_{[v_p,\ldots,v_q]}. \end{array}$$

The cap product with the orientation form of an orientable manifold X gives the isomorphism of Poincaré duality.

Remark 3.5.6. Given a map $f : X \to Y$, the cup and cap products satisfy certain identities via the induced map on cohomology groups. Let $a, b \in H^*(Y)$ and $c \in H_*(X)$ be cochain and chain classes, for which

$$f^*(a \smile b) = f^*(a) \smile f^*(b), \qquad a \frown f_* c = f_*(f^* a \frown c).$$

The first identity asserts that f^* is a ring homomorphism and the second describes the commutativity of an appropriate diagram. The cup and cap products are related by the equation

$$a(b \frown \sigma) = (a \smile b)\sigma,$$

for $a \in H^p$, $b \in H^q$ and $\sigma \in C_{p+q}$.

References: Hatcher (Algebraic topology, Chapter 2.2), Prasolov (Elements of homology theory, Chapter 2)

3.6 Covering spaces

2016-11-13

Keywords: *covering space, universal cover, normal cover, lift, fundamental group, deck transformation*

Let X, Y be topological spaces.

Definition 3.6.1. A space $\widetilde{X}$ and a map $p : \widetilde{X} \to X$ are called a *covering space* of X if either of two equivalent conditions hold:

1. There is a cover $\{U_\alpha\}_{\alpha \in A}$ of X such that $p^{-1}(U_\alpha) \cong \bigsqcup_{\beta \in B_\alpha} U_\beta$.

2. Every point $x \in X$ has a neighborhood $U \subset X$ such that $p^{-1}(U) \cong \bigsqcup_{\beta \in B} U_\beta$.

We also demand that every U_β is carried homeomorphically onto U_α (or U) by p, and the U_α (or U) are called *evenly covered*.

Some definitions require that p be surjective. A *universal cover* of X is a covering space that is universal with respect to this property, in that it covers all other covering spaces. Moreover, a cover that is simply connected is immediately a universal cover.

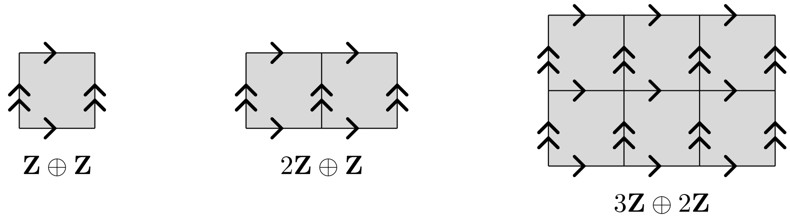
Remark 3.6.2. Every path connected (PC), locally path connected (LPC), and semi locally simply connected (SLSC) space has a universal cover.

Theorem 3.6.3. [Lifting criterion]
Let Y be PC and LPC, and $\widetilde{X}$ a covering space for X. A map $f : Y \to X$ lifts to a map $\widetilde{f} : Y \to \widetilde{X}$ iff $f_*(\pi_1(Y)) \subset p_*(\pi_1(\widetilde{X}))$.

Further, if the initial map f_0 in a homotopy $f_t : Y \to X$ lifts to $\widetilde{f_0} : Y \to \widetilde{X}$, then f_t lifts uniquely to $\widetilde{X}$. This is called the *homotopy lifting property*. Next, we will see that path connected covers of X may be classified via a correspondence through the fundamental group.

Theorem 3.6.4. Let X be PC, LPC, and SLSC. There is a bijection (up to isomorphism) between PC covers $p : \widetilde{X} \to X$ and subgroups of $\pi_1(X)$, described by $p_*(\pi_1(\widetilde{X}))$.

Example 3.6.5. Let $X = T^2$, the torus, with fundamental group $\mathbf{Z} \oplus \mathbf{Z}$. Below are some covering spaces of $p : \widetilde{X} \to X$ with the corresponding subgroups $p_*(\pi_1(\mathbf{Z} \oplus \mathbf{Z}))$.



Definition 3.6.6. Given a covering space $p : \widetilde{X} \to X$, an isomorphism g of $\widetilde{X}$ for which $\mathrm{id}_X \circ p = p \circ g$, is called a *deck transformation*, the collection of which form a group $G(\widetilde{X})$ under composition. Further, $\widetilde{X}$ is called *normal* (or *regular*) if for every $x \in X$ and every $\widetilde{x}_1, \widetilde{x}_2 \in p^{-1}(x)$, there exists $g \in G(\widetilde{X})$ such that $g(\widetilde{x}_1) = \widetilde{x}_2$.

For path connected covering spaces over path connected and locally path connected bases, being normal is equivalent to $p_*(\pi_1(\widetilde{X})) \leqslant \pi_1(X)$ being normal. In this case, $G(\widetilde{X}) \cong \pi_1(X)/p_*(\pi_1(\widetilde{X}))$. This simplifies even more for $\widetilde{X}$ a universal cover, as $\pi_1(\widetilde{X}) = 0$ then.

Theorem 3.6.7. Let G be a group, and suppose that every $x \in X$ has a neighborhood $U \ni x$ such that $g(U) \cap h(U) = \emptyset$ whenever $g \neq h \in G$. Then:
- The quotient map $q : X \to X/G$ describes a normal cover of X/G.
- If X is PC, then $G = G(X)$.

A group action satisfying the hypothesis of the previous theorem is called a *covering space action*.

Proposition 3.6.8. For any n-sheeted covering space $\widetilde{X} \to X$ of a finite CW complex, $\chi(\widetilde{X}) = n\chi(X)$.

References: Hatcher (Algebraic Topology, Chapter 1)

3.7 Čech (co)homology

2017-05-28

Keywords: *Čech, Leray, sheaf, cosheaf, cover, nerve, simplicial complex*

In this post we briefly recall the construction of Čech cohomology as well as compute a few examples. Let X be a topological space with a cover $\mathcal{U} = \{U_i\}$, $\mathcal{F}$ a C-valued sheaf on X, and $\widehat{\mathcal{F}}$ a C-valued cosheaf on X, for some category C (usually abelian groups).

Definition 3.7.1. The *nerve* N of $\mathcal{U}$ is the simplicial complex that has an r-simplex ρ for every non-empty intersection of $r+1$ opens of $\mathcal{U}$. The *support* U_ρ of ρ is this non-empty intersection. The *r-skeleton* N_r of N is the collection of all r-simplices.

Remark 3.7.2. The sheaf $\mathcal{F}$ and cosheaf $\widehat{\mathcal{F}}$ may be viewed as being defined either on the opens of $\mathcal{U}$ over X, or on the nerve N of $\mathcal{U}$. Indeed, the inclusion map $V \hookrightarrow U$ on opens is given by the forgetful map ∂. That is, $\partial_i : N_r \to N_{r-1}$ forgets the ith open defining $\rho \in N_r$, so if $U_\rho = U_0 \cap \cdots \cap U_r$, then $U_{\partial_0 \rho} = U_1 \cap \cdots \cap U_r$.

The Čech (co)homology will be defined as the (co)homology of a particular complex, whose boundary maps will be induced by, equivalently, the inclusion map on opens or ∂_i on simplices.

Definition 3.7.3. In the context above:
- a *p-chain* is a finite formal sum of elements $a_{\sigma_i} \in \widehat{\mathcal{F}}(U_{\sigma_i})$, for every σ_i a p-simplex,
- a *q-cochain* is a finite formal sum of elements $b_{\tau_j} \in \mathcal{F}(U_{\tau_j})$, for every τ_j a q-simplex,
- the *p-differential* is the map $d_p : \check{C}_p(\mathcal{U}, \mathcal{F}) \to \check{C}_{p-1}(\mathcal{U}, \mathcal{F})$ given by

$$d_p(a_\sigma) = \sum_{i=0}^{p} (-1)^i \widehat{\mathcal{F}}(\partial_i)(a_\sigma),$$

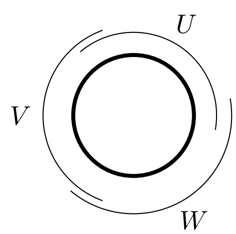
- the *q-codifferential* is the map $\delta^q : \check{C}^q(\mathcal{U}, \mathcal{F}) \to \check{C}^{q+1}(\mathcal{U}, \mathcal{F})$ given by

$$\delta^q(b_\tau) = \sum_{j=0}^{q+1} (-1)^j \mathcal{F}(\partial_j)(b_\tau).$$

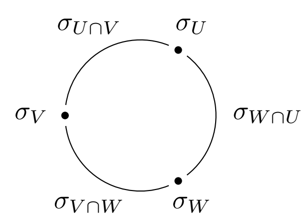
The collection of p-chains form a group $\check{C}_p(\mathcal{U}, \mathcal{F})$ and the collection of q-cochains also form a group $\check{C}^q(\mathcal{U}, \mathcal{F})$, both under the respective group operation in each coordinate. The *Čech homology* $H_*(\mathcal{U}, \mathcal{F})$ is the homology of the chain complex of $\check{C}_p$ groups, and the *Čech cohomology* $H^*(\mathcal{U}, \mathcal{F})$ is the cohomology of the cochain complex of $\check{C}^q$ groups.

Example 3.7.4. Let $X = S^1$ with a cover $\mathcal{U} = \{U, V, W\}$ and associated nerve $N_\mathcal{U}$ as below.

$X \subseteq \mathcal{U}$:



$N_\mathcal{U}$:



The cover is chosen so that all intersections are contractible. Let k be a field. Let $\widehat{\mathcal{F}}$ be a cosheaf over N and $\mathcal{F}$ a sheaf over N, with $\widehat{\mathcal{F}}(\text{0-cell}) = \mathcal{F}(\text{1-cell}) = (1,1) \in k^2$ and $\widehat{\mathcal{F}}(\text{1-cell}) = \mathcal{F}(\text{0-cell}) = 1 \in k$, so that the natural extension and restriction maps work. Then all the degree 0 and 1 chain and cochain groups are k^3. Giving a counter-clockwise orientation to X, we easily see that

$$\begin{aligned} d_1\sigma_{U\cap V} &= \sigma_V - \sigma_U, & \delta^0\sigma_U &= \sigma_{U\cap V} - \sigma_{W\cap U},\\ d_1\sigma_{V\cap W} &= \sigma_W - \sigma_V, & \delta^0\sigma_V &= \sigma_{V\cap W} - \sigma_{U\cap V},\\ d_1\sigma_{W\cap U} &= \sigma_U - \sigma_W, & \delta^0\sigma_W &= \sigma_{W\cap U} - \sigma_{V\cap W}. \end{aligned}$$

If we give an ordered basis of $(\sigma_{U\cap V}, \sigma_{V\cap W}, \sigma_{W\cap U})$ to $\check{C}_1(\mathcal{U}, \widehat{\mathcal{F}})$ and $\check{C}^1(\mathcal{U}, \mathcal{F})$, and $(\sigma_U, \sigma_V, \sigma_W)$ to $\check{C}_0(\mathcal{U}, \widehat{\mathcal{F}})$ and $\check{C}^0(\mathcal{U}, \mathcal{F})$, we find that

$$d_1 = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \delta^0 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The Čech chain and cochain complexes are then

$$0 \to \check{C}_1(\mathcal{U}, \widehat{\mathcal{F}}) \xrightarrow{d_1} \check{C}_0(\mathcal{U}, \widehat{\mathcal{F}}) \to 0, \qquad 0 \to \check{C}^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta^0} \check{C}^1(\mathcal{U}, \mathcal{F}) \to 0,$$

for which

$$\begin{aligned} H_1(\mathcal{U}, \widehat{\mathcal{F}}) &= \ker(d_1) = k, & H^0(\mathcal{U}, \mathcal{F}) &= \ker(\delta^0) = k,\\ H_0(\mathcal{U}, \widehat{\mathcal{F}}) &= k^3/\text{im}(d_1) = k^3/k^2 = k, & H^1(\mathcal{U}, \mathcal{F}) &= k^3/\text{im}(\delta^0) = k^3/k^2 = k. \end{aligned}$$

By the Čech–de Rham theorem, we know that the (co)homology groups should agree with the usual groups for S^1, as $\mathcal{U}$ was a good cover, which they do. Next we compute another example with a view towards persistent homology.

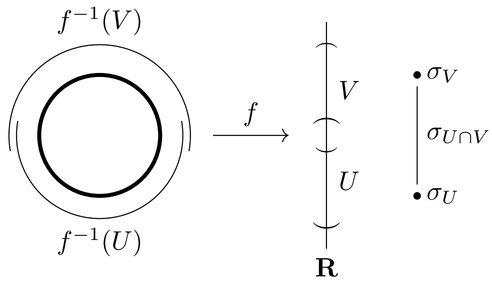
Definition 3.7.5. Let X be a topological space and $f : X \to Y$ a map with $\mathcal{U}$ covering $f(X)$. The *Leray sheaf* L^i of degree i over $N_\mathcal{U}$ is defined by $L^i(\sigma) = H^i(f^{-1}(U_\sigma))$ and $L^i(\sigma \hookrightarrow \tau) = H^i(f^{-1}(U_\tau) \hookrightarrow f^{-1}(U_\sigma))$, whenever σ is a face of τ.

Theorem 3.7.6. [Curry, Theorem 8.2.21]
In the context above, if $N_\mathcal{U}$ is at most 1-dimensional, then for any $t \in \mathbf{R}$,

$$H^i(f^{-1}(-\infty, t]) \cong H^0((-\infty, t], L^i) \oplus H^1((-\infty, t], L^{i-1}).$$

The idea is to apply this theorem in a filtration, for different values of t, but in the example below we will have t large enough so that $X \subset f^{-1}(-\infty, t]$.

Example 3.7.7. Let $f : S^1 \to \mathbf{R}$ be a projection map, and let $X = f(S^1)$ with a cover $\mathcal{U} = \{U, V\}$ as below.



Note that although $f^{-1}(U) \cap f^{-1}(V)$ is not contractible, $U \cap V$ is, and the Čech cohomology will be over $\mathcal{U} \subset \mathbf{R}$, so we are fine in applying the Čech–de Rham theorem. It is immediate that the only non-zero Leray sheaves are L^0, for which

$$L^0(\sigma_U) = k, \qquad L^0(\sigma_V) = k, \qquad L^0(\sigma_{U\cap V}) = k^2,$$

hence $\check{C}^0(\mathcal{U}, L^0) = \check{C}^1(\mathcal{U}, L^0) = k^2$. Giving $\check{C}^0(\mathcal{U}, L^0)$ the ordered basis (σ_U, σ_V) and noting the homology maps $H^0(f^{-1}(U) \hookrightarrow f^{-1}(U \cap V))$ and $H^0(f^{-1}(V) \hookrightarrow f^{-1}(U \cap V))$ are simply $1 \mapsto (1, 1)$, the Čech complex is

$$0 \to \check{C}^0(\mathcal{U}, L^0) \xrightarrow{\left[\begin{smallmatrix} -1 & -1 \\ 1 & 1 \end{smallmatrix}\right]} \check{C}^1(\mathcal{U}, L^0) \to 0.$$

Hence $H^0(\mathcal{U}, L^0) = \ker(\delta^0) = k$ and $H^1(\mathcal{U}, L^0) = k^2/\mathrm{im}(\delta^0) = k^2/k = k$, allowing us to conclude, using Curry's and the Čech–de Rham theorems, that

$$\begin{aligned} H^0(S^1) &\cong H^0(\mathcal{U}, L^0) \oplus H^1(\mathcal{U}, L^{-1}) = k \oplus 0 = k, \\ H^1(S^1) &\cong H^0(\mathcal{U}, L^1) \oplus H^1(\mathcal{U}, L^0) = 0 \oplus k = k, \\ H^2(S^1) &\cong H^0(\mathcal{U}, L^2) \oplus H^1(\mathcal{U}, L^1) = 0 \oplus 0 = 0, \end{aligned}$$

as expected.

References: Bott and Tu (Differential forms in algebraic topology, Section 10), Bredon (Sheaf theory, Section VI.4), Curry (Sheaves, cosheaves, and applications, Section 8)

3.8 Ordering simplicial complexes with unlabeled vertices

2017-12-03

Keywords: *ordering, simplicial complex, continuity, quotient, symmetric group, poset, Ran space*

The goal of this post is to describe a partial order on the collection of simplical complexes with $\leqslant n$ unlabeled vertices that is nice in the context of the space $X = \mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{>0}$.

First note that there is a natural order on (abstract) simplicial complexes, given by set inclusion. Interpreting elements of X as simplicial complexes induces a more restrictive order, as new vertices must "split off" from existing ones rather than just be introduced anywhere. Also note that the category usually denoted by SC of simplicial complexes and simplicial maps contains objects with unordered vertices. Here we assume an order on them and consider the action of the symmetric groups to remove the order.

Definition 3.8.1. Let SC_k, for some positive integer k, be the collection of simplicial complexes with k uniquely labeled vertices. This collection is a poset, with $S \leqslant T$ iff $\sigma \in T$ for every $\sigma \in S$.

The symmetric group on k elements acts on SC_k by permuting the vertices, and taking the image under this action we get SC_k/S_k, the collection of simplicial complexes with k unlabeled vertices. This set also has a partial order, with $S \leqslant T$ in SC_k/S_k iff $S' \leqslant T'$ in SC_k, for some $S' \in q_k^{-1}(S)$ and $T' \in q_k^{-1}(T)$, where $q_k : SC_k \twoheadrightarrow SC_k/S_k$ is the quotient map.

Definition 3.8.2. For all $i = 1, \ldots, k$, let $s_{k,i}$ be the *ith splitting map*, which splits the ith vertex in two. That is, if the vertices of $S \in SC_k$ are labeled $v_1, \ldots, v_k$, then $s_{k,i}$ is defined by

$$\begin{aligned} s_{k,i} \ : \ SC_k &\to SC_{k+1}, \\ S &\mapsto \left\langle S' \cup \{v_i, v_{i+1}\} \cup \bigcup_{\{v_i,w\}\in S} \{v_{i+1}, w\} \right\rangle, \end{aligned}$$

where S' is S with v_j relabeled as v_{j+1} for all $j > i$, and $\langle T \rangle$ is the simplicial complex generated by T.

By "generated by T" we mean generated in the Vietoris–Rips sense, that is, if $\{v_a, v_b\} \in T$ for all a, b in some indexing set I, then $\{v_c \ : \ c \in I\} \in \langle T \rangle$. The ith splitting map is essentially the ith face map used for simplicial sets.

Let $A = \bigcup_{k=1}^n SC_k/S_k$. The splitting maps induce a partial order on A, with $S \leqslant T$, for $S \in SC_k/S_k$ and $T \in SC_{k+1}/S_{k+1}$, iff $s_{k,i}(S') \leqslant T'$ in SC_k, for some $S' \in q_k^{-1}(S)$, $T' \in q_{k+1}^{-1}(T)$, and $i \in \{1, \ldots, k\}$. This generalizes via composition of the splitting maps to any pair $S, T \in A$, and is visually decribed by the diagram below.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & SC_{k-1} & \xrightarrow{s_{k-1,i}} & SC_k & \xrightarrow{s_{k,i}} & SC_{k+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle q_{k-1}} & & \downarrow{\scriptstyle q_k} & & \downarrow{\scriptstyle q_{k+1}} & & \\ & & SC_{k-1}/S_{k-1} & & SC_k/S_k & & SC_{k+1}/S_{k+1} & & \end{array}$$

Now, let M be a smooth, compact, connected manifold embedded in $\mathbf{R}^N$, and $X = \mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{>0}$. Let $f : X \to A$ be given by $(P, t) \mapsto VR(P, t)$, the Vietoris–Rips complex around the points of P with radius t.

Proposition 3.8.3. The map $f : X \to A$ is continuous.

Proof: Let $S \in A$ and $U_S \subseteq A$ be the open set based at S. Take any $(P, t) \in f^{-1}(U_S) \subseteq X$, for which we will show that there is an open ball $B \ni (P, t)$ completely within $f^{-1}(U_S)$.

Case 1: $t \neq d(P_i, P_j)$ for all pairs $P_i, P_j \in P$. Then set

$$\epsilon = \min \left\{ t, \min_{i<j} |t - d(P_i, P_j)|, \ \min_{i<j} d(P_i, P_j) \right\}.$$

Set $B = B_{\epsilon/4}^{\mathrm{Ran}^{\leqslant n}(M)}(P) \times B_{\epsilon/4}^{\mathbf{R}_{>0}}(t)$, which is an open neighborhood of (P, t) in X. It is immediate that $f(P', t')$, for any other $(P', t') \in B$, has all the simplices of $f(P, t)$, as $\epsilon \leqslant |t - d(P_i, P_j)|$ for all $i < j$. If P_i has split in two in P', then for every simplex containing P_i in $f(P, t)$ there are two simplices in $f(P't')$, with either of the points into which P_i split. That is, there may be new simplices in $f(P', t')$, but $f(P', t')$ will be in the image of the splitting maps. Equivalently, $f(P, t) \leqslant f(P', t')$ in A, so $B \subseteq f^{-1}(U_S)$.

Case 2: $t = d(P_i, P_j)$ for some pairs $P_i, P_j \in P$. Then set

$$\epsilon = \min \left\{ t, \min_{\substack{i<j \\ t \neq d(P_i, P_j)}} |t - d(P_i, P_j)|, \ \min_{i<j} d(P_i, P_j) \right\},$$

and define B as above. We are using the definition of Vietoris–Rips complex for which we add an edge between P_i and P_j whenever $t > d(P_i, P_j)$. Now take any $(P', t') \in B$ such that its image and the image of (P, t) under f are both in SC_k/S_k. Then any points $P_i, P_j \in P$ with $d(P_i, P_j) = t$ that have moved around to get to P', an edge will possibly be added, but never removed, in the image of f (when comparing with the image of (P, t)). This means that we have $f(P,t) \leqslant f(P',t')$ in SC_k/S_k, so certainly $f(P,t) \leqslant f(P',t')$ in A. The same argument as in the first case holds if points of P split. Hence $B \subseteq f^{-1}(U_S)$ in this case as well. ■

This proposition shows in particular that X is poset-stratified by A.

3.9 Induced orders on sets

2018-04-27

Keywords: *set, partial order, Čech, continuity*

The goal of this post is to understand when a map from a poset to an unordered set induces a partial order, and how that applies to the specific case of the set of simplicial complexes. Thanks to Yanlong Hao for spotting some mistakes in my seminar talk on the same topic yesterday.

Definition 3.9.1. Let $(A, \leqslant_A)$ be a poset and $f \colon A \to B$ a map of sets. The relation $\leqslant_B$ on B, with $a \leqslant_A a'$ implying $f(a) \leqslant_B f(a')$, is the relation *induced* by f on B. The map f is *monotonic* if whenever $b \leqslant_b b'$,

1. if $a \in f^{-1}(b)$, $a' \in f^{-1}(b)$ are comparable, then $a \leqslant_A a'$, and
2. if $a' \in f^{-1}(b')$, then there exists $a \in f^{-1}(b)$ such that $a \leqslant_A a'$.

Since f may not be surjective, there may be $b \in B$ with $f^{-1}(b) = \emptyset$. For such b we only have $b \leqslant_B b$ and b is not comparable to any other element of B.

Lemma 3.9.2. If $f \colon A \to B$ is monotonic, then the induced relation $\leqslant_B$ is a partial order on B.

Proof. For reflexivity, take any $a \in A$, which has $a \leqslant_A a$ by reflexivity of $\leqslant_A$. Then $f(a) \leqslant_B f(a)$, so every $b \in \operatorname{im}(f)$ satisfies reflexivity. Every $b \notin \operatorname{im}(f)$ also satisfies reflexivity by the comment above.

For anti-symmetry, suppose that $b \leqslant_B b'$ and $b' \leqslant_B b$. Since $b \leqslant_B b'$, there is some $a \in f^{-1}(b)$ and $a' \in f^{-1}(b')$ such that $a \leqslant_A a'$. Similarly, there is some $c' \in f^{-1}(b')$ and $c \in f^{-1}(b)$ such that $c' \leqslant_A c$. Since $c \in f^{-1}(b)$ and $c' \in f^{-1}(b')$ are comparable, and the first assumed relation is $b \leqslant_B b'$, by property 1 of Definition 3.9.1, we must have $c \leqslant_A c'$. By anti-symmetry of A, we now have that $c = c'$, so it follows that $b = f(c) = f(c') = b'$.

For transitivity, suppose that $b \leqslant_B b'$ and $b' \leqslant_B b''$. Take $a'' \in f^{-1}(b'')$, for which property 2 of Definition 3.9.1 guarantees that there exists $a' \in f^{-1}(b')$ such that $a' \leqslant_A a''$. Similarly, the first assumed relation and the same property guarantees there exists $a \in f^{-1}(b)$ such that $a \leqslant_A a'$. By transitivity of A, we have $a \leqslant_A a''$. By the definition of $\leqslant_B$, we have $b = f(a) \leqslant_B f(a'') = b''$. □

Let M be a piecewise linear, compact, connected, embedded manifold in $\mathbf{R}^N$, and SC the category of simplicial complexes. Let $A = \{1 < 2a > 2b < 3\}$. The product A^N has the product order, which we denote by $\leqslant_A$. Fix $n \in \mathbf{Z}_{>0}$ and let T be the set of all distinct 2-,3-,...,n-tuples in $\{1, \dots, n\}$, or $T := \bigcup_{k=2}^n \left(\{1, \dots, n\}^k \setminus \Delta\right)/_{S_k}$. This set has size $\sum_{k=2}^n \binom{n}{k} = 2^n - n - 1$. Assume every $v \in T$ is ordered in the canonical way. Then v induces a natural projection $\pi_v \colon M^n \to M^v$, as well as another map

$$\begin{aligned} \pi_v' \colon M^n \times \mathbf{R}_{>0} &\to A, \\ (P, t) &\mapsto \begin{cases} 1 & \forall\ i,j, \pi_v(P)_i = \pi_v(P)_j, \\ 2a & \exists\ i,j \text{ s.t. } \pi_v(P)_i \neq \pi_v(P)_j \text{ and } \bigcap_{i=1}^{|v|} B(\pi_v(P)_i, t) \neq \emptyset, \\ 2b & \exists\ i,j \text{ s.t. } \pi_v(P)_i \neq \pi_v(P)_j \text{ and } \bigcap_{i=1}^{|v|} B(\pi_v(P)_i, t) = *, \\ 3 & \exists\ i,j \text{ s.t. } \pi_v(P)_i \neq \pi_v(P)_j \text{ and } \bigcap_{i=1}^{|v|} B(\pi_v(P)_i, t) = \emptyset. \end{cases} \end{aligned}$$

Here all the balls B are closed, and M^n has the Hausdorff topology.

Lemma 3.9.3. The map π_v is continuous on $M^v \times \mathbf{R}_{>0}$.

Proof. Every $(Q,s) \in (\pi'_v)^{-1}(3)$ has an open ball of radius $\max_{i,j}\{d(\pi_v(Q)_i, \pi_v(Q)_j)\}/2 - s$ around it that is still contained within $(\pi'_v)^{-1}(3)$. Similarly, every $(Q,s) \in (\pi'_v)^{-1}(2a)$ has an open ball of radius

$$\min\left\{\frac{1}{2}\text{diam}\left(\bigcap_{i=1}^{|v|} B(\pi_v(Q)_i, s)\right), \max_{i,j}\{d(\pi_v(Q)_i, \pi_v(Q)_j)\}\right\} \tag{1}$$

around it that is still contained within $(\pi'_v)^{-1}(2a)$. The first expression in the min makes sure the intersection is non-empty, and the second expression makes sure all elements of Q are not the same.

The set $(\pi'_v)^{-1}(1 < 2a)$ is open by the same argument as for $2a \in A$, enlarging the open ball by removing the second expression in the min of expression (1). Finally, the set $(\pi'_v)^{-1}(2a > 2b < 3)$ is open by the same argument, now enlarging the ball used for $2a \in A$ by removing the first expression in the min of expression (1). □

Let $q\colon M^n \to \text{Ran}^{\leqslant n}(M)$ be the natural quotient map, and $\check{C}\colon \text{Ran}^{\leqslant n}(M) \times \mathbf{R}_{>0} \to SC$ be the Čech simplical complex map. For the next propositions, we will use two maps f and g defined as

$$\begin{array}{rcl} f\colon M^n \times \mathbf{R}_{>0} & \to & A^{2^n-n-1}, \\ (P,t) & \mapsto & \prod_{v\in T} \pi'_v(P,t), \end{array} \qquad \begin{array}{rcl} g\colon \text{im}(f) & \to & SC, \\ f(P,t) & \mapsto & \check{C}(q(P),t). \end{array}$$

The map g is well-defined because $a \in A^{2^n-n-1}$ with non-empty preimage in $M^n \times \mathbf{R}_{>0}$ specifies whether or not every k-tuple of points has a simplex spanning it, for all $k = 2, \ldots, n$. This defines a unique simplicial complex, so choosing any $(P,t) \in f^{-1}(a)$ will give the same Čech complex, up to renaming of vertices.

Proposition 3.9.4. The map $f\colon M^n \times \mathbf{R}_{>0} \to A^{2^n-n-1}$ is continuous.

Proof. Let $a \in A^{2^n-n-1}$ and suppose that $f^{-1}(a) \neq \emptyset$. Let $a_i \in A$ be in the ith factor of a, and r_i the radius of the open ball decreed by Lemma 3.9.3 to still be within $(\pi'_v)^{-1}(a_i)$, where v is the ith tuple in the chosen order on T. Then every $(P,t) \in f^{-1}(a)$ has an open ball of radius $\min_i\{r_i\}$ around it that is still contained within $f^{-1}(a)$, so f is continuous. □

Proposition 3.9.5. The map g is monotonic.

Note that any relation $S \leqslant_{SC} S'$ may be split up as a chain of relations $S = T_1 \leqslant_{SC} \cdots \leqslant_{SC} T_\ell = S'$, where the only differences between T_i and T_{i+1} are either (i) T_i has a k-simplex σ that T_{i+1} does not have, or (ii) where T_i has a single 0-simplex where a k-simplex σ and all its faces used to be in T_{i+1}. Hence it suffices to show that properties 1 and 2 of Definition 3.9.1 are satisfied in cases (i) and (ii).

Proof. Case (i): Suppose that $S \leqslant_{SC} S'$, and take $a \in g^{-1}(S)$, $a' \in g^{-1}(S')$ with $a \leqslant_A a'$. If there is $b \in g^{-1}(S)$ and $b' \in g^{-1}(S')$ such that $b' \leqslant_A b$, then $g(b)$ has the k-simplex σ that $g(b')$ does not have, but since b' is ordered lower than b, it must be that this k-simplex has collapsed to a point. Then we would be in case (ii), a contradiction, so property 1 holds in this case.

Now let $i_1, \ldots, i_\sigma$ be the indices of a' and a representing the $(k+1)$-fold intersection that describes σ, so $a'_j = 3$ and $a_j = 2b$ for all $j = i_1, \ldots, i_\sigma$. Take any $b' \in g^{-1}(S')$, which also has some indices $\ell_1, \ldots, \ell_\sigma$ representing this same $(k+1)$-fold intersection, so $b'_j = 3$ at all $j = \ell_1, \ldots, \ell_\sigma$. Let $b \in A^{2^n-n-1}$ be the element with all the same factors as b', except at indices $\ell_1, \ldots, \ell_\sigma$, which have been changed to $2b$. This element b is still in $\text{im}(f)$ as removing only this k-simplex still leaves the well-defined simplex S' we assumed at the beginning. Hence $g(b) = S'$ and property 2 holds.

Case (ii): Suppose that $S \leqslant_{SC} S'$, and take $a \in g^{-1}(S)$, $a' \in g^{-1}(S')$ with $a \leqslant_A a'$. If there is $b \in g^{-1}(S)$ and $b' \in g^{-1}(S')$ such that $b' \leqslant_A b$, then $g(b')$ has the k-simplex σ and all its faces that $g(b)$ does not have, but since b' is ordered lower than b, it must be that we have introduced σ and all its faces. Then we would be in case (i), or a chain of case (i) situations, a contradiction, so property 1 holds in this case.

Now let $i_1, \ldots, i_\sigma$ be the indices of a' and a representing the $(k+1)$-fold intersection that describes σ, and all the implied $(f+1)$-fold intersections that describe the f-faces of σ, $f > 0$. That is, $a'_j = 2a$ and $a_j = 1$ for all $j = i_1, \ldots, i_\sigma$. Take any $b' \in g^{-1}(S')$, which also has some indices $\ell_1, \ldots, \ell_\sigma$ representing this same $(k+1)$-fold (and lower) intersection, so $b'_j = 3$ at all $j = \ell_1, \ldots, \ell_\sigma$. Let $b \in A^{2^n-n-1}$ be the element with all the same factors as b', except at indices $\ell_1, \ldots, \ell_\sigma$, which have been changed to 1. This element b is still in $\mathrm{im}(f)$ as collapsing this k-simplex and all its faces to a single 0-simplex still leaves the well-defined simplex S' we assumed at the beginning. Hence $g(b) = S'$ and property 2 holds. □

Since g is monotonic, by Lemma 3.9.2 the relation $\leqslant_{SC}$ is a partial order on SC.

Part II
Extending foundations

1 Homotopy theory

1.1 The Eilenberg–Steenrod axioms

2016-02-26

Keywords: *homology theory, topological space, axioms, functor, homotopy, excision, weak equivalence*

The category Top of topological spaces may be generalized to the category Top_* of pointed topological spaces. This in turn may be generalized to the category Top_{rel} of pairs (X, A), where $X \in \text{Obj}(\text{Top})$ and A is a subspace of X. The morphisms of Top_{rel} on (X, A) are the morphisms of Top on X paired with their restrictions to A. We write (X) for $(X, \emptyset)$.

Definition 1.1.1. Let $X, Y \in \text{Obj}(\text{Top}_*)$. Then $f \in \text{Hom}_{\text{Top}_*}(X, Y)$ is an *n-equivalence* if the induced map on homotopy groups $f_* : \pi_k(X, x) \to \pi_k(Y, f(x))$ is an isomorphism for $k < n$ and an epimorphism for $k = n$. Further, f is a *weak equivalence* if it is an n-equivalence for all $n \geqslant 1$. Similarly, $f \in \text{Hom}_{\text{Top}_{rel}}((X, A), (Y, B))$ is a *weak equivalence* if $f \in \text{Hom}_{\text{Top}_*}(X, Y)$ and $f|_A \in \text{Hom}_{\text{Top}_*}(A, B)$ are weak equivalences.

Definition 1.1.2. Let C, D be two categories. A *functor* $\mathcal{F} : C \to D$ is an assignment $\mathcal{F}(X) \in \text{Obj}(D)$ for every $X \in \text{Obj}(C)$, and $\mathcal{F}(f) \in \text{Hom}_D(\mathcal{F}(X), \mathcal{F}(Y))$ for every $f \in \text{Hom}_C(X, Y)$. This assignment satisfies the following relations:
- $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$ for every $f \in \text{Hom}_C(X, Y)$ and $g \in \text{Hom}_C(Y, Z)$
- $\mathcal{F}(\text{id}_X) = \text{id}_{\mathcal{F}(X)}$ for every $X \in \text{Obj}(C)$

Definition 1.1.3. Let C be any category and $\mathcal{F} : \text{Top} \to C$ a functor. Then $\mathcal{F}$ is *homotopy invariant* if $f \simeq g$ in Top implies $\mathcal{F}(f) = \mathcal{F}(g)$ in C, where $\simeq$ is the homotopy of maps.

Definition 1.1.4. A *(relative) homology theory* of topological spaces is a collection of homotopy-invariant functors $H_n : \text{Top}_{rel} \to \text{Ab}$ and a collection of natural transformations $d_n : H_n(X, A) \to H_{n-1}(A)$.

The *Eilenberg–Steenrod axioms* are properties a relative homology theory may satisfy. The number of axioms depends on how general a view of homology theories one would like. Eilenberg and Steenrod (7), May (4), Aguilar, Gitler, and Prieto (4), Wikipedia (5), and other sources (6,8) have all different numbers of axioms. The order of the axioms below is alphabetical.

For any $(X, A) \in \text{Obj}(\text{Top}_{rel})$ and all n:

Axiom 1: Additivity. If $(X, A) = \bigoplus_i (X_i, A_i)$, then

$$H_n(X, A) \cong \bigoplus_i H_n(X_i, A_i),$$

where the isomorphism is induced by the inclusions $(X_i, A_i) \hookrightarrow (X, A)$.

Axiom 2: Exactness. There is a long exact sequence

$$\cdots \xrightarrow{d_{n+1}} H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \xrightarrow{d_n} H_{n-1}(A) \longrightarrow \cdots$$

where $H_n(A) \to H_n(X)$ and $H_n(X) \to H_n(X, A)$ are induced by the inclusions $(A) \hookrightarrow (X)$ and $(X) \hookrightarrow (X, A)$, respectively.

Axiom 3: Excision. If there exists a subset U of X with $\text{cl}(U) \subset \text{int}(A)$, then there is an isomorphism $H_n(X \setminus U, A \setminus U) \cong H_n(X, A)$ induced by the inclusion $(X \setminus U, A \setminus U) \hookrightarrow (X, A)$.

Axiom 4: Dimension. $H_n(*) = 0$ for all $n \neq 0$.

Axiom 5: Weak equivalence. If $f \in \text{Hom}_{\text{Top}_{rel}}((X,A),(Y,B))$ is a weak equivalence, then the induced map on homology $f_* : H_n(X,A) \to H_n(Y,B)$ is an isomorphism.

Singular homology is a homology theory that satisfies all the axioms above. K-theory is a homology theory that does not satisfy the dimension axiom.

References: May (A Concise course in Algebraic Topology, Chapter 13.1), Aguilar, Gitler, and Prieto (Algebraic Topology from a Homotopical Viewpoint, Chapter 5.3)

1.2 Ghost maps

2016-04-25

Keywords: *path space, loop space, spectrum, homotopy group, ghost map*

Definition 1.2.1. Let X be a topological space based at $x \in X$. Let PX be the *space of based paths* of X, that is, maps $[0,1] \to X$ with $0 \mapsto x$. Let $\Omega X \subset PX$ be the *space of based loops* of X, that is, maps $[0,1] \to X$ with $0, 1 \mapsto x$.

Note that Ω is a functor on the category of based topoloigcal spaces right-adjoint to the suspension functor Σ. Also observe there is a fibration

$$\Omega X \to PX \xrightarrow{p} X,$$

where p is evaluation at $1 \in [0,1]$. Since PX is contractible, $H_n(PX) = 0$ for $n \neq 0$, so $H_1(\Omega X) \cong H_2(X)$.

Definition 1.2.2. A *spectrum* E is a sequence of based topological spaces (E_n, x_n) and based homeomorphisms $\alpha_n : E_n \to \Omega E_{n+1}$. A *map of spectra* $f : E \to F$ is a sequence of based homeomorphisms $f_n : E_n \to F_n$ compatible with the based homeomorphisms of E and F, that is, so that the diagram

$$\begin{array}{ccc} E_n & \xrightarrow{\alpha_n} & \Omega E_{n+1} \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle \Omega f_{n+1}} \\ F_n & \xrightarrow{\beta_n} & \Omega F_{n+1} \end{array}$$

commutes for all n.

Definition 1.2.3. Let E, F be spectra. A map of spectra $f : E \to F$ is a *ghost map* if the induced map $\pi_n f : \pi_n X \to \pi_n Y$ on stable homotopy groups is the zero map.

Most commonly this term is used in spectra, but the idea of a ghost map may be generalized to other situations, where a map induces the zero map on homology, cohomology, or some similar functor.

References: Weibel (An introduction to homological algebra, Chapters 5.3, 10.9)

1.3 Spectral sequences and filtrations

2016-05-17

Keywords: *spectral sequence, filtration, good filtration, bête filtration, truncation*

Definition 1.3.1. Let $C^\bullet \in C(A)$ be a cochain complex with boundary maps $d^\bullet$ over some category A. A *filtration* of $C^\bullet$ is a sequence of objects $F^nC^\bullet$ with boundary maps $d_h^{\bullet,n}$ in the category of cochain complexes $C(A)$ of A, either a

$$\text{decreasing filtration } C^\bullet \supseteq \cdots \supseteq F^{n-1}C^\bullet \supseteq F^nC^\bullet \supseteq F^{n+1}C^\bullet \supseteq \cdots \text{ or}$$
$$\text{increasing filtration } C^\bullet \supseteq \cdots \supseteq F^{n+1}C^\bullet \supseteq F^nC^\bullet \supseteq F^{n-1}C^\bullet \supseteq \cdots,$$

where "$\supseteq$" is defined as necessary, along with maps $d_v^{\bullet,n} : F^nC^\bullet \to F^{n\pm 1}C^\bullet$. These maps are compatible, in the sense that $d_v^{k\pm 1,n} d_h^{k,n} = d_h^{k,n\mp 1} d_v^{k,n}$.

Example 1.3.2. Define "$\supseteq$" as $X \supseteq Y$ iff $\operatorname{Hom}(Y, X)$ is non-empty. The *bête* (or *brutal*) *filtration* of $C^\bullet$ is a decreasing filtration

$$(F^nC^\bullet)^i = \begin{cases} 0 & \text{if } i < n, \\ C^i & \text{if } i \geqslant n, \end{cases} \qquad \text{with} \qquad H^k(F^nC^\bullet) = \begin{cases} 0 & \text{if } k < n \\ Z^n & \text{if } k = n, \\ H^k(C^\bullet) & \text{if } k > n. \end{cases}$$

This filtration may be represented by the diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & C^{n+1} & \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & C^n & \longrightarrow & C^{n+1} & \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots & \longrightarrow & 0 & \longrightarrow & C^{n-1} & \longrightarrow & C^n & \longrightarrow & C^{n+1} & \longrightarrow \cdots \end{array},$$

which clearly commutes. The *good filtration* of $C^\bullet$ is also a decreasing filtration

$$(F^nC^\bullet)^i = \begin{cases} C^i & \text{if } i < n, \\ Z^iC^\bullet & \text{if } i = n, \\ 0 & \text{if } i > n, \end{cases} \qquad \text{with} \qquad H^k(F^nC^\bullet) = \begin{cases} H^k(C^\bullet) & \text{if } k \leqslant n, \\ 0 & \text{if } k > n. \end{cases}$$

This filtration may be represented by the diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & Z^{n-1} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots & \longrightarrow & C^{n-1} & \longrightarrow & Z^n & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots & \longrightarrow & C^{n-1} & \longrightarrow & C^n & \longrightarrow & Z^{n+1} & \longrightarrow & 0 & \longrightarrow \cdots \end{array},$$

which also commutes. Both of these are also called *truncations*. The good filtration is "better" because the cocycle groups Z^n do not appear in the cohomology groups. The same may be done for homology groups.

Definition 1.3.3. Set $F^nC^k = (F^nC^\bullet)^k = F^nC^\bullet \cap C^k$, and let the *zeroth page* of the cohomology spectral sequence of $C^\bullet$ with the filtration F be given by

$$\begin{aligned} E_0^{p,q} &= F^pC^{p+q}/F^{p+1}C^{p+q} && \text{if } F \text{ is decreasing,} \\ &= F^pC^{p+q}/F^{p-1}C^{p+q} && \text{if } F \text{ is increasing.} \end{aligned}$$

Let the *first page* of the cohomology spectral sequence of $C^\bullet$ with the filtration F be given by

$$\begin{aligned} E_1^{p,q} &= H^{p+q}(F^pC^\bullet/F^{p+1}C^\bullet) && \text{if } F \text{ is decreasing,} \\ &= H^{p+q}(F^pC^\bullet/F^{p-1}C^\bullet) && \text{if } F \text{ is increasing.} \end{aligned}$$

From now on, assume that F is an increasing filtration. Let the *second page* of the cohomology spectral sequence of $C^\bullet$ with the filtration F be given by

$$E_2^{p,q} = \frac{\ker(E_1^{p,q} \to E_1^{p+1,q})}{\operatorname{im}(E_1^{p-1,q} \to E_1^{p,q})}.$$

Continue in this manner and let the *rth page* of the cohomology spectral sequence of $C^\bullet$ with the filtration F be given by

$$E_r^{p,q} = \frac{\{x \in F^pC^{p+q} \ : \ dx \in F^{p+r}C^{p+q+1}\}}{F^{p+1}C^{p+q} + dF^{p-r+1}C^{p+q-1}}.$$

The same may be done for a homology spectral sequence. Note that a spectral sequence may also be defined without coming from a filtration.

Definition 1.3.4. A *homology spectral sequence* is a collection of objects $E^r_{p,q}$ and maps $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ with $d^r d^r = 0$ such that

$$E^{r+1}_{p,q} \cong \ker(d^r_{p,q})/\mathrm{im}(d^r_{p+r,q-r+1}).$$

Similarly, a *cohomology spectral sequence* is a collection of objects $E^{p,q}_r$ and maps $d^{p,q}_r : E^{p,q}_r \to E^{p+r,q-r+1}_r$ with $d^r d^r = 0$ such that

$$E^{p,q}_{r+1} \cong \ker(d^{p,q}_r)/\mathrm{im}(d^{p-r,q+r-1}_r).$$

References: Weibel (An introduction to homological algebra, Chapter 1.2), McCleary (A user's guide to spectral sequences, Chapter 2.2), Hutchings (Algebraic topology lecture notes, see `math.berkeley.edu/∼hutching/teach/215b-2011`)

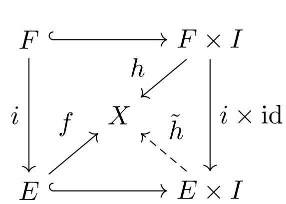
1.4 (Co)fibrations, suspensions, and loop spaces

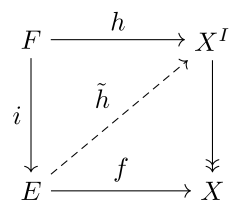
2016-07-31

Keywords: *fibration, cofibration, extension, lifting, suspension, loop space*

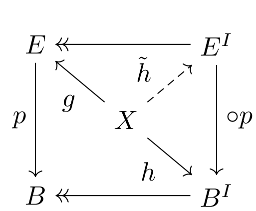
Recall the exponential object Z^Y, which, in the category of topological spaces, is the set of all continuous functions $Y \to Z$. In general, the definition involves a commuting diagram and gives an isomorphism $\mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, Z^Y)$. The subspace $F(Y,Z)$ of Z^Y consists of based functions $Y \to Z$.

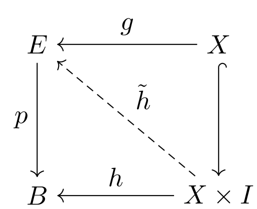
Definition 1.4.1. Let F, E, B, X be topological spaces. A map $i : F \to E$ is a *cofibration* if for every map $f : E \to X$ and every homotopy $h : F \times I \to X$, there exists a homotopy $\tilde{h} : E \times I \to X$ (extending h) making either of the equivalent diagrams below commute.





The horizontal maps on the left are the natural inclusion maps $x \mapsto (x, 0)$ and the map on the right is the natural evaluation map $\varphi \mapsto \varphi(0)$. Similarly, a map $p : E \to B$ is a *fibration* if for every map $g : X \to E$ and every homotopy $h : X \times I \to B$, there exists a homotopy $\tilde{h} : X \times I \to E$ (lifting h) making either of the equivalent diagrams below commute.





The horizontal maps on the right are the natural evaluation maps and the map on the right is the natural inclusion map.

Instead of this terminology, often we say the pair (F, E) has the *homotopy extension property* and the pair (E, B) has the *homotopy lifting property*. Now, let let (X, x) be a pointed topological space.

Definition 1.4.2. The (*reduced*) *suspension* ΣX of X is

$$\Sigma X := X \times I / X \times \{0\} \cup X \times \{1\} \cup \{x\} \times I.$$

The *unreduced suspension* SX of X is

$$SX := X \times I / X \times \{0\} \cup X \times \{1\}.$$

The *loop space* ΩX of X is

$$\Omega X := F(S^1, X).$$

Remark 1.4.3. If X is well-pointed (the inclusion $i : \{x\} \hookrightarrow X$ is a cofibration), then the natural quotient map $SX \to \Sigma X$ is a homotopy equivalence. Moreover, there is an adjunction $F(\Sigma X, Y) \cong F(X, \Omega Y)$. In the fundamental group this gives the adjunction

$$[\Sigma X, Y] \cong [X, \Omega Y],$$

where $[A, B]$ is the set of based homotopy classes of maps $A \to B$.

References: May (A concise course in algebraic toplogy, Chapters 6, 7, 8), Aguilar, Gitler, and Prieto (Algebraic topology from a homotopical viewpoint, Chapter 2.10)

1.5 Some facts about formal group laws

2016-08-08

Keywords: *formal group law, morphism, finite field*

Here we solve some problems from the 2016 West Coast Algebraic Topology Summer School (WCATSS) at The University of Oregon. Thanks to Piotr Pstragowski and Carolyn Yarnall for the solutions. First we recall some definitions.

Definition 1.5.1. Let R be a commutative ring with unit. A *formal group law* F over R is an element $F \in R[[x, y]]$ satisfying

1. $F(x, y) = F(y, x)$ (symmetry),
2. $F(x, 0) = x$ and $F(0, y) = y$ (uniticity),
3. $F(F(x, y), z) = F(x, F(y, z))$ (associativity).

It follows from these three properties that $F(x, y) = x + y +$ (higher order terms) for all F.

Proposition 1.5.2. For any formal group law $F(x, y)$ over R, x has a formal inverse. That is, there exists an element $i(x) \in R[[x]]$ such that $F(x, i(x)) = 0$.

Proof: Consider $F(x, y + z)$, with $|z| = n$. Note that

$$\begin{aligned}
F(x, y+z) &= x + y + z + \sum_{i,j \geqslant 1} a_{ij} x^i (y+z)^j \\
&= x + y + z + \sum_{i,j \geqslant 1} a_{ij} x^i \sum_{k=0}^{j} \binom{j}{k} y^k z^{j-k} \\
&= x + y + z + \sum_{i,j \geqslant 1} a_{ij} x^i \left(y^j + \sum_{k=0}^{j-1} \binom{j}{k} y^k z^{j-k} \right) \\
&= x + y + z + \sum_{i,j \geqslant 1} a_{ij} x^i y^j + \underbrace{\sum_{i,j \geqslant 1} a_{ij} x^i}_{\deg\ \geqslant\ 1} \underbrace{\sum_{k=0}^{j-1} \binom{j}{k} y^k z^{j-k}}_{\deg\ =\ k+n(j-k) \geqslant n} \\
&= F(x, y) + z + (\text{terms of deg } \geqslant\ n+1).
\end{aligned}$$

First choose z_1 to be the negative of all the degree-1 terms of $F(x, 0)$, so that $F(x, z_1)$ has terms of degree 2 and higher. Now choose z_2 to be the negative of all the degree-2 terms of $F(x, z_1)$, so $F(x, z_1 + z_2)$ has terms of degree 3 and higher. Continue in this manner ad infinitum to get a formal inverse $\sum_i z_i$ (this will be a power series) of x. ■

Recall that we call $f_a(x, y) = x + y$ the *additive* formal group law and $F_m(x, y) = x + y + xy$ the *multiplicative* formal group law. Via the universal Lazard ring of formal group laws, these turn out to be the formal group laws of ordinary singular cohomology theory (additive) and complex K-theory KU (multiplicative). Recall also nested notation: for F a formal group law, we write

$$\begin{aligned}
[1]_F(x) &= x, \\
[2]_F(x) &= F(x,x), \\
[3]_F(x) &= F(F(x,x),x), \\
[4]_F(x) &= F(F(F(x,x),x),x),
\end{aligned}$$

and so on.

Definition 1.5.3. Let F be a formal group law over R. A *morphism* of formal group laws is an element $\varphi \in R[[u]]$, giving a formal group law $\varphi F \in R[[x,y]]$ by $\varphi F(x,y) := F(\varphi(x),\varphi(y))$.

An isomorphism of formal group laws is a morphism where the formal power series φ is an isomorphism.

Proposition 1.5.4. The additive formal group law and the multiplicative formal group law are not isomorphic over F_p.

Proof: We compare $[p]_{F_m}(x)$ and $[p]_{F_a}(x)$ and show they are not the same. If there were an isomorphism φ between F_a and F_m, we should have that

$$F_m(x,x) = F_a(\varphi(x),\varphi(x)) = \varphi(F_a(x,x)) \quad \Longrightarrow \quad [p]_{F_m}(x) = \varphi([p]_{F_a}(x)),$$

since φ is a homomorphism. However, we first see that

$$[1]_{F_a}(x) = x \quad , \quad [2]_{F_a}(x) = F_a(x,x) = 2x \quad , \quad [3]_{F_a}(x) = F_a(F_a(x,x),x) = 3x,$$

and so continuing this pattern we get that $[p]_{F_a}(x) = px = 0$ in F_p. Next, for the multiplicative formal group law we find that

$$[1]_{F_m}(x) = x, \quad , \quad [2]_{F_m}(x) = F_m(x,x) = 2x + x^2 \quad , \quad [3]_{F_m}(x) = F_m(2x+x^2,x) = 3x + 3x^2 + x^3.$$

Here the pattern is not immediate, but continuing these small examples we find that $[p]_{F_m}(x) = (x+1)^p - 1 = 1 + x^p - 1 = x^p$ in F_p. An isomorphism sends only 0 to 0, but in this case φ should send $x^p \neq 0$ to 0, a contradiction. Hence no such isomorphism exists over F_p. ■

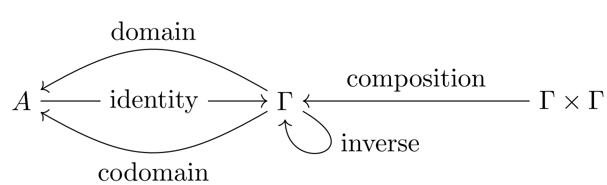
1.6 What is a stack?

2016-08-13

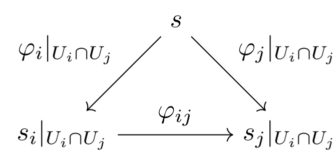
Keywords: *groupoid, sheaf, stack, Hopf, algebroid*

This is from discussions at the 2016 West Coast Algebraic Topology Summer School (WCATSS) at The University of Oregon. Thanks to Piotr Pstragowski for explaining the material.

Definition 1.6.1. A *groupoid* is a category where all the morphisms are invertible. Alternatively, a groupoid is a set of objects A, a set of morphisms Γ, and a collection of maps as described by the diagram below.

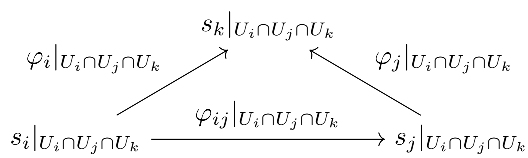


To describe stacks, we compare them with sheaves. Both start out with a space X and a topology on it, so that we may consider open sets U.

sheaf space		stack space	
$U \to \mathcal{F}(U)$	a group	$U \to \widehat{\mathcal{F}}(U)$	a groupoid
$O(X)^{op} \to \text{Set}$	open sets to groups	$O(X)^{op} \to \text{Grpd}$	open sets to groupoids
if $s_i \in \mathcal{F}(U_i)$ and $s_j \in \mathcal{F}(U_j)$ such that $s_i\|_{U_i \cap U_j} = s_j\|_{U_i \cap U_j}$, then there exists $s \in \mathcal{F}(U)$ such that $s\|_{U_i} = s_i$ and $s\|_{U_j} = s_j$		if $s_i \in \widehat{\mathcal{F}}(U_i)$ and $s_j \in \widehat{\mathcal{F}}(U_j)$ such that there an isomorphism $\varphi_{ij} : s_i\|_{U_i \cap U_j} \to s_j\|_{U_i \cap U_j}$, then there exists $s \in \mathcal{F}(U)$ and isomorphisms $\varphi_i : s\|_{U_i} \to s_i$, $\varphi_j : s\|_{U_j} \to s_j$ such that the diagram below commutes: 	

In addition to these conditions, there is a triple intersection condition for stacks that does not have an analogous one in sheaves. It is given by:

> for every U_i, U_j, U_k and $s_i, s_j, s_k \in \widehat{\mathcal{F}}(U_i), \widehat{\mathcal{F}}(U_j), \widehat{\mathcal{F}}(U_k)$, respectively, such that there exist isomorphisms $\varphi_{ij} : s_i|_{U_i \cap U_j} \to s_j|_{U_i \cap U_j}$, $\varphi_{jk} : s_j|_{U_j \cap U_k} \to s_k|_{U_j \cap U_k}$, and $\varphi_{ik} : s_i|_{U_i \cap U_k} \to s_k|_{U_i \cap U_k}$, the diagram below commutes:



Example 1.6.2. A Hopf algebroid may be viewed as a functor into groupoids, so that with the appropriate topology, it becomes a stack. Indeed, by definition a Hopf algebroid is a pair of k-algebras (A, Γ) such that $(\text{Spec}(A), \text{Spec}(\Gamma))$ is a groupoid object in affine schemes, or in other words, is a functor from affine schemes into groupoids.

References: nLab (article on groupoids)

1.7 Sheaves and cosheaves

2017-06-04

Keywords: *presheaf, sheaf, precosheaf, cosheaf, sampling*

Let X be a topological space with an open cover $\mathcal{U} = \{U_i\}$, and category $Op(X)$ of open sets of X. Let C be any abelian category, most often groups.

Definition 1.7.1. A *presheaf* $\mathcal{F}$ over X is a functor $Op(X)^{op} \to D$, and a *sheaf* if it satisfies the *gluing* axiom. A *precosheaf* $\widehat{\mathcal{F}}$ over X is a functor $Op(X) \to D$, and a *cosheaf* if it satisfies the *cutting* axiom.

The gluing axiom may be interpreted as a colimit condition and the cutting axiom (thanks to Keaton Quinn for suggesting the name) may be interpreted as a limit condition. The components of sheaves and cosheaves are compared in the table below.

	sheaf	cosheaf
functoriality	$\begin{array}{rcl} Op(S)^{op} & \to & D \\ U & \mapsto & \mathcal{F}(U) \\ (V \hookrightarrow U)^{op} & \mapsto & (\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)) \end{array}$	$\begin{array}{rcl} Op(S) & \to & D \\ U & \mapsto & \widehat{\mathcal{F}}(U) \\ (V \hookrightarrow U) & \mapsto & (\varepsilon_{VU} : \widehat{\mathcal{F}}(V) \to \widehat{\mathcal{F}}(U)) \end{array}$
gluing / cutting	if $s_i\|_{U_i \cap U_j} = s_j\|_{U_i \cap U_j}$, then $\exists s \in \mathcal{F}(U_i \cup U_j)$ s.t. $s\|_{U_i} = s_i, s\|_{U_j} = s_j$.	if $s_i\|^{U_i \cup U_j} = s_j\|^{U_i \cup U_j}$, then $\exists s \in \widehat{\mathcal{F}}(U_i \cap U_j)$ s.t. $s\|^{U_i} = s_i, s\|^{U_j} = s_j$.
colimit / limit cond.	$\mathcal{F}(U) \xrightarrow{\cong} \varprojlim_{V \subseteq U} \mathcal{F}(V)$	$\widehat{\mathcal{F}}(U) \xleftarrow{\cong} \varinjlim_{V \subseteq U} \widehat{\mathcal{F}}(V)$

The maps ρ_{UV} are called *restrictions* and ε_{VU} are called *extensions*. Above, s_i is a (co)section over U_i and s_j is a (co)section over U_j. For s a (co)section of U with $V \subset U \subset W$, write $s|_V$ for $\rho_{UV}(s)$ and $s|^W$ for $\varepsilon_{UW}(s)$. The isomorphisms with the colimits and limits are the natural maps from the respective colimit and limit diagrams.

Now we relate sheaves to persistent homology. All cohomology is be taken over a field k.

Remark 1.7.2. Suppose we have a finite point sample P and some $t > 0$, for which we can construct the nerve $N_{t,P}$, a cellular complex, of the union of balls of radius t around the points of P. If $t' < t$, then there is a natural inclusion $N_{t',P} \hookrightarrow N_{t,P}$, which induces a map $H_\ell(N_{t',P}) \to H_\ell(N_{t,P})$ on degree ℓ homology groups. Define a sheaf $\mathcal{F}^\ell$ over $\mathbf{R}$ for which

$$\mathcal{F}^\ell(U) = H^\ell(N_{\inf(U),P}), \qquad \mathcal{F}^\ell_t = H^\ell(N_{t,P}).$$

This is indeed a sheaf, as $V \subseteq U$ implies that $\inf(U) \leqslant \inf(V)$, giving a natural map $\mathcal{F}^\ell(U) \to \mathcal{F}^\ell(V)$. The gluing axiom is also satisfied: assume without loss of generality that $\inf(U_i) \leqslant \inf(U_j)$ and take $s_i \in \mathcal{F}^\ell(U_i)$, $s_j \in \mathcal{F}^\ell(U_j)$ with the assumptions as above. Then $\inf(U_i) = \inf(U_i \cup U_j)$ and $\inf(U_j) = \inf(U_i \cap U_j)$, so

$$\mathcal{F}^\ell(U_i) = \mathcal{F}^\ell(U_i \cup U_j), \qquad \mathcal{F}^\ell(U_j) = \mathcal{F}^\ell(U_i \cap U_j),$$

hence $s_i = s \in \mathcal{F}^\ell(U_i \cup U_j)$ and $s|_{U_j} = s_i|_{U_j} = s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} = s_j|_{U_j} = s_j$. Therefore sheaves capture all the persistent homology data. Note we do not take the sheaf cohomology of $\mathcal{F}^\ell$, instead the usual sequence of homology groups is induced by any increasing sequence in $\mathbf{R}$.

References: Bredon (Sheaf theory, Section VI.4), Bott and Tu (Differential forms in algebraic topology, Section 10)

1.8 Exit paths and entry paths through ∞-categories

2018-04-20

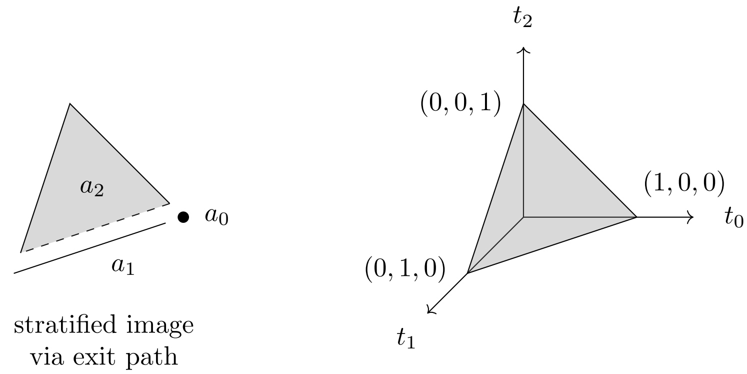
Keywords: *exit path, entry path, conical stratification, infinity category, quasi-category, Kan complex, nerve, horn, homotopy category, adjoint*

Let X be a topological space, $(A, \leqslant)$ a poset, and $f\colon X \to (A, \leqslant)$ a continuous map.

Definition 1.8.1. An *exit path* in an A-stratified space X is a continuous map $\sigma\colon |\Delta^n| \to X$ for which there exists a chain $a_0 \leqslant \cdots \leqslant a_n$ in A such that $f(\sigma(t_0, \ldots, t_i, 0, \ldots, 0)) = a_i$ for $t_i \neq 0$. An *entry path* is a continuous map $\tau\colon |\Delta^n| \to X$ for which there exists a chain $b_0 \leqslant \cdots \leqslant b_n$ in A such that $f(\tau(0, \ldots, 0, t_i, \ldots, t_n)) = b_i$ for $t_i \neq 0$.

Up to reordering of vertices of Δ^n and induced reordering of the realization $|\Delta^n|$, an exit path is the same as an entry path. The next example describes this equivalence.

Example 1.8.2. The standard 2-simplex $|\Delta^2|$ is uniquely an exit path and an entry path with a chain of 3 distinct elements, stratfied in the ways described below.



stratified image via exit path

stratified image via entry path

Recall the following algebraic constructions, through Joyal's *quasi-category* model:

- A *simplicial set* is a functor $\Delta^{op} \to \text{Set}$.
- A *Kan complex* is a simplicial set satisfying the inner horn condition for all $0 \leqslant k \leqslant n$. That is, the kth n-horn lifts (can be filled in) to a map on Δ^n.
- An ∞-*category* is a simplicial set satisfying the inner horn condition for all $0 < k < n$.

Moreover, if the lift is unique, then the Kan complex is the nerve of some category. Recall also the category $\text{Sing}(X) = \{\text{continuous } \sigma \colon |\Delta^n| \to X\}$, which can be combined with the stratification $f \colon X \to A$ of X

Remark 1.8.3. The subcategory $\text{Sing}^A(X)$ of exit paths and the subcategory $\text{Sing}_A(X)$ of entry paths are full subcategories of $\text{Sing}(X)$, with $(\text{Sing}^A(X))^{op} = \text{Sing}_A(X)$. If the stratification is *conical*, then these two categories are ∞-categories.

not conically stratified
no lift of Λ_1^2 exists

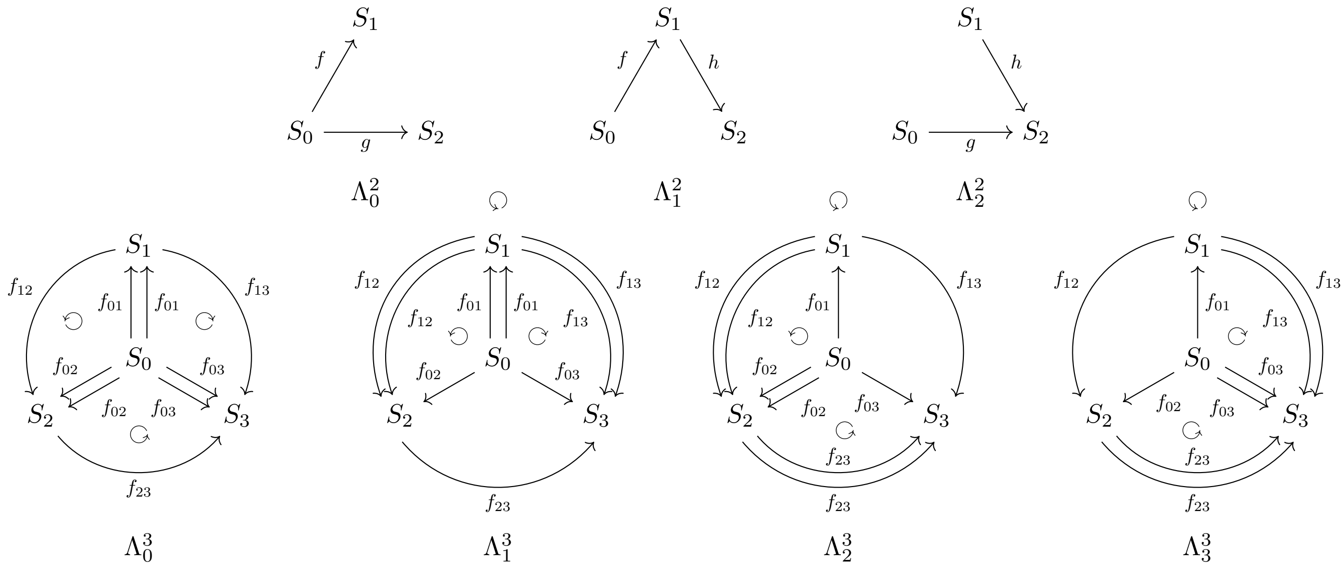
conically stratified
a lift of Λ_1^2 exists

Recall the *nerve* construction of a category. Here we are interested in the nerve of the category SC of simplicial complexes, so $N(SC)_n = \{\text{sequences of } n \text{ composable simplicial maps}\}$. Recall the kth n-horns, which are compatible diagrams of elements of $N(SC)_n$. In general, they are colimits of a diagram in the category Δ. That is,

$$\Lambda_k^n := \operatorname{colim}\left(\coprod_{0 \leqslant i < j \leqslant n} \Delta^{n-2} \rightrightarrows \coprod_{\substack{0 \leqslant i \leqslant n \\ i \neq k}} \Delta^{n-1} \right).$$

Example 1.8.4. The images of the 3 different types of 2-horns and 4 different types of 3-horns in SC are given below. Note that they are not unique, and depend on the choice of simplices S_i (equivalently, on the choice of functor

$\Delta^{op} \to SC$).



For example, the 0th 2-horn Λ_0^2 can be filled in if there exists a simplicial map $h\colon S_1 \to S_2$ in SC (that is, an element of $N(SC)_1$) such that $h \circ f = g$. Similarly, the 1st 3-horn Λ_1^3 can be filled in if there exists a functor $F\colon [0 < 1 < 2] \to SC$ for which $F(0 < 1) = f_{02}$, $F(0 < 2) = f_{03}$, and $F(1 < 2) = f_{23}$ (equivalently, a compatible collection of elements of $N(SC)_2$).

Definition 1.8.5. Let A, B be ∞-categories. A *functor* $F\colon A \to B$ is a morphism of the simplicial sets A, B. That is, $F : A \to B$ is a natural transformation for $A, B \in \text{Fun}(\Delta^{op}, \text{Set})$.

A functor of simplicial sets of a particular type can be identified with a functor of 1-categories. Recall the *nerve* of a 1-category, which turns it into an ∞-category. This construction has a left adjoint.

Definition 1.8.6. Let $\mathcal{C}$ be an ∞-category. The *homotopy category* $h\mathcal{C}$ of $\mathcal{C}$ has objects $\mathcal{C}_0$ and morphisms $\text{Hom}_{h\mathcal{C}}(X, Y) = \pi_0(\text{Map}_{\mathcal{C}}(X, Y))$.

By Lurie, h is left-adjoint to N. That is, $h\colon \text{sSet} \rightleftarrows \text{Cat} : N$, or $\text{Map}_{\text{sSet}}(\mathcal{C}, N(\mathcal{D})) \cong \text{Map}_{\text{Cat}}(h\mathcal{C}, \mathcal{D})$, for any ∞-category $\mathcal{C}$ and any 1-category $\mathcal{D}$. Our next goal is to describe a functor $\text{Sing}_A(X) \to N(SC)$, maybe through this adjunction, where SC is the 1-category of simplicial complexes and simplicial maps.

References: Lurie (Higher topos theory, Sections 1.1.3 and 1.2.3), Lurie (Higher algebra, Appendix A.6), Goerss and Jardine (Simplicial homotopy theory, Section I.3), Joyal (Quasi-categories and Kan complexes)

1.9 A functor from entry paths to the nerve of simplicial complexes

2018-04-22

Keywords: *functor, simplicial set, face map, degeneracy map, natural transformation, entry path, simplicial complex*

Fix $n \in \mathbf{Z}_{>0}$ and let $X = \text{Ran}^{\leqslant n}(M) \times \mathbf{R}_{>0}$ for M a compact, connected PL manifold embedded in $\mathbf{R}^N$. Take $\widetilde{h}\colon X \to (B, \leqslant)$ the conical stratifying map from a previous post ("Conical stratifications via semialgebraic sets," 2018-04-16) compatible with the natural stratification $h\colon X \to SC$. The goal of this post is to construct a functor $F\colon \text{Sing}_B(X) \to N(SC)$ from the ∞-category of entry paths that encodes the structure of X.

Recall that a simplicial set is a functor, an element of $\text{Fun}(\Delta^{op}, \text{Set})$. A simplicial set S is defined by its collection of n-simplices S_n, its *face maps* $s_i : S_{n-1} \to S_n$, and *degeneracy maps* $d_i : S_{n+1} \to S_k$, for all $i = 0, \ldots, n$. For the first simplicial set of interest in this post, we have

$$\begin{aligned}
\mathrm{Sing}_B(X)_n &= \mathrm{Hom}^B_{\mathrm{Top}}(|\Delta^n|, X),\\
(s_i\colon [n]\to[n-1]) &\mapsto \begin{pmatrix} \big(|\Delta^{n-1}|\to X\big)\mapsto(|\Delta^n|\to X)\\ \text{collapses } i\text{th with } (i+1)\text{th vertex, then maps as source}\end{pmatrix}\\
(d_i\colon [n]\to[n+1]) &\mapsto \begin{pmatrix} \big(|\Delta^{n+1}|\to X\big)\mapsto(|\Delta^n|\to X)\\ \text{maps as } i\text{th face of source map}\end{pmatrix}
\end{aligned}$$

We write $\mathrm{Hom}^B_{\mathrm{Top}}$ for the subset of $\mathrm{Hom}_{\mathrm{Top}}$ that respects the stratification B in the context of entry paths. For the second simplicial set, the nerve, we have

$$\begin{aligned}
N(SC)_n &= \{(S_0 \xrightarrow{f_1}\cdots\xrightarrow{f_n} S_n)\ :\ S_i\in SC,\ f_i \text{ are simplicial maps}\},\\
(s_i\colon [n]\to[n-1]) &\mapsto \Big(\big(S_0\xrightarrow{f_1}\cdots\xrightarrow{f_{n-1}} S_{n-1}\big)\mapsto\big(S_0\xrightarrow{f_1}\cdots\xrightarrow{f_i} S_i\xrightarrow{\mathrm{id}} S_i\xrightarrow{f_{i+1}}\cdots\xrightarrow{f_{n-1}} S_{n-1}\big)\Big),\\
(d_i\colon [n]\to[n+1]) &\mapsto \begin{pmatrix} i=0: & (S_0\cdots S_{n+1})\mapsto\big(S_1\xrightarrow{f_2}\cdots\xrightarrow{f_{n+1}} S_{n+1}\big)\\ 0<i<n: & (S_0\cdots S_{n+1})\mapsto\big(S_0\xrightarrow{f_1}\cdots\xrightarrow{f_{i-1}} S_{i-1}\xrightarrow{f_{i+1}\circ f_i} S_{i+1}\xrightarrow{f_{i+2}}\cdots\xrightarrow{f_{n+1}} S_{n+1}\big)\\ i=n: & (S_0\cdots S_{n+1})\mapsto\big(S_0\xrightarrow{f_1}\cdots\xrightarrow{f_n} S_n\big)\end{pmatrix}.
\end{aligned}$$

Define F on k-simplices as

$$F\left(\gamma\colon|\Delta^k|\to\mathrm{Ran}^{\leqslant n}(M)\times\mathbf{R}_{>0}\right)=\left(\widetilde{h}(\gamma(1,0,\dots,0))\xrightarrow{(\widetilde{h}\circ\gamma\circ s_k\circ\cdots\circ s_2)(|\Delta^1|)}\cdots\xrightarrow{(\widetilde{h}\circ\gamma\circ s_{k-2}\circ\cdots\circ s_0)(|\Delta^1|)}\widetilde{h}(\gamma(0,\dots,0,1))\right).$$

A morphism in $\mathrm{Sing}_B(X)$ is a composition of face maps s_i and degeneracy maps d_i, so F must satisfy the commutative diagrams

$$\begin{array}{ccc}
\mathrm{Sing}_B(X)_{n-1} & \xrightarrow{F} & N(SC)_{n-1}\\
{\scriptstyle s_i}\downarrow & & \downarrow{\scriptstyle s_i}\\
\mathrm{Sing}_B(X)_n & \xrightarrow{F} & N(SC)_n,
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{Sing}_B(X)_{n+1} & \xrightarrow{F} & N(SC)_{n+1}\\
{\scriptstyle d_i}\downarrow & & \downarrow{\scriptstyle d_i}\\
\mathrm{Sing}_B(X)_n & \xrightarrow{F} & N(SC)_n,
\end{array}$$

for all s_i, d_i. Since the maps are unwieldy when in coordinates, we opt for heuristic arguments, neglecting to trace out notation-heavy diagrams.

Commutativity of the diagram on the left is immediate, as considering a simplex $|\Delta^{n-1}|$ as the ith face of a larger simplex $|\Delta^n|$ is the same as adding a step that is the identity map in the Hamiltonian path of vertices of $|\Delta^{n-1}|$. Similarly, observing that the image of the shortest path $v_{i-1}\to v_i\to v_{i+1}$ in $|\Delta^{n+1}|$, for $v_i=(0,\dots,0,1,0,\dots,0)$ the ith standard basis vector, induced by an element $\gamma\colon|\Delta^{n+1}|\to X$ in $\mathrm{Sing}_B(X)_{n+1}$, is homotopic to the image of the shortest path $v_{i-1}\to v_{i+1}$ shows that the diagram on the right commutes. Since F is a natural transformation between the two functors $\mathrm{Sing}_B(X)$ and $N(SC)$, it is a functor on the functors as simplicial sets.

Remark 1.9.1. The particular choice of X did not seem to play a large role in the arguments above. However, the stratifying map $\widetilde{h}\colon X\to B$ has image sitting inside SC, the nerve of which is the target of F, and every morphism in $\mathrm{Sing}_B(X)$ can be interpreted as a relation in $B\subseteq SC$ (both were necessary for the commutativity of the diagrams). Hence it is not unreasonable to expect a similar functor $\mathrm{Sing}_A(X)\to N(A')$ may exist for a stratified space $X\to A\subseteq A'$.

1.10 Enriched and straightened categories

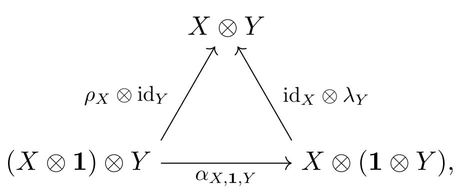
2018-06-12

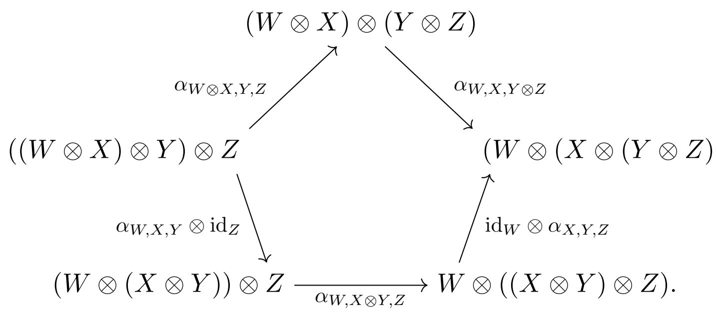
Keywords: *monoidal category, enriched category, weakly enriched category, bicategory, topological category, pseudo-functor, lax functor, cartesian morphism, fibered category, cleavage, straightening, unstraightening*

Definition 1.10.1. A category $\mathcal{C}$ is *monoidal* if it is accompanied by

- a functor $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,
- an object $\mathbf{1} \in \mathrm{Obj}(\mathcal{C})$, and
- isomorphisms
  - $\alpha_{X,Y,Z} \in \mathrm{Hom}_{\mathcal{C}}((X \otimes Y) \otimes Z, X \otimes (Y \otimes Z))$,
  - $\lambda_X \in \mathrm{Hom}_{\mathcal{C}}(\mathbf{1} \otimes X, X)$, and
  - $\rho_X \in \mathrm{Hom}_{\mathcal{C}}(X \otimes \mathbf{1}, X)$,

for all $X, Y, Z, W \in \mathrm{Obj}(\mathcal{C})$, such that $\otimes$ is unital and α is associative over $\otimes$. That is, the diagrams below commute.





"$\otimes$ is unital"

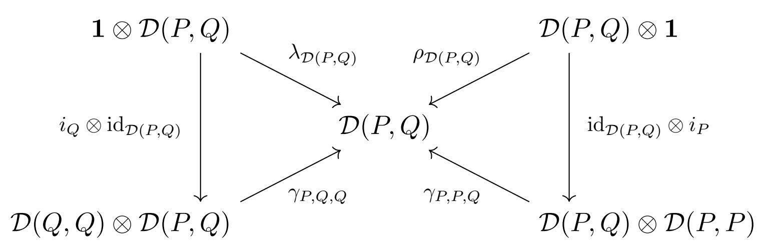
"α is associative over $\otimes$"

Definition 1.10.2. Let $\mathcal{C}$ be monoidal as above. A category $\mathcal{D}$ is *enriched over* $\mathcal{C}$ if it is accompanied by

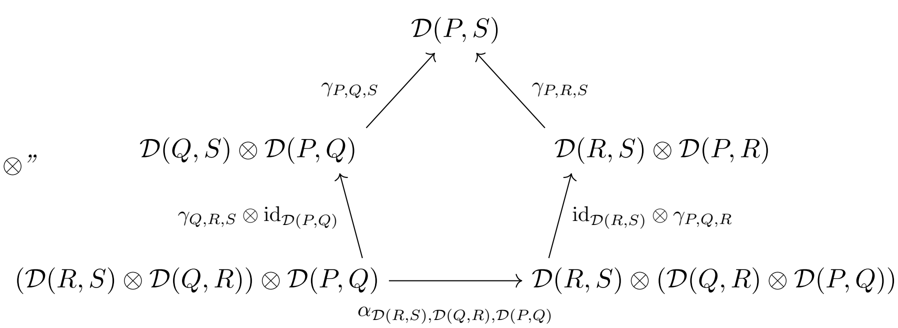
- an object $\mathcal{D}(P, Q) \in \mathrm{Obj}(\mathcal{C})$ for every $P, Q \in \mathrm{Obj}(\mathcal{D})$, and
- morphisms
  - $\gamma_{P,Q,R} \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{D}(Q, R) \otimes \mathcal{D}(P, Q), \mathcal{D}(P, R))$, and
  - $i_P \in \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \mathcal{D}(P, P))$,

for all $P, Q, R, S \in \mathrm{Obj}(\mathcal{D})$, such that γ is unital and associative over $\otimes$. The category $\mathcal{D}$ is *weakly enriched over* $\mathcal{C}$ if γ is unital and associative over $\otimes$ up to homotopy. That is, the diagrams below commute for $\mathcal{D}$ enriched, and commute up to homotopy for $\mathcal{D}$ weakly enriched.

"γ is unital"



"γ is associative over $\otimes$"



Definition 1.10.3. A topological space X is *compactly generated* if its basis of topology of closed sets is given by continuous images of compact Hausdorff spaces K whose preimages are closed in K. A topological space is *weakly Hausdorff* if the continous image of every compact Hausdorff space is closed in X.

We write $\mathcal{CG}$ for the category of compactly generated and weakly Hausdorff spaces. This is a monoidal category with the usual product of topological spaces.

Example 1.10.4. Here are some examples of enriched categories.

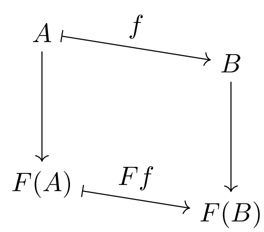
- A *topological category* is a category enriched over $\mathcal{CG}$.
- A *bicategory*, or *weak 2-category*, is a category weakly enriched over $\mathcal{C}at$, the category of small categories.

Definition 1.10.5. Let $\mathcal{C}, \mathcal{D}$ be bicategories. An assignment $F\colon \mathcal{C} \to \mathcal{D}$ is a *pseudofunctor* when it has

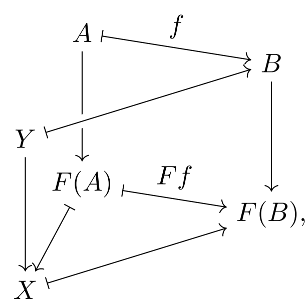
- an object $F(X) \in \mathrm{Obj}(\mathcal{D})$,
- a functor $F(X,Y)\colon \mathcal{C}(X,Y) \to \mathcal{D}(F(X), F(Y))$, and
- invertible 2-morphisms
  - $F(\mathrm{id}_X)\colon \mathrm{id}_X \Rightarrow F(X,X)(\mathrm{id}_X)$, and
  - $F(X,Y,Z)(f,g)\colon F(Y,Z)(g) \circ F(X,Y)(f) \Rightarrow F(X,Z)(g \circ f)$,

for all $X, Y, Z \in \mathrm{Obj}(C)$, such that $F(X,Y)$ is unital and associative over composition. The assignment F is a *lax functor* when the last two morphisms are not necessarily invertible.

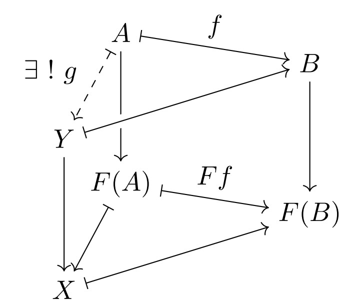
Definition 1.10.6. Let $\mathcal{C}, \mathcal{D}$ be categories and $F\colon \mathcal{C} \to \mathcal{D}$ a functor. A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ is F-*cartesian* if



fits into



then



commutes for some unique $g \in \mathrm{Hom}_{\mathcal{C}}(A,Y)$ (all the vertical arrows are F).

This definition can be rephrased in the language of simplicial sets: the morphism f is F-cartesian if whenever $Ff = d_1\Delta^2$ for some $\Delta^2 \in \mathcal{D}_2$, then every $\Lambda^2 \in \mathcal{C}$ with $\Lambda^2_1 = f$ and $F\Lambda^2_0 = d_0\Delta^2$ can be filled in by g with $Fg = d_2\Delta^2$.

Definition 1.10.7. Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor.

- The category $\mathcal{C}$ is F-*fibered over* $\mathcal{D}$ if for every morphism $h \in \mathrm{Hom}_{\mathcal{D}}(U,V)$ and every $B \in \mathrm{Obj}(\mathcal{C})$ with $F(B) = V$, there is some F-cartesian $f \in \mathrm{Hom}_{\mathcal{C}}(-,B)$ with $Ff = h$.
- A *cleavage* of an F-fibered category $\mathcal{C}$ is a class of cartesian morphisms K in $\mathcal{C}$ such that for every morphism $h \in \mathrm{Hom}_{\mathcal{D}}(U,V)$ and every $B \in \mathrm{Obj}(\mathcal{C})$ with $F(B) = V$, there is a unique F-cartesian $f \in K$ with $Ff = h$.
- A cleavage of $\mathcal{C}$ is a *splitting* if it contains all the the identity morphisms and is closed under composition.

If $\mathcal{C}$ is F-fibered over $\mathcal{D}$ and $\mathcal{C}'$ is F'-fibered over $\mathcal{D}$, then a functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}'$ is a *morphism of fibered categories* if $F = F' \circ \mathcal{F}$ and $\mathcal{F}f$ is F'-cartesian whenever f is F-cartesian.

Theorem 1.10.8. Let $\mathcal{C}$ be F-fibered over $\mathcal{D}$.

- Every cleavage of $\mathcal{C}$ defines a pseudofunctor $\mathcal{D} \to \mathcal{C}at$.
- Every pseudofunctor $\mathcal{D} \to \mathcal{C}at$ defines an F'-fibered category $\mathcal{C}'$ with a cleavage over $\mathcal{D}$.

The above result follows from sections 3.1.2 and 3.1.3 of Vistoli. Theorem 2.2.1.2 of Lurie generalizes this and provides an equivalence between the category of fibered simplicial sets over $S \in \text{sSet}$ and the category of functors $\text{sCat} \to \text{sSet}$. The forward direction is called *straightening* and the backward direction is called *unstraightening*.

References: nLab (articles "Monoidal category," "enriched category," and "pseudofunctor."), Strickland (The category of CGWH spaces), Vistoli (Notes on Grothendieck topologies, Chapter 3), Noohi (A quick introduction), Lurie (Higher Topos Theory, Section 2.2)

2 Algebraic geometry

2.1 The canonical bundle of projective space and hypersurfaces

2016-03-01

Keywords: *bundle, canonical bundle, hypersurface, sheaf, sheaf of regular functions, Serre twist*

Let $\mathbf{P}^n$ be projective n-space with coordinates $[x_0 : \cdots : x_n]$. Cover $\mathbf{P}^n$ with affine pieces $U_i = \{x_i \neq 0\}$, each of which are $\mathbf{A}^n$, in coordinates $(y_1, \dots, y_n)$, where $y_j = x_j/x_i$. Recall that the *canonical bundle* of $\mathbf{P}^n$ is the n-fold wedge of the cotangent bundle of $\mathbf{P}^n$, or $\omega_{\mathbf{P}^n} = \bigwedge^n T^*_{\mathbf{P}^n}$. The canonical bundle for an arbitrary variety is defined analogously.

Definition 2.1.1. Let X be a projective n-dimensional variety. The *sheaf of regular functions* on X is $\mathcal{O}_X$, with $\mathcal{O}_X(U) = \{f/g \ : \ f, g \in k[x_1, \dots, x_n]/I(X), g \neq 0\}$, and the restriction maps are function restriction.

There is a natural grading on $\mathcal{O}_X$, given by $\deg(f) - \deg(g)$. A shift in the grading may be applied, called a *Serre twist*, to get a differently graded (but isomorphic) module: for $\varphi \in \mathcal{O}_X$ with $\deg(\varphi) = k$, set $\varphi \in \mathcal{O}_X(\ell)$ to have $\deg(\varphi) = k - \ell$.

Let $\alpha = dy_1 \wedge \cdots \wedge dy_n \in \omega_{\mathbf{P}^n}$, which is well-defined on all of U_i. We claim this is well-defined on all of $\mathbf{P}^n$. We check this on the overlap $U_0 \cap U_n$ (for nicer notation), but the approach is analogous for $U_i \cap U_j$.

$$U_0 = \{(y_1, \dots, y_n) \ : \ y_i = x_i/x_0\} \qquad y_i = \frac{z_{i+1}}{z_i} \qquad dy_i = \frac{z_1 dz_{i+1} - z_{i+1} dz_1}{z_1^2}$$

$$U_n = \{(z_1, \dots, z_n) \ : \ z_i = x_{i-1}/x_n\} \qquad y_n = \frac{1}{z_1} \qquad dy_n = \frac{-dz_1}{z_1^2}$$

Therefore

$$\begin{aligned}
\alpha &= dy_1 \wedge \cdots \wedge dy_n \\
&= \frac{z_1 dz_2 - z_2 dz_1}{z_1^2} \wedge \cdots \wedge \frac{z_1 dz_n - z_n dz_1}{z_1^2} \wedge \frac{-dz_1}{z_1^2} \\
&= \frac{dz_2}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_1} \wedge \frac{-dz_1}{z_1^2} \\
&= \frac{(-1)^n}{z_1^{n+1}} dz_1 \wedge \cdots \wedge dz_n.
\end{aligned}$$

Since the transition function has a pole of order $n+1$ when $z_1 = 0$, which happens when $x_0 = 0$, we have that α has a pole of order $n+1$ at ∞. Therefore $\omega_{\mathbf{P}^n} \cong \mathcal{O}_{\mathbf{P}^n}(-n-1)$.

Let $X \subset \mathbf{P}^n$ be a smooth hypersurface defined by a degree d equation $F(x_0, \dots, x_n) = 0$. On the affine piece U_0 this becomes $f(y_1, \dots, y_n) = F(1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})$ with $y_i = x_i/x_0$. The total derivative is

$$\frac{\partial f}{\partial y_1} dy_1 + \cdots + \frac{\partial f}{\partial y_n} dy_n = \sum_{i=1}^n \frac{\partial f}{\partial y_i} dy_i = 0,$$

and since X is smooth, the terms never all vanish at the same time. Let $V_i = \{\frac{\partial f}{\partial y_i} \neq 0\}$, and set

$$\beta_i = \frac{(-1)^{i-1}}{\partial f/\partial y_i} dy_1 \wedge \cdots \wedge \widehat{dy_i} \wedge \cdots \wedge dy_n \in \omega_X,$$

which is well-defined on all of $V_i \subset U_0$. We claim that the choice of V_i does not matter, and indeed, assuming $i < j$,

$$\begin{aligned}
\beta_j &= \frac{(-1)^{j-1}}{\partial f/\partial y_j} dy_1 \wedge \cdots \wedge \widehat{dy_j} \wedge \cdots \wedge dy_n \\
&= \frac{(-1)^{j-1+i-1} dy_i}{\partial f/\partial y_j} \wedge dy_1 \wedge \cdots \wedge \widehat{dy_i} \wedge \cdots \wedge \widehat{dy_j} \wedge \cdots \wedge dy_n \\
&= \frac{(-1)^{j-1+i-1} \frac{-1}{\partial f/\partial y_i} \left(\frac{\partial f}{\partial y_1} dy_1 + \cdots + \widehat{\frac{\partial f}{\partial y_i} dy_i} + \cdots + \frac{\partial f}{\partial y_n} dy_n \right)}{\partial f/\partial y_j} \wedge dy_1 \wedge \cdots \wedge \widehat{dy_i} \wedge \cdots \wedge \widehat{dy_j} \wedge \cdots \wedge dy_n \\
&= \frac{(-1)^{j-1+i-1+1} \frac{1}{\partial f/\partial y_i} \cdot \frac{\partial f}{\partial y_j} dy_j}{\partial f/\partial y_j} \wedge dy_1 \wedge \cdots \wedge \widehat{dy_i} \wedge \cdots \wedge \widehat{dy_j} \wedge \cdots \wedge dy_n \\
&= \frac{(-1)^{j-1+i-1+1+j-2}}{\partial f/\partial y_i} dy_1 \wedge \cdots \wedge \widehat{dy_i} \wedge \cdots \wedge dy_n \\
&= \frac{(-1)^{i-1}}{\partial f/\partial y_i} dy_1 \wedge \cdots \wedge \widehat{dy_i} \wedge \cdots \wedge dy_n \\
&= \beta_i.
\end{aligned}$$

Hence β_i is well-defined on all of U_0, and we call it simply β. Next we claim it is well-defined on all of X. Again we only check on the overlap of $U_0 \cap U_n$. On the affine piece U_n this becomes $g(z_1, \ldots, z_n) = F(\frac{x_0}{x_n}, \ldots, \frac{x_{n-1}}{x_n}, 1) = f(\frac{z_2}{z_1}, \ldots, \frac{z_n}{z_1}, \frac{1}{z_1})$ with $z_i = x_{i-1}/x_n$. We employ the chain rule $\frac{\partial f}{\partial y_i} = \frac{\partial f}{\partial z_j} \frac{\partial z_j}{\partial y_i}$ and the results above to find that

$$\begin{aligned}
\beta &= \frac{(-1)^{i-1}}{\partial f/\partial y_i} dy_1 \wedge \cdots \wedge \widehat{dy_i} \wedge \cdots \wedge dy_n \\
&= \frac{(-1)^{i-1}}{\partial f/\partial z_j \cdot \partial z_j/\partial y_i} \frac{z_1 dz_2 - z_2 dz_1}{z_1^2} \wedge \cdots \wedge \widehat{dy_i} \wedge \cdots \wedge \frac{z_1 dz_n - z_n dz_1}{z_1^2} \wedge \frac{-dz_1}{z_1^2} \\
&= \frac{(-1)^{i-1}}{\partial f/\partial z_j \cdot \partial z_j/\partial y_i} \frac{(-1)^{n-1}}{z_1^n} dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_n \\
&= \frac{(-1)^{i+n}}{\left(\frac{1}{z_1}\right)^{d-1} (c + \cdots)\, z_1^n} dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_n \\
&= \frac{(-1)^{i+n}}{z_1^{n-d+1} (c + \cdots)} dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_n,
\end{aligned}$$

where c does not contain z_1 as a factor. This comes from expressing f in terms of the z_is and factoring. Since the transition function has a pole of order $n - d + 1$ when $z_1 = 0$, which happens when $x_0 = 0$, we have that β has a pole of order $n - d + 1$ at ∞. Therefore $\omega_X \cong \mathcal{O}_X(-n + d - 1)$.

References: Griffiths and Harris (Principles of Algebraic Geometry, Chapter 1.2)

2.2 The Hodge decomposition, diamond, and Euler characteristics

2016-03-31

Keywords: *sheaf, differential forms, structure sheaf, Hodge number, Hodge diamond, Hodge decomposition, symmetry, Euler characteristic, hypersurface*

Recall the *sheaf of r-differential forms* Ω_X^r on X (with $\Omega_X^r(U) = \{f dx_{i_1} \wedge \cdots \wedge dx_{i_r} \ :\ f \text{ is well-defined on } U\}$ and such sums) and the *structure sheaf* $\mathcal{O}_X$ on X (with $\mathcal{O}_X(U) = \{f/g \ :\ f, g \in k[U],\ g \neq 0 \text{ on } U\}$). Then we may consider the *sheaf cohomology* of X, with values in Ω_X^r or $\mathcal{O}_X$.

Definition 2.2.1. Let X be a smooth manifold of dimension n. The (p,q)*th Hodge number* is $h^{p,q} = \dim(H^{p,q})$, where $H^{p,q} = H^q(X, \Omega_X^p)$. These numbers are arranged in a *Hodge diamond* as below.

$$
\begin{array}{ccccccccc}
 & & & & h^{0,0} & & & & \\
 & & & h^{1,0} & & h^{0,1} & & & \\
 & & h^{2,0} & & h^{1,1} & & h^{0,2} & & \\
 & \iddots & & & & & & \ddots & \\
h^{n,0} & & h^{n-1,1} & & \cdots & & h^{1,n-1} & & h^{0,n} \\
 & \ddots & & & & & & \iddots & \\
 & & & h^{n,n-1} & & h^{n-1,1} & & & \\
 & & & & h^{n,n} & & & &
\end{array}
$$

The Hodge diamond has a lot of repetition - by complex conjugation, we get that $h^{p,q} = h^{q,p}$, so it is symmetric about its vertical axis. By the Hard Lefschetz theorem (or the Hodge star operator, or Poincare duality), we get that $h^{p,q} = h^{n-q,n-p}$, so it is symmetric about its horizontal axis.

Proposition 2.2.2. Let X be a Kähler manifold (note that all smooth projective varieties are Kähler) of dimension n. Then the cohomology groups of X decompose as

$$H^k(X, \mathbf{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

for all $0 \leqslant k \leqslant 2n$. This is called the *Hodge decomposition* of X.

This decomposition immediately gives all the Hodge numbers for $\mathbf{P}^n$, knowing its cohomology. For a manifold of complex dimension n, there are several numbers and polynomials that may be defined. These are:

$$\chi_{top}(X) = \sum_{i=1}^{2n} (-1)^i \dim(H^i(X, \mathbf{C})) \qquad \text{the (topological) Euler characteristic}$$

$$\chi^p(X) = \sum_{q=0}^{n-1} (-1)^q h^{p,q} \qquad \text{the chi-}p\text{ characteristic}$$

$$\chi_y(X) = \sum_{p=0}^{n-1} \chi^p y^p \qquad \text{the chi-}y\text{ characteristic}$$

Note the Euler characteristic is the alternating sum of the rows of the Hodge diamond, and the chi-p characteristic is the alternating sum of the left-right diagonals of the diamond.

Example 2.2.3. In the case X is a hypersurface in projective n-space $\mathbf{P}^n$ defined by a degree d polynomial,

$$\chi_y = [z^n] \frac{1}{(1+zy)(1-z)^2} \cdot \frac{(1+zy)^d - (1-z)^d}{(1+zy)^d + y(1-z)^d}.$$

Since every row except the middle row of the Hodge diamond of a hypersurface is known (as it comes from the Hodge diamond of $\mathbf{P}^n$ by the Lefschetz hyperplane theorem), this expression gives all the unknown numbers. This particular formula is a simplification of Theorem 22.1.1 in Hirzebruch, which itself comes from the Riemann–Roch theorem.

References: Huybrechts (Complex Geometry: An Introduction, Chapters 3.2, 3.3), Hirzebruch (Topological Methods in Algebraic Geometry, Appendix 1, Section 22)

2.3 What is a scheme?

2016-08-11

Keywords: *scheme, affine scheme, Spec, sheaf, structure sheaf, Zariski, localization, locally ringed space*

This is from a problem session at the 2016 West Coast Algebraic Topology Summer School (WCATSS) at The University of Oregon. Thanks to Tyler Lawson for explaining the material.

Definition 2.3.1. *Affine schemes* are the category Ring^{op}. An object $R \in \text{Ring}$ becomes an object $\text{Spec}(R)$ in affine schemes, and a ring map $R \to S$ becomes a map $\text{Spec}(S) \to \text{Spec}(R)$, where Spec denotes the set of prime ideals.

We try to think of $Spec(R)$ as a geometrical object.

Example 2.3.2. Let k be a field and consider the ring

$$R = k[x_1, \dots, x_n]/(f_1(x_1, \dots, x_n), \dots, f_r(x_1, \dots, x_n)).$$

$\text{Spec}(R)$ is supposed to be a substitute for the set of solutions to a system of equations

$$\begin{gathered} f_1(x_1, \dots, x_n) = 0, \\ \vdots \\ f_r(x_1, \dots, x_n) = 0. \end{gathered}$$

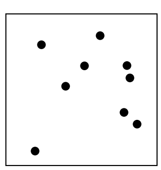
The scheme $\text{Spec}(R)$ has a more precise definition. It consists of a set, a topology, and a sheaf.

1. **Set:** The underlying set of the scheme $\text{Spec}(R)$ is the set of prime ideals of R. For example:
   · if $R = \mathbf{C}[x]$, then the prime ideals are $(x - \alpha)$ and (0);
   · if $R = \mathbf{C}[x, y]$, then the prime ideals are $(x - \alpha, y - \beta)$, irreducible polynomials $(f(x, y))$, and (0).

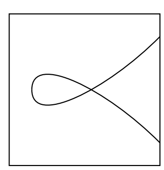
2. **Topology:** For every ideal $I \subset R$, the set $V(I) = \{P \subset R \text{ prime}, P \supset I\}$ is a closed set. Note that

$$\bigcup_{n=1}^{N} V(I_n) = V\left(\bigcap_{n=1}^{N} I_n\right) \qquad \text{and} \qquad \bigcap_{\alpha \in I} V(I_\alpha) = V\left(\sum_{\alpha \in A} I_A\right).$$

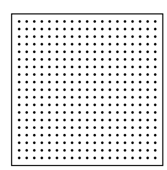
   Geometrically, the closed sets are sets of points where one or more identities (like $f(x) = 0$) can hold. For example, if $R = \mathbf{C}[x]$, then we have three different closed set types: $\text{Spec}(C[x])$, $\emptyset$, or a finite union of $(x - \alpha_1, \dots, x - \alpha_n)$. Solutions to equations can be one of the following types below.



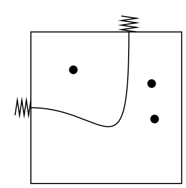
finite union of points



1-dimensional



general point



combination

3. **Sheaf:** Let X be a set with a topology. $\mathcal{O}_X$ is the sheaf for which:
   · to each open set $U \subseteq X$ we get a ring $\mathcal{O}_X(U)$;
   · to each containment $V \subseteq U \subseteq X$ of open sets, there exists a restriction map $\text{res}_{UV} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$;
   · the restriction maps are compatible, in the sense that $\text{res}_{VW} \circ \text{res}_{UV} = \text{res}_{UW}$.
   This is called the *structure sheaf* of X.

Say R is our ring, $\text{Spec}(R)$ our set of primes, and we have some open set $U \subseteq \text{Spec}(R)$. We like to think of it in the following way:
· elements of R are functions;
· elements of $\text{Spec}(R)$ are points where we can evaluate a function $f \in P$ (or where the function vanishes);
· subsets $S \subset R$ are the sets $\{f \in R \ : \ f \text{ only vanishes at points outside } U\}$.
Note that S is closed under multiplication. We *localize* R at S to get a set

$$S^{-1}R = \left\{ \left[\frac{f}{s}\right] \ : \ f \in R, s \in S \right\},$$

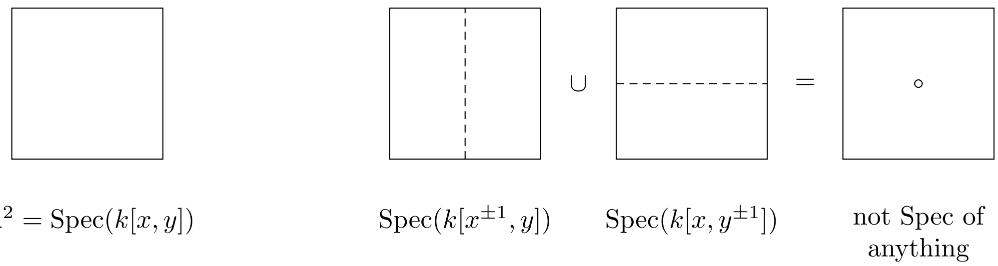
for which $\mathcal{O}_X(U) = S^{-1}R$ (good enough for today's purposes). Now we have a triple $(\text{Spec}(R), \tau, \mathcal{O}_X)$, for τ the Zariski topology, which we call a *locally ringed space*.

Definition 2.3.3. A *scheme* is a space X with a topology and a sheaf of rings that is locally isomorphic to $\mathrm{Spec}(R)$.

Since the sheaf has the space X and the topology (through the open sets) encoded in it, we may think of a scheme as a special type of sheaf. Also, isomorphism is meant in the category of locally ringed spaces.

Proposition 2.3.4. Morphisms of schemes $\mathrm{Spec}(R) \to \mathrm{Spec}(S)$ are the same as ring maps $S \to R$.

Example 2.3.5. In the Zariski topology, take $U \subseteq \mathrm{Spec}(k[x,y])$. Locally U looks like it is covered by rings, though that may not be the case globally. Indeed:



Example 2.3.6. Consider projective space $\mathbf{P}^2$, where $[x:y:z] = [\lambda x : \lambda y : \lambda z]$. We may write

$$\begin{array}{ccccccc} \mathbf{P}^2 & = & U_0 & \cup & U_1 & \cup & U_2. \\ & & [1:y:z] & & [x:1:z] & & [x:y:1] \\ & & \mathrm{Spec}(k[y,z]) & & \mathrm{Spec}(k[x,z]) & & \mathrm{Spec}(k[x,y]) \end{array}$$

How can we express $U_0 \cap U_1$? This is left as an exercise.

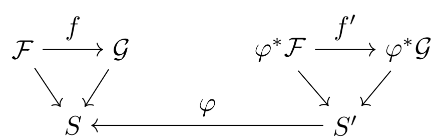
2.4 Morphisms of schemes

2016-08-13

Keywords: *scheme, sheaf, fiber product*

This is from discussions at the 2016 West Coast Algebraic Topology Summer School (WCATSS) at The University of Oregon. Thanks to Zijian Yao for explaining the material.

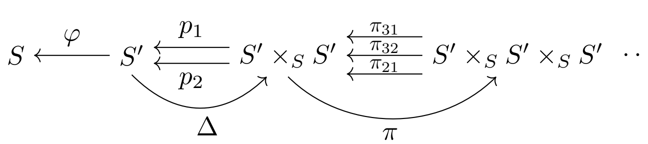
Consider a morphism of schemes $\varphi : S' \to S$ and coherent sheaves $\mathcal{F}, \mathcal{G}$ over S. Consider also a map of sheaves $f : \mathcal{F} \to \mathcal{G}$ and a map f' between the pullbacks of $\mathcal{F}$ and $\mathcal{G}$, as described by the diagram below.



There are two natural questions to ask.

1. When is $f' = \varphi^* f$?
2. If we start with $\mathcal{G}'$ over S', when is $\mathcal{G}' = \varphi^*\mathcal{G}$?

To answer these questions, consider fiber products of schemes and projections from them, as given below.



Remark 2.4.1. If **1.** is true, then $p_1^*(f') = p_2^*(f')$. If the previous statement is an equivalence, then φ is a *morphism of descent*.

Remark 2.4.2. If **2.** is true, then there exists $\alpha : p_1^*(\mathcal{G}') \to p_2^*(\mathcal{G}')$ such that $\pi_{32}^*(\alpha)\pi_{21}^*(\alpha) = \pi_{31}^*(\alpha)$ and $\pi^*(\Delta) = \alpha$. If the previous statement is an equivalence, then φ is *effective*.

2.5 Serre duality on schemes

2017-02-24

Keywords: *duality, scheme, sheaf, dualizing sheaf, delta functor, effacable functor, local ring, Cohen–Macaulay ring*

This post goes through the statement and proof of Serre duality for arbitrary projective schemes, as presented in Chapter III.7 of Hartshorne. Only the necessary tools and definitions to prove the statement are introduced.

Recall a *scheme* is a topological space X and a sheaf of rings $\mathcal{O}_X$ such that for every open set $U \subset X$, $\mathcal{O}_X(U) \cong \text{Spec}(R)$ for some ring R. Its dimension is its dimension as a topological space. A *projective scheme* is a scheme where $X \subset \mathbf{P}^n$. A *sheaf* (or *scheme*) *over a scheme* X is a sheaf (or scheme) Y and a morphism $Y \to X$. Recall also the *sheafification* $\widetilde{\mathcal{F}}$ of a presheaf $\mathcal{F}$.

Definition 2.5.1. Let $\mathcal{F}$ be a sheaf over a projective scheme X. Then $\mathcal{F}$ is

proper	if it is the image of a proper morphism (separated, finite type, universally closed),
quasi-coherent	if there exists a cover $\{U_i = \text{Spec}(A_i)\}$ of X such that $\mathcal{F}\|_{U_i} = \widetilde{M_i}$ for some A_i-module M_i,
coherent	if it is quasi-coherent and each M_i is finitely-generated as an A_i-module,
locally free	if for every $x \in X$, there exists $U \ni x$ open such that $\mathcal{F}\|_U = \bigoplus_{i\in I} \mathcal{O}_X\|_U$,
very ample	if there is an immersion $i : X \to \mathbf{P}^n$ for some n such that $i^*\mathcal{O}(1) \cong \mathcal{F}$.

Often we say $\mathcal{F}$ is very ample if it has "enough sections," as $\mathbf{P}^n$ has many sections.

Remark 2.5.2. Recall some basic definitions of the Ext functor. Let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules, and $\mathcal{L}$ a locally free sheaf of finite rank. Then:

1. $\text{Ext}^i(\mathcal{O}_X, \mathcal{F}) \cong H^i(X, \mathcal{F})$ for all $i \geqslant 0$ — Proposition III.6.3
2. $\text{Ext}^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \cong \text{Ext}^i(F, \mathcal{L}^\vee \otimes \mathcal{G})$ — Proposition III.6.7
3. $\text{Ext}^i_{\mathcal{O}_X}(\mathcal{F}_x, \mathcal{G}_x) \cong \mathcal{E}xt(\mathcal{F}, \mathcal{G})_x$ — Proposition III.6.8
4. $\text{Ext}^i(\mathcal{F}, \mathcal{G}(q)) \cong \Gamma(X, \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}(q))$ — Proposition III.6.9
5. $\mathcal{E}xt^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \cong \mathcal{E}xt^i(F, \mathcal{L}^\vee \otimes \mathcal{G}) \cong \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \otimes \mathcal{L}^\vee$
6. $\mathcal{E}xt^0(\mathcal{O}_X, \mathcal{F}) \cong \mathcal{F}$
7. $\mathcal{E}xt^i(\mathcal{O}_X, \mathcal{F}) \cong 0$ for all $i > 0$

Recall that a *local ring* of a scheme X is $\mathcal{O}_{X,x}$ for $x \in X$. It is equivalently a ring with a unique maximal left or right ideal. A *regular* local ring is a local ring R whose maximal ideal is generated by $\dim(R)$ elements.

Preliminary definitions and lemmas

Let A, B be abelian categories (recall this means kernels and cokernels exist).

Definition 2.5.3. A *δ-functor* between A and B is a collection of functors $T^i : A \to B$ that generalize derived functors, in the sense that $R^i\mathcal{F} = T^i$. A δ-functor is *universal* if for any other δ-functor U, there is a natural transformation $f : T^0 \to U^0$ that induces a unique collection of morphisms $f^{i\geqslant 0} : T^i \to U^i$ that extend f.

See Weibel for a more thorough definition (and Grothendieck for the original setting). These functors may be covariant or contravariant, homological or cohomological. Note that δ-functors are unique up to isomorphism.

Definition 2.5.4. Let $F : A \to B$ be a functor. F is *effaceable* if for every $X \in A$ there exists a monomorphism $u \in \text{Hom}_A(X, Y)$ such that $F(u) = 0$. Similarly, F is *coeffaceable* if for every $X \in A$ there exists an epimorphism $v \in \text{Hom}_A(Y, X)$ such that $F(v) = 0$.

Lemma 2.5.5. If a covariant (or contravariant) cohomological δ-functor is effaceable for every $i > 0$, then it is universal. Similarly, if a covariant (or contravariant) homological δ-functor is coeffaceable for every $i > 0$, then it is universal.

This appears as Proposition II.2.2.1 in Grothendieck and Exercise 2.4.5 in Weibel. Now let $\mathcal{F}$ be a sheaf over a projective scheme X.

Lemma 2.5.6. (Theorem III.5.2 in Hartshorne) If $\mathcal{F}$ is coherent, there is $q \gg 0$ such that $H^i(X, \mathcal{F}(q)) = 0$ all $i > 0$.

Definition 2.5.7. The *dualizing sheaf* of X is a coherent sheaf ω_X° and a *trace map* $t : H^n(X, \omega_X^\circ) \to k$ such that the isomorphism $\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \to H^n(X, \mathcal{F})^\vee$ is induced by the natural pairing

$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X^\circ)$$

composed with t.

Lemma 2.5.8. (Corollary II.5.18 in Hartshorne) If $\mathcal{F}$ is coherent, then it is a quotient of $\bigoplus_{i=1}^N \mathcal{O}_X(-q)$ for $q \gg 0$.

Next we recall some ring theory. Let A be a ring and M an A-module.

Definition 2.5.9. A sequence $a_1, \dots, a_n \in M$ is M*-regular* if a_i is not a zero divisor of $M/(a_1, \dots, a_{i-1})M$ and $M \neq (a_1, \dots, a_i)M$ for all i. The *depth* of M is the maximal length of an M-regular sequence of elements in some maximal ideal $\mathfrak{m} \leqslant M$. A local Noetherian ring is *Cohen–Macaulay* if $\mathrm{depth}(A) = \dim(A)$, where dimension is Krull dimension (maximal length of prime ideal chains). A scheme X is *Cohen–Macaulay* if every point $x \in X$ has a neighborhood U such that the local ring $\mathcal{O}_X(U)$ is Cohen–Macaulay.

Lemma 2.5.10. Let A be a regular local ring of dimension n and M, N be A-modules. Then:

1. $\mathrm{pd}(M) \leqslant n$ iff $\mathrm{Ext}^i(M, N) = 0$ for all $i > n$ — Proposition III.6.10A
2. $\mathrm{pd}(M) + \mathrm{depth}(M) = n$ if M is f.g. — Proposition III.6.12A

Main theorem and proof

First we state the duality theorem for $X = \mathbf{P}^n$, without proof. Let ω_X be the canonical sheaf of X.

Theorem 2.5.11. (Theorem III.7.1 in Hartshorne) For $\mathcal{F}$ coherent over $\mathbf{P}^n$, for $i \geqslant 0$ there are natural isomorphisms

$$\mathrm{Hom}(\mathcal{F}, \omega_X) \cong H^n(X, \mathcal{F})^\vee, \qquad \mathrm{Ext}^i(\mathcal{F}, \omega) \cong H^{n-i}(X, \mathcal{F})^\vee.$$

Now we give the duality theorem for an arbitrary projective scheme, going through the proof as in Hartshorne.

Theorem 2.5.12. (Theorem III.7.6 in Hartshorne) Let X be a projective scheme of dimension n such that $\mathcal{O}(1)$ is very ample. For $\mathcal{F}$ coherent,

$$\begin{aligned} \mathrm{Ext}^i(\mathcal{F}, \omega_X^\circ) \cong H^{n-i}(X, \mathcal{F})^\vee \quad &\iff \quad H^i(X, \mathcal{F}(-q)) = 0 \text{ for all } \mathcal{F} \text{ locally free, } i < n, q \gg 0, \\ &\iff \quad X \text{ is CM and equidimensional.} \end{aligned}$$

Proof: Natural maps $\mathrm{Ext}^i(\mathcal{F}, \omega_X^\circ) \to H^{n-i}(X, \mathcal{F})^\vee$ exist, as $\mathrm{Ext}^i(-, \omega_X^\circ) : \mathrm{Coh}(X) \to \mathrm{Mod}$ is a coeffaceable δ-functor for every $i > 0$, hence universal by Lemma 2.5.5. Indeed, by Lemma 2.5.8, we have a surjection

$$\underbrace{\bigoplus_{j=1}^N \mathcal{O}_X(-q)}_{\mathcal{E}} \xrightarrow{u} \mathcal{F} \to 0, \tag{2}$$

for which

$$\mathrm{Ext}^i(\mathcal{E}, \omega_X^\circ) = \bigoplus_{j=1}^N \mathrm{Ext}^i(\mathcal{O}_X(-q), \omega_X^\circ) = \bigoplus_{j=1}^N \mathrm{Ext}^i(\mathcal{O}_X, \omega_X^\circ(q)) = 0$$

for $i > 0$. The first equality was distributing Ext^i over the sum and the second was by applying property 2.5.2.2. Hence $\mathrm{Ext}^i(-, \omega_X^\circ)(u) = 0$ for $i > 0$, so the functor is coeffeaceable for $i > 0$, and so universal. By Definition 2.5.3 there exist maps generalizing the map Ext^0 from Definition 2.5.7.

First iff $\Leftarrow$*:* Since universal δ-functors are unique (up to isomorphism), we show $H^{n-i}(X, -)^\vee : \mathrm{Coh}(X) \to \mathrm{Mod}$ is also universal contravariant, which follows as it is coeffaceable for $i > 0$. Using the same sequence and sheaf as in equation (2), we have that

$$H^{n-i}(X, \mathcal{E}) = \bigoplus_{j=1}^N H^{n-i}(X, \mathcal{O}_X(-q)) = 0$$

whenever $n - i < n$ by hypothesis, or equivalently, when $i > 0$. The dual module is then also zero for $i > 0$, so we are done.

First iff $\Rightarrow$*:* Assume the hypothesis with index $n-i$ and a locally free sheaf $\mathcal{F}(-q)$ for $q \gg 0$, for which

$$\begin{aligned} H^i(X, \mathcal{F}(-q))^\vee &\cong \mathrm{Ext}^{n-i}(\mathcal{F}(-q), \omega_X^\circ) && \text{(hypothesis)} \\ &\cong \mathrm{Ext}^{n-i}(\mathcal{O}_X, \mathcal{F}^\vee \otimes \mathcal{O}_X(q) \otimes \omega_X^\circ) && \text{(property 2.5.2.2)} \\ &\cong H^{n-i}(X, (\mathcal{F}^\vee \otimes \omega_X^\circ) \otimes \mathcal{O}_X(q)). && \text{(property 2.5.2.1)} \end{aligned}$$

Tensoring with $\mathcal{O}_X(q)$ is twisting by q, and Lemma 2.5.6 says that $H^{n-i}(X, \mathcal{G}(q)) = 0$ for $\mathcal{G}$ coherent, for all $n-i > 0$, for q large enough. So for $i < n$ and q large enough $H^i(X, \mathcal{F}(-q))^\vee = 0$, and so its dual, the original cohomology group, is also trivial.

Second iff $\Leftarrow$*:* Embed $X \hookrightarrow \mathbf{P}^N$. As X is Cohen–Macaulay and equidimensional of dimension n, for $\mathcal{F}$ locally free on X, a stalk $\mathcal{F}_x$ of a closed point $x \in X$ has depth n. Also, $\mathcal{F}_x \subset \mathcal{O}_{\mathbf{P}^N,x}$, and $\mathcal{O}_{\mathbf{P}^n,x}$ is regular as $\mathbf{P}^N$ is smooth over k. By Lemma 2.5.10.2, we have that

$$\mathrm{pd}(\mathcal{F}_x) + n \leqslant \mathrm{pd}(\mathcal{O}_{\mathbf{P}^N,x}) + n = N,$$

so Lemma 2.5.10.1 and property 2.5.2.3 gives us that, for $i > N - n$,

$$\mathrm{Ext}^i(\mathcal{F}_x, -) = 0 \quad \Longrightarrow \quad \mathcal{E}xt^i(\mathcal{F}_x, -) = 0 \quad \Longrightarrow \quad \mathcal{E}xt^i(\mathcal{F}, -) = 0.$$

Applying Theorem 2.5.11, property 2.5.2.4, and letting the functor $\mathcal{E}xt^i(\mathcal{F}, -)$ act on $\omega_{\mathbf{P}^N}(q)$, we have

$$H^i(X, \mathcal{F}(-q))^\vee \cong \mathrm{Ext}^{N-i}_{\mathbf{P}^n}(\mathcal{F}, \omega_{\mathbf{P}^N}(q)) \cong \Gamma(\mathbf{P}^N, \mathcal{E}xt^{N-i}_{\mathbf{P}^N}(\mathcal{F}, \omega_{\mathbf{P}^n}(q))) \cong \Gamma(\mathbf{P}^N, 0) = 0$$

for $q \gg 0$ and $N - i > N - n$, or $i < n$. Since the dual is trivial, the cohomology group $H^i(X, \mathcal{F}(-q))$ is also trivial.

Second iff $\Rightarrow$*:* Omitted (techniques are similar to previous step, but use many others not used elsewhere). ■

Addendum

In certain cases, Serre duality holds for the canonical sheaf instead of the dualizing sheaf.

Proposition 2.5.13. For X a smooth projective variety over $k = \overline{k}$, $\omega_X^\circ \cong \omega_X$.

References: Grothendieck (Tohoku paper), Hartshorne (Algebraic Geometry, Section III.7), Weibel (An introduction to homological algebra, Section 2.1), Matsumura (Commutative algebra, Chapter 6)

2.6 The Fubini–Study metric and length in projective space

2017-03-05

Keywords: *metric, Fubini–Study, Hermitian, Riemannian, projective, distance, paths*

In this post we inspect how the Fubini–Study metric works and compute an example. Thanks to Professor Mihai Păun for helpful discussions. Recall that from projective space $\mathbf{P}^n$ there are natural maps

$$[x_0 : x_1 : \cdots : x_n] \xrightarrow{\varphi_i} \left(\frac{x_0}{x_i}, \ldots, \frac{\widehat{x_i}}{x_i}, \ldots, \frac{x_n}{x_i}\right)$$

for $i = 0, \ldots, n$. The maps land in $\mathbf{C}^n$ with coordinates $(z_1, z_2, \ldots, z_n)$. We use φ_0 as the main map, and conflate notation for objects in $\mathbf{P}^n$ and in $\mathbf{C}^n$ under φ_0. Most of this post deals with the $n = 2$ case.

The metric

The metric used on $\mathbf{P}^n$ is the *Fubini–Study* metric. Directly from Section 3.1 of Huybrechts, for $n = 2$ the associated differential 2-form and its image in $\mathbf{C}^2$ are

$$\begin{aligned}
\omega &= \frac{i}{2\pi}\partial\bar{\partial}\log\left(1 + \left|\frac{x_1}{x_0}\right|^2 + \left|\frac{x_2}{x_0}\right|^2\right), \\
\varphi_0(\omega) &= \frac{i}{2\pi}\partial\bar{\partial}\log\left(1 + |z_1|^2 + |z_2|^2\right) \\
&= \underbrace{\frac{i}{2\pi(1 + |z_1|^2 + |z_2|^2)^2}}_{\lambda_2} \sum_{k,\ell=1}^{2} \underbrace{(1 + |z_1|^2 + |z_2|^2)\delta_{k\ell} - \overline{z_k} z_\ell}_{\chi_{k\ell}} \, dz_k \wedge d\overline{z_\ell}.
\end{aligned} \tag{3}$$

A *Hermitian metric* on a complex manifold X may be described as a 2-tensor $h = g - i\omega$, where g is a *Riemannian metric* (also a 2-tensor) on the underlying real manifold and ω is a *Kähler form*, a 2-form. As in Lemma 3.3 of Voisin, the relationship between g and ω is given by

$$g(u, v) = \omega(u, Iv) = \omega(Iu, v), \tag{4}$$

where $I : T_xX \to T_xX$ is a tangent space endomorphism defined by

$$\begin{array}{ccc} I|_{T_x^{1,0}X} & = & i \cdot \mathrm{id}, \\ \frac{\partial}{\partial z_i} & \mapsto & i\frac{\partial}{\partial z_i}, \end{array} \qquad \begin{array}{ccc} I|_{T_x^{0,1}X} & = & -i \cdot \mathrm{id}, \\ \frac{\partial}{\partial \overline{z_i}} & \mapsto & -i\frac{\partial}{\partial \overline{z_i}}, \end{array}$$

as in Proposition 1.3.1 of Huybrechts.

An application

Let $\gamma : [0, 1] \to \mathbf{C}^2$ be a path, described as $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. Writing $\gamma_1 = u_1 + iv_1$, with $u_1 = \mathrm{Re}(\gamma_1)$ and $v_1 = \mathrm{im}(\gamma_1)$, the derivative of γ_1 with respect to t is given by

$$\frac{d\gamma_1}{dt} = \frac{du_1}{dt}\frac{\partial}{\partial x_1} + i\frac{dv_1}{dt}\frac{\partial}{\partial y_1} = \frac{du_1}{dt}\left(\frac{\partial}{\partial \overline{z_1}} + \frac{\partial}{\partial z_1}\right) + i\frac{dv_1}{dt}\left(\frac{\partial}{\partial \overline{z_1}} - \frac{\partial}{\partial z_1}\right) = \underbrace{\left(\frac{du_1}{dt} + i\frac{dv_1}{dt}\right)}_{\gamma_1'}\frac{\partial}{\partial \overline{z_1}} + \underbrace{\left(\frac{du_1}{dt} - i\frac{dv_1}{dt}\right)}_{\overline{\gamma}_1'}\frac{\partial}{\partial z_1},$$

and analogously for γ_2. Hence

$$\frac{d\gamma}{dt} = \overline{\gamma}_1'\frac{\partial}{\partial z_1} + \gamma_1'\frac{\partial}{\partial \overline{z_1}} + \overline{\gamma}_2'\frac{\partial}{\partial z_2} + \gamma_2'\frac{\partial}{\partial \overline{z_2}}. \tag{5}$$

The length of γ is

$$\int_0^1 \sqrt{g\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right)}\, dt = \int_0^1 \sqrt{\omega\left(\frac{d\gamma}{dt}, I\frac{d\gamma}{dt}\right)}\, dt,$$

using equation (4). Recall that the pairing of vectors with covectors is given by

$$(d\alpha_1 \wedge \cdots \wedge d\alpha_n)\left(\frac{\partial}{\partial \beta_1}, \ldots, \frac{\partial}{\partial \beta_n}\right) = \det \begin{bmatrix} d\alpha_1 \frac{\partial}{\partial \beta_1} & d\alpha_1 \frac{\partial}{\partial \beta_2} & \cdots & d\alpha_1 \frac{\partial}{\partial \beta_n} \\ d\alpha_2 \frac{\partial}{\partial \beta_1} & d\alpha_2 \frac{\partial}{\partial \beta_2} & \cdots & d\alpha_2 \frac{\partial}{\partial \beta_n} \\ \vdots & \vdots & \ddots & \vdots \\ d\alpha_n \frac{\partial}{\partial \beta_1} & d\alpha_n \frac{\partial}{\partial \beta_2} & \cdots & d\alpha_n \frac{\partial}{\partial \beta_n} \end{bmatrix} = \det\left(d\alpha_i \frac{\partial}{\partial \beta_j}\right),$$

for α_i, β_j a basis of the underlying real manifold (as in the previous post "Vector fields," 2016-10-10). The components of the vector (5) may be viewed as given in directions $z_1, \overline{z_1}, z_2, \overline{z_2}$, respectively, which also indicates how the coefficient functions $\chi_{k\ell}$ act on (5). Apply the definition of ω from equation (3), and note that we are always at the tangent space to the point $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, to get that

$$\begin{aligned}
&\omega\left(\frac{d\gamma}{dt}, I\frac{d\gamma}{dt}\right)\\
&= \lambda_2(\gamma(t)) \sum_{k,\ell=1}^{2} \chi_{k\ell}(\gamma(t)) dz_k \wedge d\overline{z_\ell} \left(\overline{\gamma}_1' \frac{\partial}{\partial z_1} + \gamma_1' \frac{\partial}{\partial \overline{z_1}} + \overline{\gamma}_2' \frac{\partial}{\partial z_2} + \gamma_2' \frac{\partial}{\partial \overline{z_2}}, i\overline{\gamma}_1' \frac{\partial}{\partial z_1} - i\gamma_1' \frac{\partial}{\partial \overline{z_1}} + i\overline{\gamma}_2' \frac{\partial}{\partial z_2} - i\gamma_2' \frac{\partial}{\partial \overline{z_2}}\right)\\
&= \lambda_2(\gamma(t)) \sum_{k,\ell=1}^{2} \chi_{k\ell}(\gamma(t)) \det \begin{bmatrix} \overline{\gamma}_k'(t) & i\overline{\gamma}_k'(t) \\ \gamma_\ell'(t) & -i\gamma_\ell'(t) \end{bmatrix}\\
&= \frac{(1+|\gamma_2(t)|^2)|\gamma_1'(t)|^2 - \overline{\gamma}_1(t)\gamma_2(t)\overline{\gamma}_1'(t)\gamma_2'(t) - \overline{\gamma}_2(t)\gamma_1(t)\overline{\gamma}_2'(t)\gamma_1'(t) + (1+|\gamma_1(t)|^2)|\gamma_2'(t)|^2}{\pi\left(1+|\gamma_1(t)|^2+|\gamma_2(t)|^2\right)^2}.
\end{aligned}$$

Unfortunately this expression does not simplify too much. In $\mathbf{P}^n$, with $\gamma = (\gamma_1, \dots, \gamma_n) : [0,1] \to \mathbf{C}^n$, we have that

$$g\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right) = \lambda_n(\gamma(t)) \sum_{k,\ell=1}^{n} \chi_{k\ell}(\gamma(t)) \det \begin{bmatrix} \overline{\gamma}_k'(t) & i\overline{\gamma}_k'(t) \\ \gamma_\ell'(t) & -i\gamma_\ell'(t) \end{bmatrix}.$$

An example

Here we compute the distance between two points in $\mathbf{P}^2$. Let γ be the straight line segment connecting $p = [p_0 : p_1 : p_2]$ and $q = [q_0 : q_1 : q_2]$. The word "straight" is used loosely, and means the segment may be parametrized as

$$\gamma(t) = [(1-t)p_0 + tq_0 : (1-t)p_1 + tq_1 : (1-t)p_2 + tq_2],$$

so $\gamma(0) = p$ and $\gamma(1) = q$. The image of γ under φ_0 and its derivative are given by

$$\varphi_0(\gamma(t)) = \left(\frac{(1-t)p_1 + tq_1}{(1-t)p_0 + tq_0}, \frac{(1-t)p_2 + tq_2}{(1-t)p_0 + tq_0}\right) = (\gamma_1, \gamma_2), \qquad \gamma_i' = \frac{q_i p_0 - q_0 p_i}{((1-t)p_0 + tq_0)^2}.$$

If, for example, $p = [1 : 1 : 0]$ and $q = [1 : 0 : 1]$, then

$$\text{length}(\gamma) = \frac{3}{4\pi} \int_0^1 \frac{1}{(t^2 - t + 1)^2}\, dt = \frac{9 + 2\pi\sqrt{3}}{18\pi}.$$

A further goal is to consider the path γ as lying on a projective variety, beginning with a complete intersection. This would allow some of the dz_i to be expressed in terms of other dz_j.

References: Huybrechts (Complex geometry, Section 3.1), Voisin (Hodge theory and complex algebraic geometry 1, Chapter 3.1), Wells (Differential analysis on complex manifolds, Chapter V.4)

2.7 Lengths of paths on projective varieties

2017-03-15

Keywords: *metric, Fubini–Study, curve, variety, projective, distance, paths, complete intersection*

This post contains calculations that continue on the ideas from the previous post "Fubini–Study metric," 2017-03-05. First we suppose that γ lies on a curve $C \subset \mathbf{P}^2$, with the curve defined as the zero locus of a polynomial P. Taking the derivative of P on $\mathbf{C}^2$ gives $P_{z_1} dz_1 + P_{z_2} dz_2 = 0$, which can be manipulated to give

$$\begin{aligned}
dz_2 &= \frac{-P_{z_1}}{P_{z_2}} dz_1, & \frac{\partial}{\partial z_2} &= \frac{-P_{z_2}}{P_{z_1}} \frac{\partial}{\partial z_1},\\
d\overline{z_2} &= \frac{-\overline{P_{z_1}}}{\overline{P_{z_2}}} d\overline{z_1}, & \frac{\partial}{\partial \overline{z_2}} &= \frac{-\overline{P_{z_2}}}{\overline{P_{z_1}}} \frac{\partial}{\partial \overline{z_1}}.
\end{aligned}$$

Using the above and equation (5) from the previous post, for $e = \frac{\partial}{\partial z_1} + \frac{\partial}{\partial \overline{z_1}} + \frac{\partial}{\partial z_2} + \frac{\partial}{\partial \overline{z_2}}$, we get

$$\frac{d\gamma}{dt} = \left(\overline{\gamma}_1' - \frac{P_{z_2}}{P_{z_1}}\overline{\gamma}_2'\right)\frac{\partial}{\partial z_1} + \left(\gamma_1' - \frac{\overline{P_{z_2}}}{\overline{P_{z_1}}}\gamma_2'\right)\frac{\partial}{\partial \overline{z_1}}$$

$$\left(\sum_{k,\ell=1}^{2}\chi_{k\ell}(\gamma)dz_k\wedge d\overline{z_\ell}\right)(e,e) = 1 + |\gamma_2|^2 + \frac{\overline{P_{z_1}}}{\overline{P_{z_2}}}\overline{\gamma}_1\gamma_2 + \frac{P_{z_1}}{P_{z_2}}\gamma_1\overline{\gamma}_2 + \left|\frac{P_{z_1}}{P_{z_2}}\right|^2\left(1+|\gamma_1|^2\right) = 1 + \left|\frac{P_{z_1}}{P_{z_2}}\right|^2 + \left|\frac{P_{z_1}}{P_{z_2}}\gamma_1 + \gamma_2\right|^2,$$

$$(dz_1\wedge d\overline{z_1})\left(\frac{d\gamma}{dt}, I\frac{d\gamma}{dt}\right) = \det\begin{bmatrix}\overline{\gamma}_1' - \frac{P_{z_2}}{P_{z_1}}\overline{\gamma}_2' & i\left(\overline{\gamma}_1' - \frac{P_{z_2}}{P_{z_1}}\overline{\gamma}_2'\right)\\ \gamma_1' - \frac{\overline{P_{z_2}}}{\overline{P_{z_1}}}\gamma_2' & -i\left(\gamma_1' - \frac{\overline{P_{z_2}}}{\overline{P_{z_1}}}\gamma_2'\right)\end{bmatrix} = -2i\left|\gamma_1' - \frac{\overline{P_{z_2}}}{\overline{P_{z_1}}}\gamma_2'\right|^2.$$

Hence

$$g\left(\frac{d\gamma}{dt},\frac{d\gamma}{dt}\right) = \frac{\left(1 + \left|\frac{P_{z_1}}{P_{z_2}}\right|^2 + \left|\frac{P_{z_1}}{P_{z_2}}\gamma_1+\gamma_2\right|^2\right)\left|\gamma_1' - \frac{\overline{P_{z_2}}}{\overline{P_{z_1}}}\gamma_2'\right|^2}{\pi\left(1+|\gamma_1|^2+|\gamma_2|^2\right)^2}.$$

Now we move to $\mathbf{P}^n$, and consider $X \subset \mathbf{P}^n$ a complete intersection of codimension r, or the zero set of polynomials $P_1 = 0, \ldots, P_r = 0$. Expressing some covectors in terms of others reduces the number of determinants we calculated above from $2n$ to $2(n-r)$. Then

$$\begin{aligned}
P_{1,z_1}dz_1 + \cdots + P_{1,z_n}dz_n &= 0, & dz_n &= c_{n,1}dz_1 + \cdots + c_{n,n-r}dz_{n-r},\\
&\vdots & &\vdots\\
P_{r,z_1}dz_1 + \cdots + P_{r,z_n}dz_n &= 0, & dz_{n-r+1} &= c_{n-r+1,1}dz_1 + \cdots + c_{n-r+1,n-r}dz_{n-r},
\end{aligned}$$

for the $c_{i,j}$ some combinations of the P_{k,z_ℓ}. By orthonormality of the basis vectors, and assuming that the $c_{i,j}$ are all non-zero, we find

$$\frac{\partial}{\partial z_i} = \sum_{j=1}^{n-r}\frac{1}{(n-r)c_{i,j}}\frac{\partial}{\partial z_j}, \qquad \frac{\partial}{\partial \overline{z_i}} = \sum_{j=1}^{n-r}\frac{1}{(n-r)\overline{c_{i,j}}}\frac{\partial}{\partial \overline{z_j}},$$

for all integers $n - r < i \leqslant n$. This allows us to rewrite the path derivative as

$$\begin{aligned}
\frac{d\gamma}{dt} &= \sum_{i=1}^{n}\overline{\gamma}_i'\frac{\partial}{\partial z_i} + \gamma_i'\frac{\partial}{\partial \overline{z_i}}\\
&= \sum_{i=1}^{n-r}\left(\overline{\gamma}_i'\frac{\partial}{\partial z_i} + \gamma_i'\frac{\partial}{\partial \overline{z_i}}\right) + \sum_{i=n-r+1}^{n}\left(\sum_{j=1}^{n-r}\frac{\overline{\gamma}_i'}{(n-r)c_{i,j}}\frac{\partial}{\partial z_j} + \sum_{j=1}^{n-r}\frac{\gamma_i'}{(n-r)\overline{c_{i,j}}}\frac{\partial}{\partial \overline{z_j}}\right)\\
&= \sum_{i=1}^{n-r}\left(\overline{\gamma}_i' + \sum_{j=n-r+1}^{n}\frac{\overline{\gamma}_j'}{(n-r)c_{j,i}}\right)\frac{\partial}{\partial z_i} + \left(\gamma_i' + \sum_{j=n-r+1}^{n}\frac{\gamma_j'}{(n-r)\overline{c_{j,i}}}\right)\frac{\partial}{\partial \overline{z_i}}.
\end{aligned}$$

In the case of a curve in $\mathbf{P}^n$, when $r = n-1$, let $c_{1,1} = 1$ and $e = \frac{\partial}{\partial z_1} + \frac{\partial}{\partial \overline{z_1}} + \cdots + \frac{\partial}{\partial z_n} + \frac{\partial}{\partial \overline{z_n}}$ to get

$$\frac{d\gamma}{dt} = \left(\sum_{j=1}^{n}\frac{\overline{\gamma}_j'}{c_{j1}}\right)\frac{\partial}{\partial z_1} + \left(\sum_{j=1}^{n}\frac{\gamma_j'}{\overline{c_{j1}}}\right)\frac{\partial}{\partial \overline{z_1}},$$

$$\left(\sum_{k,\ell=1}^{n}\chi_{k\ell}(\gamma)dz_k\wedge d\overline{z_\ell}\right)(e,e) = \sum_{k,\ell=1}^{n}\left(1 + \sum_{i=1}^{n}|\gamma_i|^2\right)\delta_{k\ell} - \overline{\gamma_k c_{\ell 1}}\gamma_\ell c_{k1},$$

$$(dz_1\wedge d\overline{z_1})\left(\frac{d\gamma}{dt}, I\frac{d\gamma}{dt}\right) = \det\begin{bmatrix}\sum_{j=1}^{n}\frac{\overline{\gamma}_j'}{c_{j1}} & i\sum_{j=1}^{n}\frac{\overline{\gamma}_j'}{c_{j1}}\\ \sum_{j=1}^{n}\frac{\gamma_j'}{\overline{c_{j1}}} & -i\sum_{j=1}^{n}\frac{\gamma_j'}{\overline{c_{j1}}}\end{bmatrix} = -2i\left|\sum_{j=1}^{n}\frac{\gamma_j'}{\overline{c_{j1}}}\right|^2.$$

Hence

$$g\left(\frac{d\gamma}{dt},\frac{d\gamma}{dt}\right) = \frac{\left(\sum_{k,\ell=1}^{n}\left(1+\sum_{i=1}^{n}|\gamma_i|^2\right)\delta_{k\ell} - \overline{\gamma_k c_{\ell 1}}\gamma_\ell c_{k1}\right)\left|\sum_{j=1}^{n}\frac{\gamma_j'}{\overline{c_{j1}}}\right|^2}{\pi\left(1+\sum_{i=1}^{n}|\gamma_i|^2\right)^2}.$$

The terms $\overline{\gamma_k c_{\ell 1}}\gamma_\ell c_{k1}$ may be rearranged into terms $|\gamma_k c_{\ell 1} - \gamma_\ell c_{k1}|^2$, but it does not provide any enlightening results, similarly to the rest of this post.

2.8 Sheaves, derived and perverse

2017-12-05

Keywords: *sheaf, direct image, inverse image, support, derived category, derived functor, derived sheaf, cohomology sheaf, constructible sheaf, perverse sheaf, complex*

Let X, Y be topological spaces and $f : X \to Y$ a continuous map. We let $\mathrm{Shv}(X)$ be the category of sheaves on X, $D(\mathrm{Shv}(X))$ the derived category of sheaves on X, and $D_b(\mathrm{Shv}(X))$ the bounded variant. Recall that $D(\mathcal{A})$ for an abelian category $\mathcal{A}$ is constructed first by taking $C(\mathcal{A})$, the category of cochains of elements of $\mathcal{A}$, quotienting by chain homotopy, then quotienting by all acylic chains.

Remark 2.8.1. Let $\mathcal{F} \in \mathrm{Shv}(X)$. Recall:

- a *section* of $\mathcal{F}$ is an element of $\mathcal{F}(U)$ for some $U \subseteq X$,
- a *germ* of $\mathcal{F}$ at $x \in X$ is an equivalence class in $\{s \in \mathcal{F}(U) \ : \ U \ni x\}/\sim_x$,
- $s \sim_x t$ iff every neighborhood W of x in $U \cap V$ has $s|_W = t|_W$, for $s \in \mathcal{F}(U)$, $t \in \mathcal{F}(V)$,
- the *support of the section* $s \in \mathcal{F}(U)$ is $\mathrm{supp}(s) = \{x \in U \ : \ s \nsim_x 0\}$,
- the *support of the sheaf* $\mathcal{F}$ is $\mathrm{supp}(\mathcal{F}) = \{x \in X \ : \ \mathcal{F}_x \neq 0\}$.

Definition 2.8.2. The map f induces functors between categories of sheaves, called

$$\begin{array}{rrcl} \textit{direct image} & f_* \ : \ \mathrm{Shv}(X) & \to & \mathrm{Shv}(Y), \\ & (U \mapsto \mathcal{F}(U)) & \mapsto & (V \mapsto \mathcal{F}(f^{-1}(V))), \\ \\ \textit{inverse image} & f^* \ : \ \mathrm{Shv}(Y) & \to & \mathrm{Shv}(X), \\ & (V \mapsto \mathcal{G}(V)) & \mapsto & \mathrm{sh}\left(U \mapsto \operatorname*{colim}_{V \supseteq f(U)} \mathcal{G}(V)\right), \\ \\ \textit{direct image with compact support} & f_! \ : \ \mathrm{Shv}(X) & \to & \mathrm{Shv}(Y), \\ & (U \mapsto \mathcal{F}(U)) & \mapsto & \left(V \mapsto \left\{s \in \mathcal{F}(f^{-1}(V)) \ : \ f|_{\mathrm{supp}(s)} \text{ is proper}\right\}\right). \end{array}$$

Above we used that $f : X \to Y$ is *proper* if $f^{-1}(K) \subseteq X$ is compact, for every $K \subseteq Y$ compact. Next, recall that a functor $\varphi : \mathcal{A} \to \mathcal{B}$ induces a functor $R\varphi : D(\mathcal{A}) \to D(\mathcal{B})$, called the (first) *derived functor* of φ, given by $R\varphi(A^\bullet) = H^1(\varphi(A)^\bullet)$.

Remark 2.8.3. Each of the maps $f_*, f^*, f_!$ have their derived analogues $Rf_*, Rf^*, Rf_!$, respectively. For reasons unclear, $Rf_!$ has a right adjoint, denoted $Rf^! : D(\mathrm{Shv}(Y)) \to D(\mathrm{Shv}(X))$. This is called the *exceptional inverse image.*

We are now ready to define perverse sheaves.

Definition 2.8.4. Let $A^\bullet \in D(\mathrm{Shv}(X))$. Then:

- the *ith cohomology sheaf* of $A^\bullet$ is $H^i(A^\bullet) = \ker(d^i)/\mathrm{im}(d^i)$,
- $A^\bullet$ is a *constructible complex* if $H^i(A^\bullet)$ is a constructible sheaf for all i,
- $A^\bullet$ is a *perverse sheaf* if $A^\bullet \in D_b(\mathrm{Shv}(X))$ is constructible and $\dim(\mathrm{supp}(H^{-i}(P))) \leqslant i$ for all $i \in \mathbf{Z}$ and for $P = A^\bullet$ and $P = (A^\bullet)^\vee = (A^\vee)^\bullet$ the dual complex of sheaves.

We finish off with an example.

Example 2.8.5. Let $X = \mathbf{R}$ be a stratified space, with $X_0 = 0$ the origin and $X_1 = \mathbf{R} \setminus 0$. Let $\mathcal{F} \in \text{Shv}(X)$ be an $\mathbf{R}$-valued sheaf given by $\mathcal{F}(U) = \inf_{x \in U} |x|$, and define a chain complex $A^\bullet$ in the following way:

$$0 \longrightarrow A^{-1} = \mathcal{F} \xrightarrow{d^{-1}=\text{id}} A^0 = \mathcal{F} \xrightarrow{d^0=0} 0.$$

Note that for any $U \subseteq \mathbf{R}$, we have $H^{-1}(A^\bullet)(U) = \ker(d^{-1})(U) = \ker(\text{id} : \mathcal{F}(U) \to \mathcal{F}(U)) = \emptyset$ if $0 \notin U$, and 0 otherwise. Hence $\text{supp}(H^{-1}(A^\bullet)) = \mathbf{R} \setminus 0$, whose dimension is 1. Next, $H^0(A^\bullet)(U) = \ker(d^0)(U)/\text{im}(d^{-1})(U) = \ker(0 : \mathcal{F}(U) \to 0)/\text{im}(\text{id} : \mathcal{F}(U) \to \mathcal{F}(U)) = \mathcal{F}(U)/\mathcal{F}(U) = 0$, and so $\dim(\text{supp}(H^0(A^\bullet))) = 0$. Note that $A^\bullet$ is self-dual and constructible, as the cohomology sheaves are locally constant. Hence $A^\bullet$ is a perverse sheaf.

References: Bredon (Sheaf theory, Chapter II.1), de Catalado and Migliorini (What is... a perverse sheaf?), Stacks project (Articles "Supports of modules and sections" and "Complexes with constructible cohomology")

3 Differential geometry

3.1 Smooth projective varieties as Kähler manifolds

06-16-2016
Keywords: *manifold, variety, complex, metric, structure, fundamental form, Riemannian, Hermitian, Kähler*

Definition 3.1.1. Let k be a field and $\mathbf{P}^n$ projective n-space over k. An *algebraic variety* $X \subset \mathbf{P}^n$ is the zero locus of a collection of homogeneous polynomials $f_i \in k[x_0, \dots, x_n]$.

Here we let $k = \mathbf{C}$, the complex numbers. Complex projective space $\mathbf{CP}^n$ may be described as a complex manifold, with open sets $U_i = \{(x_0 : \cdots : x_n) \ : \ x_i \neq 0\}$ and maps

$$\begin{array}{rcl} \varphi_i \ : \ U_i & \to & \mathbf{C}^n, \\ (x_0 : \cdots : x_n) & \mapsto & \left(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i}\right), \end{array}$$

which can be quickly checked to agree on overlaps. In this context we assume all varieties are smooth, so they are submanifolds of $\mathbf{CP}^n$.

Definition 3.1.2. An *almost complex manifold* is a real manifold M together with a vector bundle endomorphism $J : TM \to TM$ (called a *complex structure*) with $J^2 = -\text{id}$.

Note that every complex manifold admits an almost complex structure on its underlying real manifold. Indeed, given standard coordinates $z_i = x_i + y_i$ for $i = 1, \dots, n$ on $\mathbf{C}^n$, we get a basis $\partial/\partial x_1, \dots, \partial/\partial x_n$, $\partial/\partial y_1, \dots, \partial/\partial y_n$ on the underlying real tangent space T_pU, for $p \in M$ and $U \ni p$ a neighborhood. Then J is defined by

$$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i} \qquad , \qquad J\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i}.$$

Write $T_{\mathbf{C}}M = TM \otimes_{\mathbf{R}} \mathbf{C}$ for the complexification of the tangent bundle, which admits a canonical decomposition $T_{\mathbf{C}}M = T^{1,0}M \oplus T^{0,1}M$, where $J|_{T^{1,0}} = i \cdot \text{id}$ and $J|_{T^{0,1}} = (-i) \cdot \text{id}$. We call $T^{1,0}M$ the *holomorphic* tangent bundle of M and $T^{0,1}M$ the *antiholomorphic* tangent bundle of M, even though it is extraneous to consider any related map here as holomorphic. Define vector bundles (or sheaves, to consider sections on open sets)

$$A^k_M = \bigwedge\nolimits^k (T_{\mathbf{C}}M)^*, \qquad A^{p,q}_M = \bigwedge\nolimits^p (T^{1,0}M)^* \otimes_{\mathbf{C}} \bigwedge\nolimits^q (T^{0,1}M)^*,$$

where we drop the subscript M when the context makes it clear. There is a canonical decomposition $A^k = \bigoplus_{p+q=k} A^{p,q}$, which yields projection maps $\pi^{p,q} : A^k \to A^{p,q}$. The exterior differential d on T^*M may be extended $\mathbf{C}$-linearly to $(T_{\mathbf{C}}M)^*$, and hence also to A^k. Define two new maps

$$\begin{aligned} \partial = \pi^{p+1,q} \circ d|_{A^{p,q}} \ &: \ A^{p,q} \to A^{p+1,q}, \\ \bar\partial = \pi^{p,q+1} \circ d|_{A^{p,q}} \ &: \ A^{p,q} \to A^{p,q+1}. \end{aligned}$$

These satisfy the Leibniz rule and (under mild assumptions) $\partial^2 = \bar\partial^2 = 0$ and $\partial\bar\partial = -\bar\partial\partial$.

From now on, the manifold M will be complex with the natural complex structure described above.

Definition 3.1.3. A *Riemannian metric* on M is a function $g : TM \times TM \to C^\infty(M)$ such that for all $V, W \in TM$,
· $g(V, W) = g(W, V)$, and
· $g_p(V_p, V_p) \geqslant 0$ for all $p \in M$, with equality iff $V = 0$.
A *Riemannian manifold* is a pair (M, g) where g is Riemannian.

Locally we write $g_p : T_pM \times T_pM \to \mathbf{R}$, defined as $g_p(V_p, W_p) = g(V, W)(p)$. If $x_1, \dots, x_n$ are local coordinates on some open set $U \subset M$, then $g = \sum_{i,j} g_{ij} dx_i \wedge dx_j \in A^2(M)$, for $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) \in C^\infty(U)$. Writing $V = \sum_i f_i \frac{\partial}{\partial x_i}$ and $W = \sum_j g_j \frac{\partial}{\partial x_j}$, we get the local expression

$$g_p(V_p, W_p) = \sum_{i,j} g_{ij}(p) f_i(p) g_j(p).$$

Definition 3.1.4. A *Hermitian metric* on a complex manifold M is a Riemannian metric g such that $g(JV, JW) = g(V, W)$ for all $V, W \in TM$. A *Hermitian manifold* is a pair (M, g) where g is Hermitian.

There is an induced form $\omega : TM \times TM \to C^\infty(M)$ given by $\omega(V, W) = g(JV, W)$, called the *fundamental form*. From g being Hermitian it follows that $\omega \in A^{1,1}(M) \subset A^2(M)$. Note also that any two of the structures J, g, ω determine the remaining one.

Definition 3.1.5. A *Kähler metric* on a complex manifold M is a Hermitian metric whose fundamental form is closed (that is, $d\omega = 0$). A Kähler manifold is a pair (M, g) where g is Kähler.

Example 3.1.6. Recall the atlas given to $\mathbf{CP}^n$ above. There is a metric (canonical in some sense) on each U_j given by

$$\omega_j = \frac{i}{2\pi}(\partial \circ \bar{\partial}) \left(\log \left(\sum_{\ell=0}^{n} \left| \frac{x_\ell}{x_j} \right|^2 \right) \right),$$

called the *Fubini–Study metric*. Each ω_j is a section of $A^{1,1}(U_j)$, and as a quick calculation shows that $\omega_j|_{U_j \cap U_k} = \omega_k|_{U_j \cap U_k}$, there is a global metric $\omega_{FS} \in A^{1,1}(\mathbf{CP}^n)$ such that $\omega_{FS}|_{U_j} = \omega_j$ for all j.

Hence $\mathbf{CP}^n$ is a Kähler manifold. If we have a smooth projective variety $X \subset \mathbf{CP}^n$, then it is a submanifold of $\mathbf{CP}^n$, so by restricting ω_{FS} to X, we get that X is also a Kähler manifold. Therefore all smooth projective varieties are Kähler.

References: Huybrechts (Complex Geometry, Chapters 1.3, 2.6, 3.1), Lee (Riemannian manifolds, Chapter 3)

3.2 Connections, curvature, and Higgs bundles

07-25-2016

Keywords: *manifold, connection, curvature, curvature tensor, holomorphic vector bundle, sheaf, differential forms, cotangent sheaf, Higgs bundle, Higgs, Riemannian, Hermitian, Kähler, Ricci, Einstein*

Recall (from a previous post) that a Kähler manifold M is a complex manifold (with natural complex structure J) with a Hermitian metic g whose fundamental form ω is closed. In this context M is Kähler. Previously we used upper-case letters V, W to denote vector fields on M, but here we use lower-case letters s, u, v and call them sections (to consider vector bundles more generally as sheaves).

Definition 3.2.1. A *connection* on M is a $\mathbf{C}$-linear homomorphism $\nabla : A^0_M \to A^1_M$ satisfying the Leibniz rule $\nabla(fs) = (df) \wedge s + f\nabla(s)$, for s a section of TM and $f \in C^\infty(M)$.

For ease of notation, we often write $\nabla_u s$ for $\nabla(s)(u)$, where s, u are sections of TM. On Kähler manifolds there is a special connection that we will consider.

Proposition 3.2.2. On M there is a unique connection ∇ that is (for any $u, v \in A^0_M$)

1. Hermitian (satisfies $dg(u, v) = g(\nabla(u), v) + g(u, \nabla(v)))$,
2. torsion-free (satisfies $\nabla_u v - \nabla_v u - [u, v] = 0)$, and
3. compatible with the complex structure J (satisfies $\nabla_u v = \nabla_{Ju}(Jv))$.

If ∇ satisfies the first two conditions, it is called the *Levi-Civita* connection, and if it satisfies the first and third conditions, it is called the *Chern* connection. If g is not necessarily Hermitian, ∇ is called *metric* if it satisfies the first condition. From here on out ∇ denotes the unique tensor described in the proposition above.

Definition 3.2.3. The *curvature tensor* of M is defined by

$$R(u, v) = \nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]}.$$

It may be viewed as a map $A^2 \to A^1$, or $A^3 \to A^0$, or $A^0 \to A^0$. The *Ricci tensor* of M is defined by

$$r(u, v) = \text{trace}(w \mapsto R(u, v)w) = \sum_i g(R(a_i, u)v, a_i),$$

for the a_i a local orthonormal basis of $A^0 = TM$. This is a map $A^2 \to A^0$. The *Ricci curvature* of M is defined by

$$\text{Ric}(u, v) = r(Ju, v).$$

This is a map $A^2 \to A^0$.

Definition 3.2.4. An *Einstein manifold* is a pair (M,g) that is Riemannian and for which the Ricci curvature is directly proportional to the Riemannian metric. That is, there exists a constant $\lambda\in\mathbf{R}$ such that $\mathrm{Ric}(u,v)=\lambda g(u,v)$ for any $u,v\in A^1$.

Recall that a *holomorphic vector bundle* $\pi : E\to M$ has complex fibers and holomorphic projection map π. Here we consider two special vector bundles (as sheaves), defined on open sets $U\subset M$ by

$$\begin{aligned}\mathrm{End}(E)(U) &= \{f:\pi^{-1}(U)\to\pi^{-1}(U)\ :\ f|_{\pi^{-1}(x)}\text{ is a homomorphism}\},\\ \Omega_M(U) &= \left\{\sum_{i=0}^n f_i dz_1\wedge\cdots\wedge dz_i\ :\ f_i\in C^\infty(U)\right\},\end{aligned}$$

where $z_1,\dots,z_n$ are local coordinates on U. The first is the endomorphism sheaf of E and the second is the sheaf of differential forms of M, or the holomorphic cotangent sheaf. The cotangent sheaf as defined is a presheaf, so we sheafify to get Ω_M.

Definition 3.2.5. A *Higgs vector bundle* over a complex manifold M is a pair (E,θ), where $\pi : E\to M$ is a holomorphic vector bundle and θ is a holomorphic section of $\mathrm{End}(E)\otimes\Omega_M$ with $\theta\wedge\theta=0$, called the *Higgs field*.

References: Huybrechts (Complex Geometry, Chapters 4.2, 4.A), Kobayashi and Nomizu (Foundations of Differential Geometry, Volume 1, Chapter 6.5)

3.3 Higgs fields of principal bundles

2016-08-24

Keywords: *principal bundle, fiber bundle, adjoint representation, associated bundle, Lie group, Lie algebra, differential forms, conjugate, Higgs field, Higgs*

The goal here is to understand the setting of Higgs fields on Riemannian manifolds, in the manner of Hitchin. First we consider general topological spaces X and groups G.

Definition 3.3.1. Let X be a topological space and G a group. A *principal bundle* (or *principal G-bundle*) P over X is a fiber bundle $\pi : P\to X$ together with a continuous, free, and transitive right action $P\times G\to P$ that preserves the fibers. That is, if $p\in\pi^{-1}(x)$, then $pg\in\pi^{-1}(x)$ for all $g\in G$ and $x\in X$.

Now suppose we have a principal bundle $\pi : P\to X$, a representation ρ of G, and another space Y on which G acts on the left. Define an equivalence relation $(p,y)\sim(p',y')$ on $P\times Y$ iff there is some $g\in G$ for which $p'=pg$ and $y'=\rho(g^{-1})y$. This is an equivalence relation. We will be interested in the adjoint representation (induced by conjugation).

Proposition 3.3.2. The projection map $\pi' : P\times_\rho Y := (P\times Y)/\sim\ \to X$, where $\pi'([p,y])=\pi(p)$, defines a vector bundle over X, called the *associated bundle* of P.

Recall a *Lie group* G is a group that is also a topological space, in the sense that there is a continuous map $G\times G\to G$, given by $(g,h)\mapsto gh^{-1}$. The *Lie algebra* $\mathfrak{g}$ of the Lie group G is the tangent space T_eG of G at the identity e. We will be interested in principal G-bundles $P\to\mathbf{R}^2$ and associated bundles $P\times_{\mathrm{ad}}\mathfrak{g}\to\mathbf{R}^2$, where ad is the adjoint representation of G.

Next, recall we had the space $\mathcal{A}_M^k$ of k-differential forms on M (see post "Smooth projective varieties as Kähler manifolds," 2016-06-16), defined in terms of wedge products of elements in the cotangent bundle $(TM)^*=T^*M$ of M. Now we generalize this to get differential forms over arbitrary vector bundles.

Definition 3.3.3. Let $E\to M$ be a vector bundle. Let

$$\begin{aligned}\mathcal{A}_M^k(E) &:= \Gamma(E\otimes\textstyle\bigwedge^k T^*M)=\Gamma(E)\otimes_{\mathcal{A}_M^0}\mathcal{A}_M^k,\\ \mathcal{A}_M^{p,q}(E) &:= \Gamma(E\otimes\textstyle\bigwedge^p(T^{1,0}M)^*\otimes\bigwedge^q(T^{0,1}M)^*)=\Gamma(E)\otimes_{\mathcal{A}_M^0}\mathcal{A}_M^{p,q}\end{aligned}$$

be the spaces of k- and (p,q)-differential forms, respectively, over M with values in E.

Equality above follows by functoriality. Now we are close to understanding where exactly the Higgs field lives, in Hitchin's context.

Definition 3.3.4. Given a function $f : \mathbf{C} \to \mathbf{C}$, the *conjugate* of f is $\bar{f}$, defined by $\bar{f}(z) = \overline{f(\bar{z})}$.

Hitchin denotes this as f^*, but we will stick to $\bar{f}$. Finally, let P be a G-principal bundle over $\mathbf{R}^2$ and $P \times_{\text{ad}} \mathfrak{g}$ the associated bundle of P. Given $f \in \mathcal{A}^0_{\mathbf{R}^2}((P \times_{\text{ad}} \mathfrak{g}) \otimes \mathbf{C})$, set

$$\begin{aligned} \theta &= \tfrac{1}{2} f(dx + i\ dy) \in \mathcal{A}^{1,0}_{\mathbf{R}^2}((P \times_{\text{ad}} \mathfrak{g}) \otimes \mathbf{C}), \\ \theta^* &= \tfrac{1}{2} \bar{f}(dx - i\ dy) \in \mathcal{A}^{0,1}_{\mathbf{R}^2}((P \times_{\text{ad}} \mathfrak{g}) \otimes \mathbf{C}), \end{aligned}$$

called a *Higgs field* over $\mathbf{R}^2$ and (presumably) a *dual* (or *conjugate*) Higgs field over $\mathbf{R}^2$. Note this agrees with the definition in a previous post ("Connections, curvature, and Higgs bundles," 2016-07-25).

References: Hitchin (Self-duality equations on a Riemann surface), Wikipedia (article on associated bundles, article on vector-valued differential forms)

3.4 Equations on Riemann surfaces

2016-08-25

Keywords: *Riemann surface, connection, curvature, Hodge star, Hitchin, Yang–Mills, Higgs, Higgs field, manifold*

Recall that a *Riemann surface* is a complex 1-manifold M with a complex structure Σ (a class of analytically equivalent atlases on X). Here we consider equations that relate connections and Higgs fields with solutions on Riemann surfaces. Let $G = SU(2)$ (complex 2-matrices with determinant 1) or $SO(3)$ (real 3-matrices with determinant 1), θ a Higgs field over M, and P a principal G-bundle over M.

Definition 3.4.1. The *curvature* of a principal G-bundle P is the map

$$\begin{array}{rcl} F_\nabla \ : \ \mathcal{A}^0_M(P) & \to & \mathcal{A}^2_M(P), \\ \omega s & \mapsto & (d_\nabla \circ \nabla)(\omega s), \end{array}$$

where the extension $d_\nabla : \mathcal{A}^k_M(P) \to \mathcal{A}^{k+1}_M(P)$ is defined by the Leibniz rule, that is $d_\nabla(\omega \otimes s) = (d\omega) \otimes s + (-1)^k \omega \wedge \nabla s$, for ω a k-form and s a smooth section of P.

Since we may write $\mathcal{A}^1 = \mathcal{A}^{1,0} \oplus \mathcal{A}^{0,1}$ as the sum of its holomorphic and anti-holomorphic parts, respectively (see post "Smooth projective varieties as Kähler manifolds," 2016-06-16), we may consider the restriction of d_∇ to either of these summands.

Definition 3.4.2. For a vector space V, define the *Hodge star* $*$ by

$$\begin{array}{rcl} * \ : \ \bigwedge^k(V^*) & \to & \bigwedge^{n-k}(V^*), \\ e^{i_1} \wedge \cdots \wedge e^{i_k} & \mapsto & e^{j_1} \wedge \cdots \wedge e^{j_{n-k}}, \end{array}$$

so that $e^{i_1} \wedge \cdots \wedge e^{i_k} \wedge e^{j_1} \wedge \cdots \wedge e^{j_{n-k}} = e^1 \wedge \cdots \wedge e^n$. Extend by linearity from the chosen basis.

The dual of the generalized connection d_∇ is written $d^*_\nabla = (-1)^{m+mk+1} * d_\nabla *$, where $\dim(M) = m$ and the argument of d^*_∇ is in $\mathcal{A}^k_M$ (this holds for manifolds M that are not necessarily Riemann surfaces as well).

Now we may understand some equations on Riemann surfaces. They all deal with the connection ∇, its generalization d_∇, its curvature F_∇, and the Higgs field θ. Below we indicate their names and where they are mentioned (and described in further detail).

Hitchin equations	$d_\nabla\|_{A^{0,1}}\,\theta = 0$ $F_\nabla + [\theta,\theta^*] = 0$	[2], Introduction
Yang–Mills equations	$d_\nabla^* d_\nabla\theta + *[*F_\nabla,\theta] = 0$ $d_\nabla^*\theta = 0$	[1], Section 4
self-dual Yang–Mills equation	$F_\nabla - *F_\nabla = 0$	[2], Section 1
Yang–Mills–Higgs equations	$d_\nabla * F_\nabla + [\theta, d_\nabla\theta] = 0$ $d_\nabla * d_\nabla\theta = 0$	[4], equation (1)

Recall the definitions of θ and $\theta*$ from a previous post ("Higgs fields of principal bundles," 2016-08-24). Now we look at these equations in more detail. The first of the Hitchin equations says that θ has no anti-holomorphic component, or in other words, that θ is holomorphic. In the second equation, the Lie bracket $[\cdot,\cdot]$ of the two 1-forms is

$$\begin{aligned}[\theta,\theta^*] &= \left[\tfrac{1}{2}f(dz+i\ dy), \tfrac{1}{2}\bar{f}(dz-i\ dy)\right]\\ &= -\tfrac{i}{4}f\bar{f}\ dx\wedge dy + \tfrac{i}{4}f\bar{f}\ dy\wedge dx - \tfrac{i}{4}f\bar{f}\ dx\wedge dy + \tfrac{i}{4}f\bar{f}\ dy\wedge dx\\ &= -i|f|^2\ dx\wedge dy.\end{aligned}$$

In the Yang–Mills and Yang–Mills-Higgs equations, we can simplify some parts by noting that, for a section s of the complexification of $P\times_{\mathrm{ad}}\mathfrak{g}$,

$$\begin{aligned}d_\nabla(\theta\otimes s) &= \tfrac{1}{2}d_\nabla(fdx\otimes s) + \tfrac{i}{2}d_\nabla(fdy\otimes s)\\ &= \tfrac{1}{2}(df\wedge dx\otimes s - fdx\wedge\nabla s) + \tfrac{i}{2}(df\wedge dy - fdy\wedge\nabla s)\\ &= \left(\frac{i}{2}\frac{\partial f}{\partial x} - \frac{1}{2}\frac{\partial f}{\partial y}\right)dx\wedge dy\otimes s - \underbrace{\tfrac{1}{2}f(dx+idy)}_{\theta}\wedge\nabla s.\end{aligned}$$

The Hodge star of θ is $*\theta = \frac{1}{2}f(dy - idx)$, so

$$\begin{aligned}d_\nabla * (\theta\otimes s) &= \tfrac{1}{2}d_\nabla(fdy\otimes s) - \tfrac{i}{2}d_\nabla(fdx\otimes s)\\ &= \tfrac{1}{2}(df\wedge dy\otimes s - fdy\wedge\nabla s) - \tfrac{i}{2}(df\wedge dx - fdx\wedge\nabla s)\\ &= \left(\frac{1}{2}\frac{\partial f}{\partial x} + \frac{i}{2}\frac{\partial f}{\partial y}\right)dx\wedge dy\otimes s + \underbrace{\tfrac{1}{2}f(idx-dy)}_{i\theta}\wedge\nabla s.\end{aligned}$$

We could express $\nabla s = (s_1dx + s_2dy)\otimes s^1$, but that would not be too enlightening. Next, note the self-dual Yang–Mills equation only makes sense over a (real) 4-dimensional space, since the degrees of the forms have to match up. In that case, with a basis $dz_1 = dx_1 + idy_1, dz_2 = dx_2 + idy_2$ of $\mathcal{A}^1$, we have

$$\begin{aligned}F_\nabla &= F_{12}dx_1\wedge dy_1 + F_{13}dx_1\wedge dx_2 + F_{14}dx_1\wedge dy_2 + F_{23}dy_1\wedge dx_2 + F_{24}dy_1\wedge dy_2 + F_{34}dx_2\wedge dy_2,\\ *F_\nabla &= F_{12}dx_2\wedge dy_2 - F_{13}dy_1\wedge dy_2 + F_{14}dy_1\wedge dx_2 + F_{23}dx_1\wedge dy_2 - F_{24}dx_1\wedge dx_2 + F_{34}dx_1\wedge dy_1.\end{aligned}$$

Then the self-dual equation simply claims that

$$F_{12} = F_{34}\qquad,\qquad F_{13} = -F_{24}\qquad,\qquad F_{14} = F_{23}.$$

Remark 3.4.3. This title of this post promises to talk about equations on Riemann surfaces, yet all the differential forms are valued in a principal G-bundle over $\mathbf{R}^2$ (or $\mathbf{R}^4$). However, since the given equations are conformally invariant (this is not obvious), and as a Riemann surface locally looks like $\mathbf{R}^2$, we may consider the solutions to the equations as living on a Riemann surface.

References:
[1] Atiyah and Bott (The Yang–Mills equations over Riemann surfaces)
[2] Hitchin (Self-duality equations on a Riemann surface)
[3] Huybrechts (Complex Geometry, Chapter 4.3)
[4] Taubes (On the Yang–Mills–Higgs equations)

3.5 The Grassmannian is a complex manifold

2016-09-22

Keywords: *Grassmannian, manifold, complex*

Let $Gr(k, \mathbf{C}^n)$ be the space of k-dimensional complex subspaces of $\mathbf{C}^n$, also known as the *complex Grassmannian.* We will show that it is a complex manifold of dimension $k(n-k)$. Thanks to Jinhua Xu and professor Mihai Păun for explaining the details.

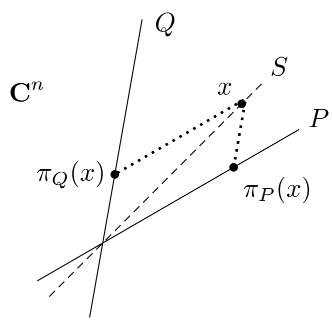
To begin, take $P \in Gr(k, \mathbf{C}^n)$ and an $n-k$ subspace Q of $\mathbf{C}^n$, such that $P \cap Q = \{0\}$. Then $P \oplus Q = \mathbf{C}^n$, so we have natural projections

$$\begin{array}{ccc} & \mathbf{C}^n & \\ {\scriptstyle \pi_P} \swarrow & & \searrow {\scriptstyle \pi_Q} \\ P & & Q. \end{array}$$

A neighborhood of P, depending on Q may be described as $U_Q = \{S \in Gr(k, \mathbf{C}^n) \ : \ S \cap Q = \{0\}\}$. We claim that $U_Q \cong \mathrm{Hom}(P, Q)$. The isomorphism is described by

$$\begin{array}{rcl} \mathrm{Hom}(P,Q) & \to & U_Q, \\ A & \mapsto & \{v + Av \ : \ v \in P\}, \\ (\pi_Q|_S) \circ (\pi_P|_S)^{-1} & \leftarrow\!\shortmid & S. \end{array}$$

The reverse map, call it φ_Q, is also the chart for the manifold structure. The idea of decomposing $\mathbf{C}^n$ into P and Q and constructing a homomorphism from P to Q may be visualized in the following diagram.



Then $\mathrm{Hom}(P,Q) \cong \mathrm{Hom}(\mathbf{C}^k, \mathbf{C}^{n-k}) \cong \mathbf{C}^{k(n-k)}$, so $Gr(k, \mathbf{C}^n)$ is locally of complex dimension $k(n-k)$. To show that there is a complex manifold structure, we need to show that the transition functions are holomorphic. Let $P, P' \in Gr(k, \mathbf{C}^n)$ and $Q, Q' \in Gr(n-k, \mathbf{C}^n)$ such that $P \cap Q = P' \cap Q' = \{0\}$. Let $X \in \mathrm{Hom}(P,Q)$ such that $X \in \varphi_Q(U_Q \cap U_{Q'})$, with $\varphi_Q(S) = X$ and $\varphi_{Q'}(S) = X'$ for some $S \in U_Q \cap U_{Q'}$. Define $I_X(v) = v + Xv$, and note the transition map takes X to

$$\begin{aligned} X' &= \varphi_{Q'} \circ \varphi_Q^{-1}(X) && \text{(definition)} \\ &= \varphi_{Q'}(S) && \text{(assumption)} \\ &= (\pi_{Q'}|_S) \circ (\pi_{P'}|_S)^{-1} && \text{(definition)} \\ &= (\pi_{Q'}|_S) \circ I_X \circ I_X^{-1} \circ (\pi_{P'}|_S)^{-1} && \text{(creative identity)} \\ &= (\pi_{Q'}|_S \circ I_X) \circ (\pi_{P'}|_S \circ I_X)^{-1} && \text{(redistribution)} \\ &= (\pi_{Q'}|_P + \pi_{Q'}|_Q \circ X) \circ (\pi_{P'}|_P + \pi_{P'}|_Q \circ X)\,. && \text{(definition)} \end{aligned}$$

At this last step we have compositions and sums of homomorphisms and linear maps, which are all holomorphic. Hence the transition functions of $Gr(k, \mathbf{C}^n)$ are holomorphic, so it is a complex manifold.

Part III
Topological data analysis

0.0 New directions in TDA

2017-08-03

Keywords: *informal, TDA, persistence, functor, interleaving*

This post is informal, meant as a collection of (personally) new things from the workshop "Topological data analysis: Developing abstract foundations" at the Banff International Research Station, July 31 - August 4, 2017. New actual questions:

1. Does there exist a constructible sheaf valued in persistence modules over $\mathrm{Ran}^{\leqslant n}(M)$?
   - On the stalks it should be the persistence module of $P \in \mathrm{Ran}^{\leqslant n}(M)$. What about arbitrary open sets?
   - Is there such a thing as a colimit of persistence modules?
   - Uli Bauer suggested something to do with ordering the elements of the sample and taking small open sets.
2. Can framed vector spaces be used to make the TDA pipeline functorial? Does Ezra Miller's work help?
   - Should be a functor from $(\mathbf{R}, \leqslant)$, the reals as a poset, to Vect or Vect_{fr}, the category of (framed) vector spaces. Filtration function $f : \mathbf{R}^n \to \mathbf{R}$ is assumed to be given.
   - Framed perspective should not be too difficult, just need to find right definitions.
   - Does this give an equivalence of categories (category of persistence modules and category of matchings)? Is that what we want? Do we want to keep only specific properties?
   - Ezra's work is very dense and unpublished. But it seems to have a very precise functoriality (which is not the main thrust of the work, however).
3. Can the Bubenik–de Silva–Scott interleaving categorification be viewed as a (co)limit? Diagrams are suggestive.
   - Reference is 1707.06288 on the arXiv.
   - Probably not a colimit, because that would be very large, though the arrows suggest a colimit.
   - Have to be careful, because the (co)limit should be in the category of posets, not just interleavings.

New things to learn about:

1. Algebraic geometry / homotopy theory: the etale space of a sheaf, Kan extensions, model categories, symmetric monoidal categories.
2. TDA related: Gromov–Hausdorff distance, the universal distance (Michael Lesnick's thesis and papers), merge trees, Reeb graphs, Mapper (the program).

1 Sampling and statistics

1.1 Reconstructing a manifold from sample data, with noise

2016-05-26

Keywords: *TDA, manifold, sampling, statistics, probability, measure, normal distribution, multivariable, nerve*

We follow the article [3] and add more background and clarifications. Some assumptions are made that are not explicitly mentioned in the article, to make calculations easier.

Background in probability, measure theory, topology

Let X be a random variable over a space A. Recall that the expression $P(X)$ is a number in $[0,1]$ describing the probability of the event X happening. This is called a *probability distribution*. Here we will consider continuous random variables, so $P(X=x)=0$ for any single element $x \in A$.

Definition 1.1.1. The *probability density function* of X is the function $f : A \to \mathbf{R}$ satisfying
· $f(x) \geqslant 0$ for all $x \in A$, and
· $\int_B f(x)\ dx = P(X \in B)$ for any $B \subseteq A$.
The second condition implies $\int_A f(x)\ dx = 1$.

Often authors use just P instead of f, and write $P(x)$ instead of $P(X=x)$.

Definition 1.1.2. Let $Y = g(X)$ be another random variable. The *expected value* of Y is

$$E[Y] = E[g(X)] = \int_A g(x)f(x)\ dx.$$

The *mean* of X is $\mu := E[X]$, and the *variance* of X is $\sigma^2 := E[(X-\mu)^2]$. If $\vec{X} = (X_1\ \cdots\ X_n)^T$ is a multivariate random variable, then $\vec{\mu} = E[\vec{X}]$ is an n-vector, and the variance is an $(n \times n)$-matrix given as

$$\Sigma = E[(\vec{X} - E[\vec{X}])(\vec{X} - E[\vec{X}])^T] \qquad \text{or} \qquad \Sigma_{ij} = E[(X_i - E[X_i])(X_j - E[X_j])].$$

The *covariance* of X and Y is $E[(X-E[X])(Y-E[Y])]$. Note that the covariance of X with itself is just the usual variance of X.

Example 1.1.3. One example of a probability distribution is the *normal* (or *Gaussian*) distribution, and we say a random variable with the normal distribution is *normally distributed*. If a random variable X is normally distributed with mean μ and variance σ^2, then the probability density function of X is

$$f(x) = \frac{\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}{\sigma\sqrt{2\pi}}.$$

If $\vec{X} = (X_1\ \cdots\ X_n)^T$ is a normally distributed multivariate random variable, then $\vec{\mu} = (E[X_1]\ \cdots\ E[X_n])^T$ and the probability density function of $\vec{X}$ is

$$f(\vec{x}) = \frac{\exp\left(-\frac{1}{2}(\vec{x}-\vec{\mu})^T\Sigma^{-1}(\vec{x}-\vec{\mu})\right)}{\sqrt{(2\pi)^n \det(\Sigma)}}.$$

Definition 1.1.4. A *measure* on $\mathbf{R}^D$ is a function m : {subsets of $\mathbf{R}^D$} $\to [0,\infty]$ such that $m(\emptyset) = 0$ and $m(\bigcup_{i\in I} E_i) = \sum_{i \in I} m(E_i)$ for $\{E_i\}_{i\in I}$ a countable sequence of disjoint subsets of $\mathbf{R}^D$. A *probability measure* on $\mathbf{R}^D$ is a measure m on $\mathbf{R}^D$ with the added condition that $m(\mathbf{R}^D) = 1$.

A probability distribution is an example of a probability measure.

Definition 1.1.5. Let $U = \{U_i\}_{i\in I}$ be a covering of a topological space M. The *nerve* of the covering U is a set N of subsets of I given by

$$N = \left\{ J \subset I\ :\ \bigcap_{j\in J} U_j \neq \emptyset \right\}.$$

Note that this makes N into an *abstract simplicial complex*, as $J \in N$ implies $J' \in N$ for all $J' \subseteq J$.

Let M be a smooth compact submanifold of $\mathbf{R}^D$. By the tubular neighborhood theorem (see Theorem 2.11.4 in [3]), every smooth compact submanifold M of $\mathbf{R}^D$ has a tubular neighborhood for some $\epsilon > 0$.

Definition 1.1.6. For a particular embedding of M, let the *condition number* of M be $\tau = \sup\{\epsilon\ : M$ has an $\epsilon-$tubular neighborhood$\}$.

Distributions on a manifold

Let M be a d-dimensional manifold embedded in $\mathbf{R}^D$, with $D > d$. Recall that every element in $NM \subseteq \mathbf{R}^D$, the normal bundle of M, may be represented as a pair $(\vec{x}, \vec{y})$, where $\vec{x} \in M$ and $\vec{y} \in T^\perp$ (since M is a manifold, all the normal spaces are isomorphic). Hence we may consider a probability distribution P on NM, with $\vec{X}$ the d-multivariate random variable representing points on M and $\vec{Y}$ the $(D-d)$-multivariate random variable representing points on the space normal to M at a point on M. We make the assumption that $\vec{X}$ and $\vec{Y}$ are independent, or that

$$P(\vec{X}, \vec{Y}) = P_M(\vec{X}) P_{T^\perp}(\vec{Y}).$$

That is, $P_{T^\perp}$ is a probability distribution that is the same at any point on the manifold.

Definition 1.1.7. Let P be a probability distribution on NM and f_M the probability density function of P_M. In the context described above, P satisfies the *strong variance condition* if

· there exist $a, b > 0$ such that $f_M(\vec{x}) \in [a, b]$ for all $\vec{x} \in M$, and

· $P_{T^\perp}(\vec{Y})$ is normally distributed with $\vec{\mu} = 0$ and $\Sigma = \sigma^2 I$.

The second condition implies that the covariance of Y_i with Y_j is trivial iff $i \neq j$, and that the vairance of all the Y_is is the same. From the normally distributed multivariate example above, this also tells us that the probability density function $f^\perp$ of $\vec{Y}$ is

$$f^\perp(\vec{y}) = \frac{\exp\left(-\frac{\sigma^2}{2}\sum_{i=1}^{D-d} y_i^2\right)}{\sigma^{D-d}\sqrt{(2\pi)^{D-d}}}.$$

Theorem 1.1.8. In the context described above, let P be a probability distribution on NM satisfying the strong variance condition, and let $\delta > 0$. If there is $c > 1$ such that

$$\sigma < \frac{c\tau(\sqrt{9} - \sqrt{8})}{9\sqrt{8(D-d)}},$$

then there is an algorithm that computes the homology of M from a random sample of n points, with probability $1 - \delta$. The number n depends on τ, δ, c, d, D, and the diameter of M.

The homology computing algorithm

Below is a broad view of the algorithm described in sections 3, 4, and 5 of [1]. Let M be a d-manifold embedded in $\mathbf{R}^D$, and P a probability measure on NM satisfying the strong variance condition.

1. Calculate the following numbers:

$$\begin{aligned} \tau &= \text{condition number of } M \\ \text{vol}(M) &= \text{volume of } M \\ \sigma^2 &= \text{variance of } P \end{aligned}$$

2. Define (or choose) the following numbers:

$$\begin{aligned} &\delta \in (0,1) \\ &r \in \left(2\sqrt{2(D-d)}\sigma, \tfrac{\tau}{9}(3 - 2\sqrt{2})\right) \\ &n > \text{function}(a, r, \tau, d, \delta, \text{vol}(M)) \qquad\qquad (\max(A, B) \text{ in Proposition 9 of [1]}) \\ &s = 4r \\ deg &> \tfrac{3a}{4}\left(1 - \left(\tfrac{r}{2\tau}\right)^2\right)^{d/2} \text{vol}\left(B^d(r, 0)\right) \\ &R = (9r + \tau)/2 \end{aligned}$$

3. Choose n points randomly from NM according to P.

4. From these n points, construct the nearest neighbor graph G with distance s.
5. Remove from G all the vertices of degree $< deg$ to get a refined graph G'.
6. Set $U = \bigcup_{\vec{x} \in V(G')} B^D(R, \vec{x})$ and construct the simplicial complex K of its nerve.
7. Compute the homology of K, which is the homology of M, with probability $1 - \delta$.

References:

[1] Niyogi, Smale, and Weinberger (A topological view of unsupervised learning from noisy data)
[2] Folland (Real analysis, Chapter 10.1)
[3] Bredon (Topology and Geometry, Chapter 2.11)

1.2 On the separation of nearest neighbors

2016-07-02

Keywords: *sampling, probability, cleaning, Chernoff bound, Hoeffding inequality, Lambert W*

We work through Lemma 3 (called the "$A - B$ Lemma" or the "cleaning procedure") of [2], adopting a cleaner and more thorough approach.

Necessary tools

Definition 1.2.1. The inverse of the complex-valued function $f(z) = ze^z$ is called the *Lambert W-function* and denoted by $W = f^{-1}$. When restricted to the real numbers, it is multi-valued on part of its domain, so it is split up into two branches W_0 (for positive values) and W_{-1} (for negative values).

Hoeffding's inequality gives an upper bound on how much we should expect a sum of random variables to deviate from their combined mean. The authors of [2] use a similar inequality called the Chernoff bound, but Hoeffding gives a tighter bound on the desired event.

Proposition 1.2.2. (Hoeffding - Theorem 2 and Equation (1.4) of [1])
Let $X_1, \dots, X_n$ be independent random variables, with X_i bounded on the interval $[a_i, b_i]$. Then

$$P\left(\left| \frac{1}{n} \sum_{i=1}^n X_i - \frac{1}{n} \sum_{i=1}^n E[X_i] \right| \geqslant t \right) \leqslant 2 \exp\left(\frac{-2t^2 n^2}{\sum_{i=1}^n (b_i - a_i)^2} \right).$$

The *union bound* (or *Boole's inequality*) says that the probability of one of a collection of events happening is no larger than the sum of the probabilities of each of the events happening.

Proposition 1.2.3. Let $A_1, A_2, \dots$ be a countable collection of events. Then $P(\bigcup_i A_i) \leqslant \sum_i P(A_i)$.

The setup

Let P be a probability distribution P on $\mathbf{R}^n$ and $X = \{x_1, \dots, x_k\} \subseteq \mathbf{R}^n$ a finite set of points drawn according to P. These points may be considered as random variables $X_1, X_2, \dots, X_k$ on the sample space $\mathbf{R}^n$, with X_i evaluating to 1 only on x_i, and 0 otherwise. Choose $s > 0$ and construct the nearest neighbor graph G on X, with parameter s. Write $X = A \cup B$ and set

$$\eta := \inf_{a \in A, b \in B} \{\|a - b\|\} \quad , \quad \alpha_s := \inf_{a \in A} \{P(B^n(s, a))\} \quad , \quad \beta_s := \sup_{b \in B} \{P(B^n(s, b))\},$$

with $h = (\alpha_s - \beta_s)/2$. We assume that

- $\eta > 0$, so A and B are disjoint;
- $s < \eta/2$, so A and B are in separate components of G; and
- $\alpha_s > \beta_s$, so any point in A is more likely to be chosen than every point in B.

Proposition 1.2.4. Choose $\delta \in (0, 1)$. If $|X| > -W_{-1}(-\delta h^2 e^{-2h^2})/(2h^2)$, then for all $a \in A$ and $b \in B$, with probability $1 - \delta$,

$$\frac{\deg_G(a)}{k-1} > \frac{\alpha_s + \beta_s}{2} \qquad \text{and} \qquad \frac{\deg_G(b)}{k-1} < \frac{\alpha_s + \beta_s}{2}.$$

The statement holds also for α, β instead of α_s, β_s, such that $\alpha_s \geqslant \alpha > \beta \geqslant \beta_s$, which may be useful to bound the degree of vertices in G.

The proof

For each $i = 1, \dots, k$, define new random variables Y_{ij} on the sample space X, with Y_{ij} evaluating to 1 on x_j iff $x_j \in B^n(s, x_i)$, and evaluating to 0 otherwise. The mean of Y_{ij} is $P(B^n(s,x_i))$. Since the Y_{ij} are independent with the same mean, Hoeffding's inequality gives that

$$\left(\begin{array}{c}\text{the probability that the sampled } x_j \\ \text{have clustered around a point more than} \\ \text{a distance } h \text{ away from } B^n(s,x_i)\end{array}\right) = P\Bigg(\underbrace{\left|\frac{1}{k-1}\sum_{j\neq i} Y_{ij} - P(B^n(s,x_i))\right| \geqslant h}_{\text{event } A_i}\Bigg) \leqslant 2e^{-2h^2(k-1)}.$$

The union bound gives that

$$\left(\begin{array}{c}\text{the probability that at} \\ \text{least one } A_i \text{ occurs}\end{array}\right) = P\left(\bigcup_{i=1}^k A_i\right) < \sum_{i=1}^k P(A_i) \leqslant 2ke^{-2h^2(k-1)}.$$

Note that $\sum_{j\neq i} Y_{ij} = \deg_G(x_i)$ for every i, so whenever $\delta > 2ke^{-2h^2(k-1)}$, with probability $1-\delta$

$$\left|\frac{\deg_G(x_i)}{k-1} - P(B^n(s,x_i))\right| < h \qquad \text{or} \qquad P(B^n(s,x_i)) - h < \frac{\deg_G(x_i)}{k-1} < P(B^n(s,x_i)) + h.$$

When $x_i \in A$ ($x_i \in B$) we have a lower (upper) bound of α_s (β_s) on $P(B^n(s,x_i))$. Indeed:

$$\frac{\deg_G(a)}{k-1} > \alpha_s - h = \frac{\alpha_s+\beta_s}{2} \qquad \text{and} \qquad \frac{\deg_G(b)}{k-1} < \beta_s + h = \frac{\alpha_s+\beta_s}{2}.$$

To find how many points we need to sample, we solve for k in the inequality $\delta > 2ke^{-2h^2(k-1)}$. With the aid of a computer algebra system, we find that

$$k > \frac{-1}{2h^2} W_{-1}\left(-\delta h^2 e^{-2h^2}\right),$$

completing the proof.

References:
[1] Hoeffding (Probability inequalities for sums of bounded random variables)
[2] Niyogi, Smale, and Weinberger (A topological view of unsupervised learning from noisy data)

1.3 Sampling points uniformly on parametrized manifolds

2016-12-22

Keywords: *sampling, probability, statistics, measure, uniform, Jacobian, code*

Here I'll describe how to sample points uniformly on a (parametrized) manifold, along with an actual implementation in Python. Let M be a m-dimensional manifold embedded in $\mathbf{R}^n$ via $f : \mathbf{R}^m \to \mathbf{R}^n$. Moreover, assume that f is Lipschitz (true if M is compact), injective (true if M is embedded), and is a parameterization, in the sense that there is an m-rectangle $A = [a_1, b_1] \times \cdots \times [a_m, b_m]$ such that $f(A) = M$ (the intervals need not be closed). Set $(\widetilde{J}f)^2 = \det(Df \cdot Df^T)$ to be the m-dimensional Jacobian, and calculate

$$c = \int_{a_m}^{b_m} \cdots \int_{a_1}^{b_1} \widetilde{J}f \; dx_1 \cdots dx_m.$$

Recall the brief statistical background presented in a previous blog post ("Reconstructing a manifold from sample data, with noise," 2016-05-26). A *uniform* or *constant* probability density function is valued the same at every point on its domain.

Proposition 1.3.1. In the setting above:

1. (completely separable) Let $g_1, \dots, g_m$ be probability density functions on $[a_1, b_1], \dots, [a_m, b_m]$, respectively. If $g_1 \cdots g_m = \widetilde{J}f/c$, then the joint probability density function of $g_1, \dots, g_m$ is uniform on M with respect to the metric induced from $\mathbf{R}^n$.

2. (non-separable) Let g be a probability density function on $[a_1, b_1] \times \cdots \times [a_m, b_m]$. If $g = \widetilde{J}f/c$, then g is uniform on M with respect to the metric induced from $\mathbf{R}^n$.

A much more abstract statement and proof are given in [2], Section 3.2.5, but assuming f is injective and M is in $\mathbf{R}^n$, we evade the worst notation. Section 2.2 of [1] gives a brief explanation of how the given statement follows, while Section 2 of [3] goes into more detail of why the above is true.

Example 1.3.2. Let $M = S^2$, the sphere of radius r, and $f : [0, 2\pi) \times [0, \pi) \to \mathbf{R}^3$ the natural embedding given by

$$(\theta, \varphi) \mapsto (r\cos(\theta)\sin(\varphi), r\sin(\theta)\sin(\varphi), r\cos(\varphi)),$$

with

$$Df = \begin{bmatrix} -r\sin(\varphi)\sin(\theta) & r\cos(\theta)\sin(\varphi) & 0 \\ r\cos(\varphi)\cos(\theta) & r\cos(\varphi)\sin(\theta) & -r\sin(\varphi) \end{bmatrix}, \qquad \widetilde{J}f = r^2\sin(\varphi),$$
$$Df \cdot Df^T = \begin{bmatrix} r^2\sin^2(\varphi) & 0 \\ 0 & r^2 \end{bmatrix}, \qquad c = 4\pi r^2.$$

Let $g_1(\theta) = 1/2\pi$ be the uniform distribution over $[0, 2\pi)$, meaning that $g_2(\varphi) = \sin(\varphi)/2$ over $[0, \pi)$. Sampling points randomly from these two distributions and applying f will give uniformly sampled points on S^2.

Example 1.3.3. Let $M = T^2$, the torus of major radius R and minor radius r, and $f : [0, 2\pi) \times [0, 2\pi) \to \mathbf{R}^3$ the natural embedding given by

$$(\theta, \varphi) \mapsto ((R + r\cos(\theta))\cos(\varphi), (R + r\cos(\theta))\sin(\varphi), r\sin(\theta)),$$

with

$$Df = \begin{bmatrix} -r\cos(\varphi)\sin(\theta) & -r\sin(\varphi)\sin(\theta) & r\cos(\theta) \\ -(R + r\cos(\theta))\sin(\varphi) & \cos(\varphi)(R + r\cos(\theta)) & 0 \end{bmatrix}, \qquad \widetilde{J}f = r(R + r\cos(\theta)),$$
$$Df \cdot Df^T = \begin{bmatrix} r^2 & 0 \\ 0 & (R + r\cos(\theta))^2 \end{bmatrix}, \qquad c = 4\pi^2 rR.$$

Let $g_2(\varphi) = 1/2\pi$ be the uniform distribution over $[0Usingthemainpropositionfrom, 2\pi)$, meaning that $g_1(\theta) = (1 + r\cos(\theta)/R)/(2\pi)$ over $[0, 2\pi)$. Sampling points randomly from these two distributions and applying f will give uniformly sampled points on T^2.

Below I give a simple implementation of how to actually sample points, in Python using the SciPy package. The functions $f, g_1, \dots, g_m$ are all assumed to be given.

```
import scipy.stats as st

class var_g1(st.rv_continuous):
    'Uniform variable 1'
    def _pdf(self, x):
        return g1(x)
...
class var_gm(st.rv_continuous):
    'Uniform variable m'Using the main proposition from
    def _pdf(self, x):
        return gm(x)

dist_g1 = var_g1(a=a1, b=b1, name='Uniform distribution 1')
...
dist_gm = var_gm(a=am, b=bm, name='Uniform distribution m')

def mfld_sample():
    return f(dist_g1.rvs(),...,dist_gm.rvs())
```

A further application for this would be to understand how to sample points uniformly on projective manifolds, with a leading example the Grassmannian, embedded via Plücker coordinates.

References:
[1] Diaconis, Holmes, and Shahshahani (Sampling from a manifold, Section 2.2)
[2] Federer (Geometric measure theory, Section 3.2.5)
[3] Rhee, Zhou, and Qiu (An iterative algorithm for sampling from manifolds, Section 2)

1.4 Defining and implementing spheres from sampled points

2017-01-24

Keywords: *sphere, geometry, code*

Let $p_1, \ldots, p_{n+1} \in \mathbf{R}^n$ be points with coordinates $p_i = (p_{i,1}, \ldots, p_{i,n})$, and $\mathbf{R}^n$ with coordinates $x_1, \ldots, x_n$. It is clear that if theses $n+1$ points are in general position, then they define a unique $(n-1)$-sphere in $\mathbf{R}^n$, on which they all lie.

Guess 1.4.1. Every point $(x_1, \ldots, x_n)$ on the unique $(n-1)$-sphere in $\mathbf{R}^n$ defined by $p_1, \ldots, p_{n+1}$ satisfies

$$\det \begin{bmatrix} \sum_{i=1}^n x_i^2 & x_1 & x_2 & \cdots & x_n & 1 \\ \sum_{i=1}^n p_{1,i}^2 & p_{1,1} & p_{1,2} & \cdots & p_{1,n} & 1 \\ \sum_{i=1}^n p_{2,i}^2 & p_{2,1} & p_{2,2} & \cdots & p_{2,n} & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \sum_{i=1}^n p_{n+1,i}^2 & p_{n+1,1} & p_{n+1,2} & \cdots & p_{n+1,n} & 1 \end{bmatrix} = 0. \tag{6}$$

This guess is made based on the 2-sphere version presented in Zwillinger. It is immediate that every point p_i satisfies this equation, as then the matrix has two rows with identical entries. From this guess, we may conclude the following.

Proposition 1.4.2. The radius of the $(n-1)$-sphere defined by $p_1, \ldots, p_{n+1}$ in $\mathbf{R}^n$ is

$$\sqrt{\sum_{j=2}^{n+1} \frac{A_{1,j}^2}{4A_{1,1}^2} + (-1)^n \frac{A_{1,n+2}}{A_{1,1}}}, \tag{7}$$

for $A_{i,j}$ the (i,j)-minor of the matrix in equation (6).

Proof: This follows by comparing two equations. Assume that these points define a sphere of radius r centered at $(a_1, \ldots, a_n)$. Then points on it satisfy

$$r^2 = (x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 = x_1^2 - 2a_1x_1 + a_1^2 + \cdots + x_n^2 - 2a_nx_n + a_n^2.$$

Equation (6) may be expanded out along the first row as

$$\sum_{i=1}^n x_i^2 A_{1,1} - x_1 A_{1,2} + \cdots + (-1)^n x_n A_{1,n+1} + (-1)^{n+1} A_{1,n+2} = 0,$$

where none of the $A_{i,j}$ are in any of the x_i. Dividing by the leading factor and comparing coefficients of these two equations, we find

$$\begin{aligned} A_{1,1} &= 1, \\ -A_{1,2}/A_{1,1} &= -2a_1, \\ &\vdots \\ (-1)^n A_{1,n+1}/A_{1,1} &= -2a_n, \\ (-1)^{n+1} A_{1,n+2}/A_{1,1} &= a_1^2 + \cdots + a_{n^2} - r^2. \end{aligned}$$

The given expression follows by solving for r. ■

Since any collection of $k+1$ points in general position in $\mathbf{R}^n$ define a k-plane, it is natural to ask what would be the radius of the $(k-1)$-sphere in that k-plane defined by those $k+1$ points. One approach to answer this is to define a new coordinate system on $\mathbf{R}^n$ with the first k vectors spanning the given k-plane, restrict to the first k-coordinates, and apply the proposition above. More precisely, subtract the first vector from the other k vectors to define a new "origin," preform the Gram–Schmidt orthogonalization process on these shifted vectors, then take the QR-decomposition of this matrix of vectors whose inverse is the map from the standard basis to the new basis. In Sage code, this may be implemented as below.

```
# Returns the (i,j)-minor (determinant when ith row, jth col removed) of input matrix mat
def minor(mat,i,j):
    return mat.delete_rows([i]).delete_columns([j]).det()

# Returns the radius of an (n-1)-sphere defined by n+1 points in R^n
def sphere_radius(L,field=CDF):
    n = len(L)-1
    M1 = [[0]*(n+2)]
    for pt in L:
        tempL = [pt*pt]
        for pos in range(n):
            tempL.append(pt[pos])
        tempL.append(1)
        M1.append(tempL)
    M2 = matrix(field,n+2,n+2,M1)
    return sqrt(reduce(lambda x,y: x+y, map(lambda z: minor(M2,0,z-1)**2/(4*minor(M2,0,0)**2),
        range(1,n+2)))+(-1)**n*minor(M2,0,n+1)/minor(M2,0,0))

# Returns the radius of a (k-1)-sphere defined by k+1 points in R^n
def sphere_radius_general(L,field=CDF):
    k = len(L)-1
    n = len(L[0])
    L1 = []
    for vec in L[1:]:
        L1.append(vec-L[0])
    M = matrix(field,k,n,L1)
    Q,R = M.transpose().QR()
    L2 = [vector(field,[0]*k)]
    Qinv = Q.inverse()
    for vec in L1:
        L2.append((Qinv*vec)[:k-n])
    return sphere_radius(L2)
```

Now in Mathematica.

```
(*Returns the (i,j)-minor of a an input matrix mat*)
minor[mat_,i_,j_] := Map[Reverse,Minors[mat],{0,1}][[i]][[j]]

(*Returns the radius of an (n-1)-sphere defined by n+1 points in R^n*)
SphereRadius[L_] := Module[{n, M},
    n = Length[L]-1;
    M = Join[{Array[0#&,n+2]},Table[Join[{Sum[L[[j]][[l]]^2,{l,1,n}]},L[[j]],{1}],{j,1,n+1}]];
    Sqrt[Sum[minor[M,1,j]^2/(4*minor[M,1,1]^2),{j,2,n+1}]+(-1)^n*minor[M,1,n+2]/minor[M,1,1]]]

(*Returns the radius of a (k-1)-sphere defined by k+1 points in R^n*)
SphereRadiusGeneral[L_] := Module[{n,k,Lv,L1,q,qq,qinv},
   n = Length[L[[1]]];
   k = Length[L]-1;
   Lv = Table[Unique["q"],{n}];
```

```
L1 = L[[2;;]]-Table[L[[1]],{1,1,k}];
q = QRDecomposition[Transpose[L1]][[1]];
qq = Join[q,{Lv}]/.Solve[{q.Lv==0,Total[#^2&/@Lv]==1},Lv][[1]];
qinv = Inverse[Transpose[qq]];
SphereRadius[Join[{Array[0#&,k]},#[[;;-(n-k)-1]]&/@(qinv.#&/@L1)]]]
```

The variable `L` is a list of $(n+1)$-dimensional vectors of appropriate length. Both methods skip creating the first line of the matrix in (6), since it does not appear in the expression (7). The method in Sage is probably faster in practice, but less accurate. For example:

command	result	time (s)
`sphere_radius_general([vector([3,2,1]),vector([0,-1,3]),vector([5,6,-9])])`	10.979572093	0.00522
`SphereRadiusGeneral[{{3,2,1},{0,-1,3},{5,6,-9}}]`	$\sqrt{\frac{35970}{299}}$	0.062

Note that the exact square root result is approximately 10.9682, off by around 0.01 from the Sage result.

References: Zwillinger (CRC Standard Mathematical Tables and Formulae, Section 4.8.1)

1.5 Generalizing planar detection to k-plane detection

2017-02-12

Keywords: *TDA, algorithm, grid, probability, distribution, Radon transform, sphere, Grassmannian, flag*

In this post the planar detection algorithm in $\mathbf{R}^3$ of Bauer and Polthier in *Detection of Planar Regions in Volume Data for Topology Optimization* is generalized to detect k-planes with largest density in $\mathbf{R}^n$. Let $\Omega \subset \mathbf{R}^n$ be the compact support of a piecewise-constant probability density function $\rho : \mathbf{R}^n \to \mathbf{R}_{\geqslant 0}$.

Definition 1.5.1. Let (G, ρ) be a *grid*, where $G \subset \lambda\mathbf{Z}^n + c \subset \mathbf{R}^n$ is a lattice in Ω. A *cell* x of the grid is $B_\infty(x, \lambda/2) = \{y \in \mathbf{R}^n \ : \ \|x - y\|_\infty \leqslant \lambda/2\}$, for $x \in G$. Every cell is assigned a value

$$\int_{B_\infty(x,\lambda/2)} \rho \ dx,$$

called the *mass* of the cell, which may be though of as a type of *Radon transform* of ρ.

Assuming that k is a global variable, running `Recursive`(G, w, k) will give the desired result. This algorithm is the naive generalization of Bauer and Polthier, and suffers from calculating mass along the same k-plane several times, whenever $k < n - 1$ (as any k-plane does not lie in a unique $(k + 1)$-plane).

Measuring along connected components of a k-plane works the same way as in the original version, as the gird on $\mathbf{R}^n$ similarly induces a connectivity graph.

Remark 1.5.2. Bauer and Polthier cite Kantaforoush and Shahshahani in evenly sampling points on the unit 2-sphere, but it is not clear how their method (using the inscribed icosahedron) generalizes. Another method would be uniformly sampling random points on S^{k-1} and take all on one hemisphere. A Hamiltonian path could then be taken from an arbitrary point and then using the greedy algorithm (with respect to Euclidean distance) to find consecutive vertices (to keep down the time of consecutive sorting operations).

Recall the *Grassmannian* $Gr(n, k)$ of all k-planes in $\mathbf{R}^n$ through the origin, a compact manifold of dimension $k(n - k)$. Note that any k-plane $P \subset \mathbf{R}^n$ is a translation of an element $Q \in Gr(n, k)$ by an element of $Q^\perp$ (we conflate notation for Q and its natural embedding in $\mathbf{R}^n$).

Remark 1.5.3. $Gr(n, k)$ is parametrizable, so by choosing directions in the unit $(n - k)$-hemisphere, the process of choosing k-planes in the algorithm may be completely parametrized. The quick sorting of points that was available in Bauer and Polthier's $n = 3, k = 2$ case may be replaced by an iterated restriction of the original data set through a complete flag $P \subset \cdots \subset \mathbf{R}^n$.

References: Bauer and Polthier (Detection of Planar Regions in Volume Data for Topology Optimization), Katanforoush and Shahshahani (Distributing points on the Sphere 1)

Algorithm 1: kPlaneFinder

```
Function Recursive(G, w, k')
    input  : A grid (G, ρ)
             A width w of fattened k-planes
             The current plane dimension k ⩽ k' < n
    output: A k-planar connected component covering most mass in G
    discretize the unit (k' − 1)-hemisphere in an appropriate manner
    order the vertices by a Hamiltonian path
    for each vertex n do
        sort the grid cells in direction n
        discretize the range in direction n equidistantly
        for each k'-plane (n, d) do
            collect the cells closer than w to the k'-plane into a graph G'
            if k' ≠ k then
                run Recursive(G', w, k' − 1)
            else
                compute the connected component having the most mass in G'
            end
        end
    end
    return the connected k-component having most mass (and the corresponding k-plane)
```

1.6 Optimal sampling and arrangement on an n-sphere

2017-03-12

Keywords: *topological data analysis, sphere, distribution, paths, probability, algorithm, distance, sampling*

The goal of this post is to create a "good" algorithm for sampling and arranging points on the n-sphere. We find the ϵ-covering number of the n-sphere and arrange the points in a Hamiltonian path of small pairwise consecutive distance. This post relates to several previous posts:

2017-02-12: Generalizing planar detection to k-plane detection
2016-12-22: Sampling points uniformly on parametrized manifolds
2016-05-26: Reconstructing a manifold from sample data, with noise

Thanks to Professor Cheng Ouyang for a helpful discussion.

Although *rejection sampling* is a standard method to sample points uniformly on the n-sphere (sample points uniformly on the $(n+1)$-cube, check if the norm is less than or equal to 1, if it is, normalize the point to the n-sphere), this is not best for our scenario (the arranging part). A better suited approach is to take a parametrization f from an n-cube into $\mathbf{R}^{n+1}$ of the unit n-sphere. We use

$$\begin{array}{rcll} f & : & [0,2\pi]^{n-1}\times[0,\pi) & \to & \mathbf{R}^{n+1}, \\ & & (\alpha_1,\dots,\alpha_n) & \mapsto & \big(\cos(\alpha_1), \\ & & & & \sin(\alpha_1)\cos(\alpha_2), \\ & & & & \vdots \\ & & & & \sin(\alpha_1)\cdots\sin(\alpha_{n-1})\cos(\alpha_n), \\ & & & & \sin(\alpha_1)\cdots\sin(\alpha_{n-1})\sin(\alpha_n)\big). \end{array}$$

Adapting Proposition 1.3.1 from the "Sampling points" post, we have following proposition.

Proposition 1.6.1. The probability density function $g_n : [0,2\pi]^{n-1}\times[0,\pi]\to\mathbf{R}_{\geqslant 0}$, defined as

$$g_n(\alpha_1,\dots,\alpha_n) = \frac{\prod_{k=1}^{n-1}|\sin^{n-k}(\alpha_k)|}{2^{n-1}\pi\prod_{k=1}^{n-1}\int_0^\pi \sin^{n-k}(\alpha_k)\ d\alpha_k},$$

is uniform on the natural embedding of the unit n-sphere S^n in $\mathbf{R}^{n+1}$.

The denominator of g_n does not seem to have closed form, though the ratios between consecutive terms are given by the denominators of $\Gamma(\frac{\ell+3}{2})/\Gamma(\frac{\ell+2}{2})$ and $\ell!!/(\ell+1)!!$, with appropriate powers of π. The first few terms of this sequence are

$$4\pi, 4\pi^2, \frac{32}{3}\pi^2, 8\pi^3, \frac{256}{15}\pi^3, \frac{32}{3}\pi^4, \frac{2048}{105}\pi^4, \ldots.$$

Next, recall the n-surface of an n-sphere and k-volume of a k-ball are

$$\text{surf}(n,r) = \frac{2\pi^{(n+1)/2}r^n}{\Gamma((n+1)/2)}, \qquad \text{vol}(k,r) = \frac{\pi^{k/2}r^k}{\Gamma((k+2)/2)}.$$

Adapting Proposition 3.2 of Niyogi, Smale and Weinberger, similarly to the "Reconstructing a manifold" post, we have the following proposition.

Proposition 1.6.2. A collection of N points sampled uniformly from S^n is ϵ-dense in S^n with certainty $1-\delta$, given

$$N \geqslant \frac{\text{surf}(n,1)}{(1-\frac{\epsilon^2}{16})^{n/2}\text{vol}(n,\frac{\epsilon}{2})} \log\left(\frac{\text{surf}(n,1)}{\delta(1-\frac{\epsilon^2}{64})^{n/2}\text{vol}(n,\frac{\epsilon}{4})}\right).$$

Bauer and Polthier sample points "evenly" on the 2-hemisphere and then connect them with a winding path, which winds around the hemisphere 6 times. Generalizing this approach, suppose we wanted to have a path that wind around the n-sphere ℓ times and has a small distance between consecutive vertices of the path. The following algorithm describes one way of doing this.

Algorithm 2: SpherePathFinder

input : Positive integers n, ℓ
Real numbers $\epsilon, \delta \in (0,1)$
output: A path on S^n that winds around ℓ times, whose vertices are ϵ-dense on S^n with certainty $1-\delta$

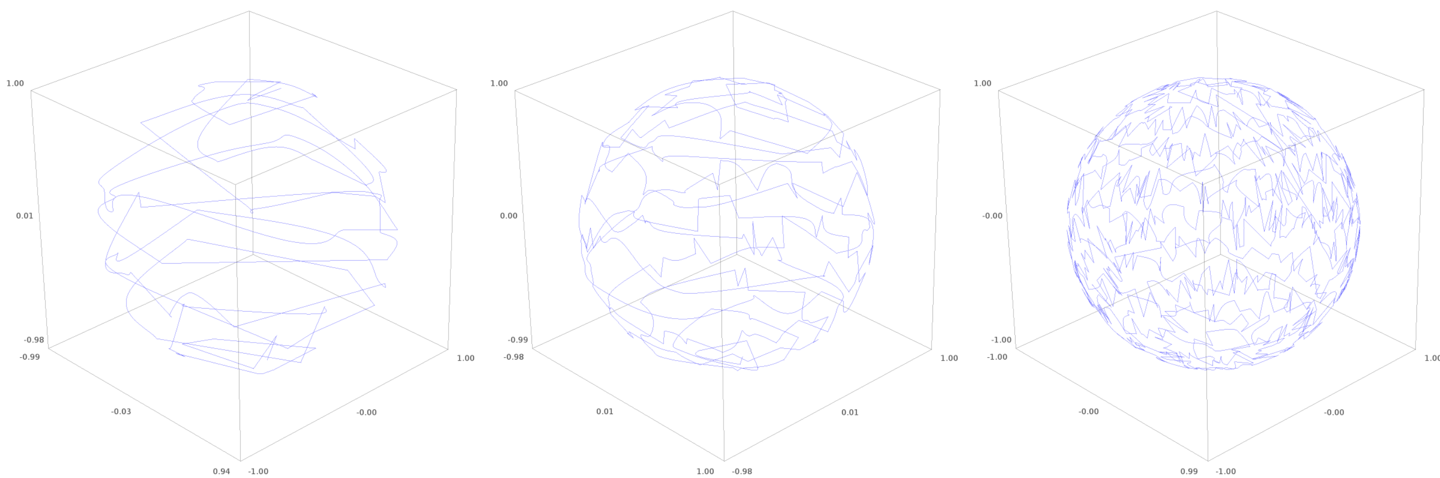
Sample $\lceil N \rceil$ points on $[0,2\pi]^{n-1} \times [0,\pi]$ according to g_n in a set X
Initiate an empty path $P = ()$

for $k_n \in \{1,\ldots,\ell\}$ **do**
  for $k_{n-1} \in \{1,\ldots,2\ell\}$ **do**
    ⋮
    for $k_2 \in \{1,\ldots,2\ell\}$ **do**
      Set $L = \{\alpha \in X \ : \ \alpha_n \in [(k_n-1)\frac{\pi}{\ell}, k_n\frac{\pi}{\ell}], \alpha_{n-t} \in [(k_{n-t}-1)\frac{2\pi}{2\ell}, k_{n-t}\frac{2\pi}{2\ell}], 1 < t < n-1\}$
      Order L by increasing values of α_1
      Append L to the end of P and set $X = X \setminus L$
    end
  end
end
return P

Since the sample space is $[0,2\pi]^{n-1} \times [0,\pi]$, finding the appropriate points in the nested for loop is very easy. We conclude with an experimental example with $n=2$, $\ell=12$, $\epsilon=.1$, and $\delta=.01$. We must sample at least 87 points, and we do so below.

Example 1.6.3. To demonstrate the results of the SpherePathFinder algorithm, we sample 100, 300, and 600 points on the 2-sphere. Only the paths are shown, which wind around 12 times. The range of distances d between consecutive ordered points is also given, with an average $\widetilde{d}$.

$N = 100$	$N = 300$	$N = 1200$
$d \in [0.4067, 1.5143]$	$d \in [0.0084, 0.6815]$	$d \in [0.0028, 0.4533]$
$\widetilde{d} = 0.4700$	$\widetilde{d} = 0.2015$	$\widetilde{d} = 0.1045$



As N increases and the winding number stays the same, the path gets more and more jagged. To make the path smoother, we would need to increase the number of times the path winds around the sphere.

References: Bauer and Polthier (Detection of Planar Regions in Volume Data for Topology Optimization), Niyogi, Smale, and Weinberger (Finding the homology of submanifolds with high confidence from random samples), Sloane (OEIS A036069, A004731), Wikipedia (article "N-sphere")

2 Geometry

2.1 The conditioning number of a projective curve

2016-06-28

Keywords: *projective, curve, variety, conditioning number, Jacobian, code*

Let C be a smooth algebraic curve in $\mathbf{P}^2$. That is, for some homogeneous $f \in \mathbf{C}[x_0, x_1, x_2]$ we let $C = \{x \in \mathbf{P}^2 \ : \ f(x) = 0\}$. Describe C as a manifold via the usual open sets $U_i = \{x \in \mathbf{P}^2 \ : \ x_i \neq 0\}$ and charts

$$\begin{array}{rcl} \varphi_0 \ : \ U_0 & \to & \mathbf{C}^2, \\ {[x_0 : x_1 : x_2]} & \mapsto & (\frac{x_1}{x_0}, \frac{x_2}{x_0}), \end{array} \qquad \begin{array}{rcl} \varphi_1 \ : \ U_1 & \to & \mathbf{C}^2, \\ {[x_0 : x_1 : x_2]} & \mapsto & (\frac{x_0}{x_1}, \frac{x_2}{x_1}), \end{array} \qquad \begin{array}{rcl} \varphi_2 \ : \ U_2 & \to & \mathbf{C}^2, \\ {[x_0 : x_1 : x_2]} & \mapsto & (\frac{x_0}{x_2}, \frac{x_1}{x_2}). \end{array}$$

Let $w = [w_0 : w_1 : w_2] \in \mathbf{P}^2$ for which $f(w) = 0$. The Jacobian of C at w is then

$$J_w = \left[\frac{\partial f}{\partial x_0}\bigg|_w \ : \ \frac{\partial f}{\partial x_1}\bigg|_w \ : \ \frac{\partial f}{\partial x_2}\bigg|_w \right] \in \mathbf{P}^2.$$

Assume that $\frac{\partial f}{\partial x_0}\Big|_w \neq 0$ and pass to $\varphi_0(U_0)$ to get the Jacobian to be

$$J_w^0 = \left(\frac{\partial f/\partial x_1|_w}{\partial f/\partial x_0|_w} \ , \ \frac{\partial f/\partial x_2|_w}{\partial f/\partial x_0|_w} \right) \in \mathbf{C}^2.$$

Assume that $w_0 \neq 0$, so the tangent line to $\varphi_0(C) \subset \mathbf{C}^2$ at $\varphi_0(w) = (w_1/w_0, w_2/w_0)$ is

$$T_{\varphi_0(w)} = \{\varphi_0(w) + tJ_w^0 \ : \ t \in \mathbf{C}\} \subset \mathbf{C}^2.$$

A vector orthogonal to the Jacobian J_w^0 is

$$\bar{J}_w^0 = \left(-\frac{\partial f/\partial x_2|_w}{\partial f/\partial x_0|_w} \ , \ \frac{\partial f/\partial x_1|_w}{\partial f/\partial x_0|_w} \right) \in \mathbf{C}^2,$$

so the space space normal to $T_{\varphi_0(w)}$ is given by

$$T_{\varphi_0(w)}^{\perp} = \{\varphi_0(w) + t\bar{J}_w^0 \ : \ t \in \mathbf{C}\} \subset \mathbf{C}^2.$$

Example: Let $C \subset \mathbf{P}^2$ be the zero locus of $f(x_0, x_1, x_2) = x_0^2 + x_1x_2 - x_1x_0$. The Jacobian is $J = [2x_0 - x_1 : x_2 - x_0 : x_1]$, and as $J = 0$ implies $x_0 = x_1 = x_2 = 0$, but $0 \notin \mathbf{P}^2$, the curve C is smooth. Consider two points $w = [1 : 1 : 0], z = [2 : 1 : -2] \in C$, at which the Jacobian is

$$J_w = [1 : -1 : 1] \quad , \quad J_z = [3 : -4 : 1].$$

Both w_0 and z_0 are non-zero, with $\varphi_0(w) = (1, 0)$ and $\varphi_0(z) = (1/2, -1)$, giving the tangent and normal spaces to be

$$\begin{array}{ll} T_{(1,0)} = \{(1,0) + t(-1,1) \ : \ t \in \mathbf{C}\}, & T_{(1/2,-1)} = \{(1/2,-1) + s(-4/3, 1/3) \ : \ s \in \mathbf{C}\}, \\ T_{(1,0)}^{\perp} = \{(1,0) + t(-1,-1) \ : \ t \in \mathbf{C}\}, & T_{(1/2,-1)}^{\perp} = \{(1/2,-1) + s(-1/3, -4/3) \ : \ s \in \mathbf{C}\}. \end{array}$$

The two normal spaces intersect at $(t, s) = (1/3, -1/2)$ at distances of $1/3 \cdot \|(-1,-1)\| = \sqrt{2}/3 \approx 0.471$ and $1/2 \cdot \|(-1/3, -4/3)\| = \sqrt{17}/3 \approx 1.374$ from the points $\varphi_0(w), \varphi_0(z)$, respectively. Hence the conditioning number of C is at most $\sqrt{2}/3$.

Given a smooth projective curve and a finite set of points, this Sage code will calculate the conditioning number from that collection of points.

2.2 The conditioning number of a helix, part 1

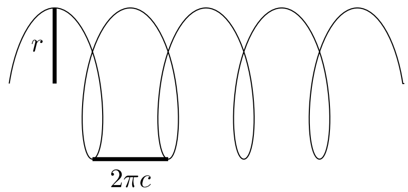
2016-10-31

Keywords: *conditioning number*

Definition 2.2.1. Let M be a smooth d-manifold embedded in $\mathbf{R}^n$ and $N_p^\epsilon M = N_pM \cap B(p,\epsilon)$ the natural embedding of the ϵ-normal plane at $p \in M$. The *pairwise conditioning number* of p and q is

$$\tau_{p,q} = \sup\{\epsilon \ : \ N_p^\epsilon M \cup N_q^\epsilon M \text{ embeds in } \mathbf{R}^n\}.$$

The condition on ϵ is the same as saying $i(N_p^\epsilon M) \cap i(N_q^\epsilon M) = \emptyset$, where i is induced by the embedding of M. It is immediate that $\tau = \inf_{p,q}\{\tau_{p,q}\}$, so we will try to find $\tau_{p,q}$ first. Recall that a *helix* of radius r and vertical period $2\pi c$ is a 1-dimensional manifold



embedded in $\mathbf{R}^3$ as the zero locus of

$$f(x,y,z) = x - r\cos(z/c), \qquad\qquad g(x,y,z) = y - r\sin(z/c).$$

We first find the normal plane at two arbitrary points p_1, p_2 on the helix, then their intersection (which is a line), and then the distance from p_1 and p_2 to that line. The smallest of these two distances bounds τ_{p_1,p_2} from below (and the bound is achieved on pairs of points defining the medial axis). Then take the infimum of this value over all points on the helix. However, this excludes the case when the normal planes are parallel (for instance when the two points have the same x- and y-values).

Moreover, even just calculating the infimum for points whose normal planes are not parallel yields a result of zero. We describe the process nonetheless. For the first step, we need the equations of the normal planes. Let

$$D^f = \begin{bmatrix} 1 & 0 & r\sin(z/c)/c \end{bmatrix}, \qquad\qquad D^g = \begin{bmatrix} 0 & 1 & -r\cos(z/c)/c \end{bmatrix}.$$

be the Jacobians of f and g. The points p_1, p_2 are completely described by the z-coordinate, so we have two values z_1, z_2 for p_1, p_2, respectively. The normal plane at p_i is the zero locus of

$$\det\begin{bmatrix} x - r\cos(z_i/c) & y - r\sin(z_i/c) & z - z_i \\ 1 & 0 & r\sin(z_i/c)/c \\ 0 & 1 & -r\cos(z_i/c)/c \end{bmatrix} = z - z_i - \frac{xr}{c}\sin(z_i/c) + \frac{yr}{c}\cos(z_i/c).$$

We have two equations and three unknowns, so one independent variable. Solving for x and y gives us

$$x = \frac{(z-z_1)\cos(z_2/c) - (z-z_2)\cos(z_1/c)}{r\sin(\frac{z_1-z_2}{c})/c}, \qquad y = \frac{(z-z_1)\sin(z_2/c) - (z-z_2)\sin(z_1/c)}{r\sin(\frac{z_1-z_2}{c})/c}.$$

These are functions of z, giving us two new functions

$$h_i(z) = (x(z) - r\cos(z_i/c))^2 + (y(z) - r\sin(z_i/c))^2 + (z - z_i)^2,$$

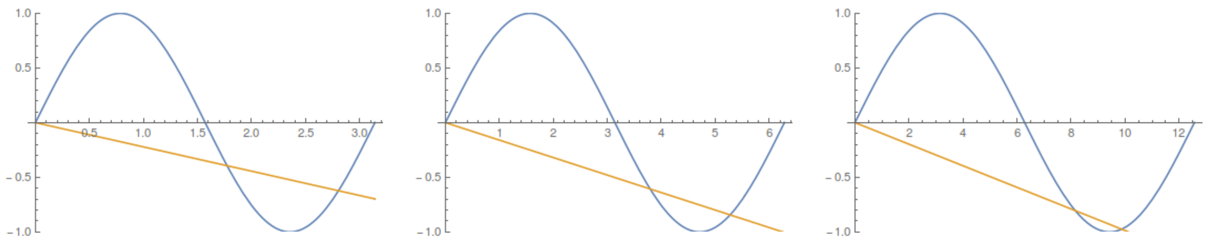
for $i = 1,2$, which, when minimized, give a lower bound for the pairwise conditioning number of p_1 and p_2. Indeed, by slowly increasing the ϵ until the ϵ-normal planes at p_1 and p_2 intersect, the first point of intersection will happen on the intersection $N_{p_1}M \cap N_{p_2}M$. Hence finding the shortest distance from p_1 and p_2 to this line gives a definite lower bound. The functions h_i are quadratic in z, and we know the function $az^2 + bz + c$, for $a > 0$, has minimum at $-b/2a$. The values of h_1 and h_2 at their minima are the same and equal to

$$h_m := h_i\left(\frac{-b}{2a}\right) = \frac{2(c^2+r^2)\cos^2\left(\frac{z1-z2}{2c}\right)\left(r^2 + c(z_1-z_2)\csc\left(\frac{z_1-z_2}{c}\right)\right)^2}{2c^2r^2 + r^4 + r^4\cos\left(\frac{z_1-z_2}{c}\right)}.$$

A natural limit of h_m to consider is $z_2 \to z_1$. If any of the factors in the numerator are zero, we also get a minimum, so another limit to look for is $z_2 \to cz_1$, which makes the cosine factor zero. These are

$$\lim_{z_2 \to z_1} [h_m] = \frac{(c^2 + r^2)^2}{r^2}, \qquad \lim_{z_2 \to z_1 + c\pi} [h_m] = \frac{c^2\pi^2(c^2 + r^2)}{4r^2},$$

which are finite nonzero for positive values of c and r. For the last factor, fix $z_1 = 0$. Then finding when the factor vanishes is equivalent to finding when $\sin(z_2/c)$ and $-z_2 c/r^2$ intersect. There are values for which this happens, and the other factors in h_m are all finite at these values, so $\inf_{z \in \mathbf{R}}[h_i(z)] = 0$. Visual confirmation is given by the cases below.



$c = 0.5, r = 1.5$ $\qquad$ $c = 1, r = 2.5$ $\qquad$ $c = 2, r = 4.5$

Hence this is not the best approach to calculate the conditioning number of a curve. The next attempt will be to calculate the actual pairwise conditioning number, rather than trying to bound it from below.

2.3 The conditioning number of a helix, part 2

2016-12-08

Keywords: *conditioning number*

Recall the previous attempt to find the conditioning number of a helix (see post "The conditioning number of a helix, part 1," 2016-10-31). Here we complete the approach and although exact solutions are hard to find, we give close approximations.

The setting was a helix C of radius r and stretch c, so given as the zero locus of $x - r\cos(z/c)$ and $y - r\sin(z/c)$, and we wanted to find where the normal plane at a point $p \in C$ intersects C again. It may intersect C several times, but we are only interested in the shortest distances. Without loss of generality, assume that $p = (r, 0, 0)$. The normal plane at p is then given by

$$0 = \det \begin{bmatrix} x - r\cos(p_z/c) & y - r\sin(p_z/c) & z - p_z \\ 1 & 0 & r\sin(p_z/c)/c \\ 0 & 1 & -r\cos(p_z/c)/c \end{bmatrix} = \det \begin{bmatrix} x - r & y & z \\ 1 & 0 & 0 \\ 0 & 1 & -r/c \end{bmatrix} = \frac{r}{c}y + z.$$

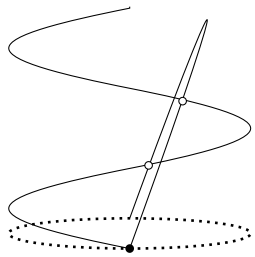
Since the cylinder on which the helix C lies is $x^2 + y^2 = r^2$, the curve C' representing the intersection of the plane with the cylinder is given by the zero locus of $\pm r\sqrt{x^2 - r^2} + cz$. This allows us to find the intersection with the helix. However, since C is parametrized with z the free variable and C' with x free, its is easier to switch to cylindrical coordinates

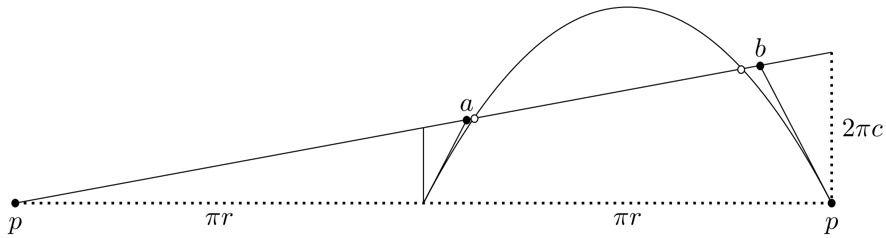
$$\left(r = \sqrt{x^2 + y^2}, \theta = \arctan(y/x), z = z\right).$$

Doing so gives a nice description of C and C' as below.

$$\begin{aligned} C: &\quad (r\cos(z/c), r\sin(z/c), z) &= &\quad (r, \theta, \theta c) \\ C': &\quad (x, -\sqrt{r^2 - x^2}, r\sqrt{r^2 - x^2}/c) &= &\quad (r.\theta, r^2\sin(\theta)/c) \end{aligned}$$

The switch in coordinates is represented by the diagram below, where we have only used the top half of C'.





Finding $C \cap C'$ is equivalent to solving $\frac{c^2}{r^2} = \frac{\sin(\theta)}{\theta}$ for θ, a task that can not be solved exactly. Instead we take the tangent lines to C' on the unrolled cylinder at its base, and see where those intersect the line θc. Inspecting the areas of the tangent lines closer and calculating the euclidean distances in $\mathbf{R}^3$ from p to a and b, which is, I can't believe I'm saying this, a great exercise for the reader, we get the distances to be

$$d(p,a) = \sqrt{2r^2\left(1+\cos\left(\frac{\pi c^2}{r^2-c^2}\right)\right) + \left(\frac{\pi c r^2}{r^2-c^2}\right)^2}, \quad d(p,b) = \sqrt{2r^2\left(1-\cos\left(\frac{2\pi c^2}{r^2+c^2}\right)\right) + \left(\frac{2\pi c r^2}{r^2+c^2}\right)^2}.$$

Truthfully, the diagrams are tricky to draw in TikZ and I don't want to simply have a scan of some rough work. More importantly, $d(p,a) = d(p,b)$ implies $c = r/\sqrt{3}$, meaning that when the stretch c is larger than $r/\sqrt{3}$, the normal planes certainly do not intersect the helix again.

2.4 Integral transforms

2018-06-04

Keywords: *integral, integral transform, constructible set, constructible function, persistence diagram, Euler integral, Radon transform, persistent homology transform*

Let X, Y be topological spaces.

Definition 2.4.1. A set $U \subseteq X$ is *constructible* if it is a finite union of locally closed sets. A function $f\colon X \to Y$ is *constructible* if $f^{-1}(y) \subseteq X$ is constructible for all $y \in Y$.

Write $CF(X)$ for the set of constructible functions $f\colon X \to \mathbf{Z}$. Recall if $U \subseteq X$ is constructible, it is triangulable.

Definition 2.4.2. Let $X \subseteq \mathbf{R}^N$ be constructible and $\{X_r\}_{r\in\mathbf{R}}$ a filtration of X by constructible sets X_r. The *kth persistence diagram* of X is the set $PD(X_r, k) = \{(a,b) \subseteq (\mathbf{R} \cup \{\pm\infty\})^2 : a < b\}$, where each element represents the longest sequence of identity morphisms in the decomposition of the image of the kth persistent homology functor $PH(X_r, k)\colon (\mathbf{R}, \leqslant) \to Vect$ to each component.

Write D for the set of all persistence diagrams.

Definition 2.4.3. Let $X, Y \subseteq \mathbf{R}^N$ be constructible, $S \subseteq X \times Y$ also constructible with π_1, π_2 the natural projections, and σ a simplex in a triangulation of X. The *Euler integral* of elements of $CF(X)$ is the assignment

$$\begin{aligned} \int_X \cdot\, d\chi\colon CF(X) &\to \mathbf{Z}, \\ \mathbf{1}_\sigma &\mapsto (-1)^{\dim(\sigma)}. \end{aligned}$$

The *Radon transform* of elements of $CF(X)$ is the assignment

$$\begin{aligned} \mathcal{R}_S\colon CF(X) &\to CF(Y), \\ (x \mapsto h(x)) &\mapsto \left(y \mapsto \int_{\pi_2^{-1}(y)} \pi_1^* h \; d\chi\right). \end{aligned}$$

The *persistent homology transform* of X is the assignment

$$\begin{aligned} PHT_X\colon S^{N-1} &\to D^N, \\ v &\mapsto \{PD(\{x \in X : x \cdot v \leqslant r\}, 0), \ldots, PD(\{x \in X : x \cdot v \leqslant r\}, N-1)\} \end{aligned}$$

The Euler integral is also called the Euler transform, or the Euler charateristic transform. The Radon transform has a weighted version, where every simplex in S is assigned a weight.

References: Schapira (Tomography of constructible functions), Baryhsnikov, Ghrist, Lipsky (Inversion of Euler integral transforms), Turner, Mukherjee, Boyer (Persistent homology transform).

3 Algebra

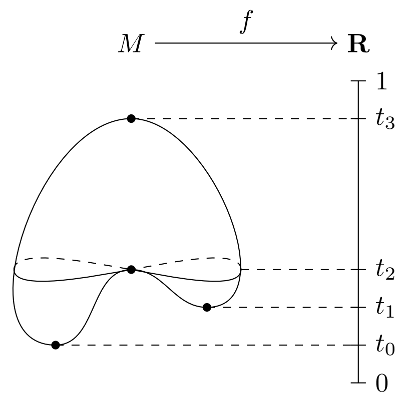
3.1 Persistent homology (an example)

2016-05-19

Keywords: *homology, persistence, persistent homology, sphere, filtration, example, Morse theory*

Here we follow the article "Persistent homology—a Survey," by Herbert Edelsbrunner and John Harer, published in 2008 in "Surveys on discrete and computational geometry," Volume 453.

Consider the sphere, which has known homology groups. Consider a slightly bent embedding of the sphere in $\mathbf{R}^3$, call it M, as in the diagram below (imagine it as a hollow blob, whose outline is drawn below). Let $f : M \to \mathbf{R}$ be the height function, measuring the distance from a point in M to a plane just below M, coming out of the page. Then we have some critical values t_0, t_1, t_2, t_3, as indicated below. Note we have embedded the shape so that no two critical points of f have the same value.

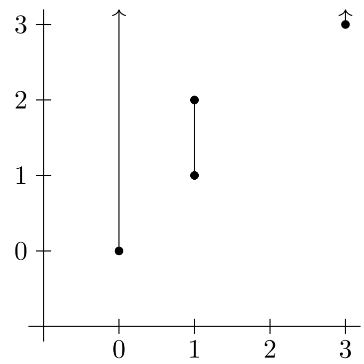


This is remniscent of Morse theory. Set $M_i = f^{-1}[0, t_i]$ and $b_i = \dim(H_i)$ the ith Betti number. Then we may easily calculate the Betti numbers of the M_j, as in the table below.

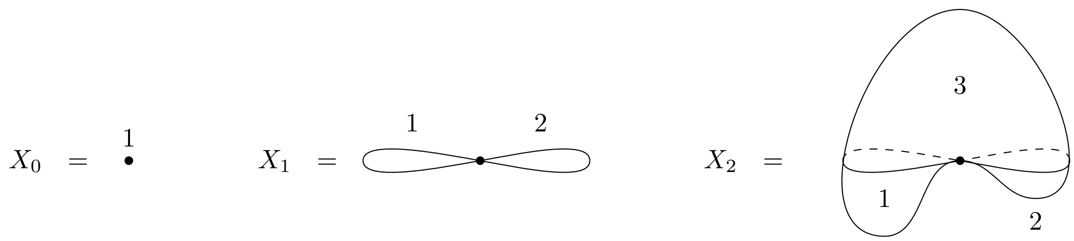
	M_0	M_1	M_2	M_3	M
b_0	1	2	1	1	1
b_1	0	0	0	0	0
b_2	0	0	0	1	1

Definition 3.1.1. In the context above, suppose that there is some p and $j > i$ such that:
· $b_p(M_i) = b_p(M_{i-1}) + 1$,
· $b_p(M_j) = b_p(M_{j-1}) - 1$, and
· the generator of H_p introduced at t_i is the same generator of H_p that disappears at t_j.
Then (i, j) (or (t_i, t_j)) is called a *persistence pair* and the *persistence* of (i, j) is $j - i$ (or $f(j) - f(i)$).

For i not in a persistence pair, we say that i represents an *essential cycle*, or that the persistence of i is infinite. In the example considered, the only persistence pair is $(1, 2)$. This may be presented in a *persistence diagram*, with the indices of critical points on both axes, and the persistence measured as a vertical distance.



If we put a simplicial complex structure on M, we may also calculate the homology (and persistence pairs, although they may be different than the ones found above). To make calculations easier, we instead describe a CW structure on our embedded sphere M (with X_i the i-skeleton, and the ordering of the i-cells as indicated). The results will be the same as for a simplicial complex structure.



This gives one 0-cell, two 1-cells, and three 2-cells (with the obvious gluings), allowing us to construct the chain groups C_p as well as maps between them. The map $d_p : C_p \to C_{p-1}$ as a matrix has size $\dim(C_{p-1}) \times \dim(C_p)$, and has entry (i, j) equal to the number of times, counting multiplicity, that the ith $(p-1)$-cell is a face of the jth p-cell. Calculations are done in $\mathbf{Z}/2\mathbf{Z}$.

$$d_2 \;:\; C_2 \to C_1 \quad \text{is} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad\qquad d_1 \;:\; C_1 \to C_0 \quad \text{is} \quad \begin{bmatrix} 0 & 0 \end{bmatrix}$$

The Betti numbers are then $b_p = \dim(C_p) - \operatorname{rk}(d_p) - \operatorname{rk}(d_{p+1})$. From above, it is immediate that $\operatorname{rk}(d_1) = 0$, $\operatorname{rk}(d_2) = 2$, and $\operatorname{rk}(d_p) = 0$ for all other p. This tells us that

$$\begin{aligned} b_0 &= \dim(C_0) - \operatorname{rk}(d_0) - \operatorname{rk}(d_1) = 1 - 0 - 0 = 1, \\ b_1 &= \dim(C_1) - \operatorname{rk}(d_1) - \operatorname{rk}(d_2) = 2 - 0 - 2 = 0, \\ b_2 &= \dim(C_2) - \operatorname{rk}(d_2) - \operatorname{rk}(d_3) = 3 - 2 - 0 = 1, \end{aligned}$$

as expected. To find the persistence pairs, we introduce a filtration on the simplices (equivalently, on the cells) by always having the faces of a cell precede the cell, as well as lower-dimensional cells preceding higher-dimensional cells. Using the same ordering as described above, consider the following filtration:

$$\begin{aligned} K_0 &= \{\}, \\ K_1 &= \{e_1^0\}, \\ K_2 &= \{e_1^0, e_1^1, e_2^1\}, \\ K_3 &= \{e_1^0, e_1^1, e_2^1, e_1^2, e_2^2, e_3^2\}, \end{aligned}$$

so $\emptyset = K_0 \subset K_1 \subset K_2 \subset K_3 = M$. This gives an ordering on all the cells of M, namely

$$\sigma_1 = e_1^0,\; \sigma_2 = e_1^1,\; \sigma_3 = e_2^1,\; \sigma_4 = e_1^2,\; \sigma_5 = e_2^2,\; \sigma_6 = e_3^2.$$

Construct the *boundary matrix* D, with the (i, j) entry of D equal to the number of times, counting multiplicity, modulo 2, that σ_i is a codimension 1 face of σ_j. In the case of our example sphere, we get the matrix

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

in its reduced form (call it $\tilde{D}$). With respect to the matrix $\tilde{D}$, define the following numbers:

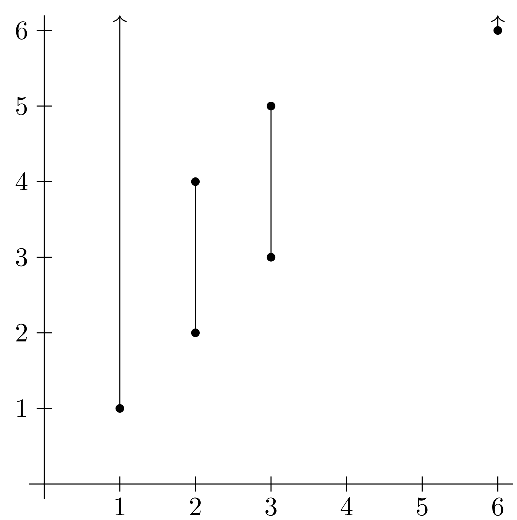
$$\begin{aligned} low(j) &= \text{the row number of the lowest non-zero entry in column } j, \\ zero(p) &= \text{the number of zero columns that correspond to } p\text{-simplices}, \\ one(p) &= \text{the number of 1s in rows that correspond to } p\text{-simplices}. \end{aligned}$$

We calculate all the relevant values of these expressions to be as below.

$$\begin{array}{lll} low(1) = 0 & zero(0) = 1 & one(0) = 0 \\ low(2) = 0 & zero(1) = 2 & one(1) = 2 \\ low(3) = 0 & zero(2) = 1 & one(2) = 0 \\ low(4) = 2 & & \\ low(5) = 3 & & \\ low(6) = 0 & & \end{array}$$

For persistence, we have
- if $low(j) = i \neq 0$, then (i, j) is a persistence pair,
- if $low(j) = 0$ and there is no k such that $low(k) = j$, then j is an essential cycle.

For our sphere example, we get two persistence pairs $(2, 4)$ and $(3, 5)$, and two essential cycles 1 and 6. Note that this is different from the persistence pairs found by the height function $f : M \to \mathbf{R}$ earlier (but there are still two essential cycles), because there we were comparing the homologies $H_p(M_j)$, but here we are comparing $H_p(K_\ell)$. The persistence diagram is as below.



As an added feature, from the numbers above we may calculate the homology and relative homology groups. Construct the relative chain groups $C_p(M, K_\ell) = C_p(M)/C_p(K_\ell)$ and set $zero(p, \ell)$ to be $zero(p)$ for the lower right submatrix of $\tilde{D}$ corresponding to the cells in $M - K_\ell$ (and similarly for $one(p, \ell)$). We find these numbers for the bent sphere to be as below.

$$\begin{array}{llll} zero(0,0)=1 & zero(0,1)=0 & zero(0,2)=0 & zero(0,3)=0 \\ zero(1,0)=2 & zero(1,1)=2 & zero(1,2)=0 & zero(1,3)=0 \\ zero(2,0)=1 & zero(2,1)=1 & zero(2,2)=1 & zero(2,3)=0 \end{array}$$

$$\begin{array}{llll} one(0,0)=0 & one(0,1)=0 & one(0,2)=0 & one(0,3)=0 \\ one(1,0)=2 & one(1,1)=2 & one(1,2)=0 & one(1,3)=0 \\ one(2,0)=0 & one(2,1)=0 & one(2,2)=0 & one(2,3)=0 \end{array}$$

Note that $zero(p,0) = zero(p)$ and $one(p,0) = one(p)$, as well as $zero(p,3) = one(p,3) = 0$. The above numbers are useful in calculating

$$\dim(H_p(M)) = zero(p) - one(p),$$
$$\dim(H_p(M, K_\ell)) = zero(p,\ell) - one(p,\ell).$$

References: Edelsbrunner and Harer (Persistent homology - a Survey)

3.2 Revisiting persistent homology

2017-03-27

Keywords: *persistent homology, filtration, persistence module, extended persistence, zigzag persistence, categorification, multidimensional persistence, barcode*

Here we revisit and expand on persistent homology, previously in the post “Persistent homology (an example),” 2016-05-19. All homology, except where noted, will be over a field k, and X will be a topological space. Often a Morse-type function $f : X \to \mathbf{R}$ is introduced along with X, but we will try to take a more abstract view.

Definition 3.2.1. The space X may be described as a *filtered space* with a *filtration* of *sublevel* sets

$$\emptyset = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_m = X,$$

whose *kth persistence module* is the (not necessarily exact) sequence

$$0 = H_k(X_0) \to H_k(X_1) \to \cdots \to H_k(X_m) = H_k(X)$$

of homology groups of the filtration.

Remark 3.2.2. Every persistence module may be uniquely decomposed as a direct sum of sequences $0 \to k \to \cdots \to k \to 0$, where every map is id, except the first and last. The indices at which each sequence in the summand has its first and last non-zero map are called the *birth* and *death* of the homology class represented by the sequence.

In some cases a homology class may not die, so we consider the *extended* persistence module to make everything finite. We introduce the *superlevel* sets $X^i = X \setminus X_i$. If f was our Morse-type function for X, with critical points $p_1 < \cdots < p_m$, then for $t_0 < p_1 < t_1 < \cdots < p_m < t_m$, we set $X_i = f^{-1}(-\infty, t_i]$ and $X^i = f^{-1}[t_i, \infty)$. The extended persistence module of X is

$$0 = H_k(X_0) \to H_k(X_1) \to \cdots \to H_k(X_m) \to H_k(X, X^m) \to H_k(X, X^{m-1}) \to \cdots \to H_k(X, X^0) = 0.$$

Definition 3.2.3. The *persistence* of a homology class in a persistence module conveys the idea of how long it is alive, presented by a *persistence pair*.

first alive at	last alive at	persistence pair
X_i	X_j	$(i, j+1)$
X_i	(X, X^j)	(i, j)
(X, X^i)	(X, X^j)	$(i+1, j)$

The persistence of all homology classes in a persistence module is often presented in a *persistence diagram*, the collection of persistence pairs (i,j), or (p_i,p_j) or $(f(p_i),f(p_j))$, as desired; or a linear *barcode*, the collection of persistence pairs (i,j) as intervals $[i,j]$, ordered vertically.

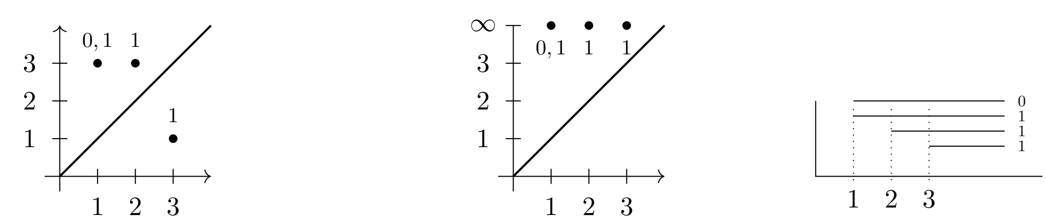
Example 3.2.4. Let $X = T^n = (S^1)^n$ be the n-torus. One filtration of X is $X_0 = \emptyset$ and $X_i = T^i$ for $1 \leqslant i \leqslant n$. Note that $H_k(T^n, T^n \setminus X_n) = H_k(T^n)$ and $H_k(T^n, T^n \setminus X_0) = H_k(\emptyset)$. The first $n+1$ modules of the extended persistence module at level k split into $\binom{n}{k}$ sequences, as $H_k(T^n) = \mathbf{Z}^{\binom{n}{k}}$. Geometric considerations allow $X^i = T^n \setminus T^i$ to be simplified in some cases. For instance, when $n=3$ and $k=0,1$ we have that $\widetilde{H}_k(T^3, T^3 \setminus T^2) \cong \widetilde{H}_k(T^3, T^2) \cong \widetilde{H}_k(T^3/T^2)$, and knowing that $X^1 = T^3 \setminus T^1 \simeq (S^1 \vee S^1) \times S^1$, the relevant part of the long exact sequence for relative homology is

$$\begin{array}{ccccccc} \widetilde{H}_1(X^1) & \xrightarrow{f} & \widetilde{H}_1(T^3) & \xrightarrow{g} & \widetilde{H}_1(T^3, X^1) & \longrightarrow & \widetilde{H}_0(X^1). \\ \| & & \| & & \| & & \| \\ \mathbf{Z}^3 & & \mathbf{Z}^3 & & A & & 0 \end{array}$$

The two 1-cycles from $S^1 \vee S^1 \subset X^1$ map via f to the same 1-cycle in T^3, hence $\mathrm{im}(g) = \mathbf{Z}^2$. By exactness, $\ker(g) = \mathbf{Z}^2$, and as g is surjective, $A = \mathbf{Z}$. Hence the extended persistence k-modules decompose as

$$\begin{array}{lcccccccccccccccc} & H_k(\emptyset) & \to & H_k(T^1) & \longrightarrow & H_k(T^2) & \longrightarrow & H_k(T^3) & \longrightarrow & H_k(T^3) & \longrightarrow & \widetilde{H}_k(T^3, X^2) & \longrightarrow & \widetilde{H}_k(T^3, X^1) & \to & H_k(\emptyset) \\ k=0: & & & \mathbf{Z} & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathbf{Z}, & & & & & & \\ k=1: & & & \mathbf{Z} & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathbf{Z} & & & & & & \\ & \oplus & & & & \mathbf{Z} & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathbf{Z} & & & & & & \\ & \oplus & & & & & & \mathbf{Z} & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathbf{Z}. & & \end{array}$$

The persistence pairs are $(1,3)$ with multiplicity 2 and $(2,3)$, $(3,1)$ with multiplicity 1. The persistence diagrams and barcodes of the degree 0 and 1 homology classes are given below.



extended persistence diagram persistence diagram barcode

The diagonal $y = x$ is often given to indicate how short a lifespan a class has. Barcodes are usually not given for extended persistence diagrams, as length of a class (birth to death) is less important than position (above or below the diagonal).

Now we consider some generalizations of the ideas presented above.

Remark 3.2.5. A filtration can also be viewed as a diagram $X_0 \to X_1 \to \cdots \to X_m$, where each arrow is the inclusion map. We could generalize and consider a *zigzag diagram*, a sequence $X_0 \leftrightarrow X_1 \leftrightarrow \cdots \leftrightarrow X_m$, where $\leftrightarrow$ represents either $\to$ or $\leftarrow$. Homology can be applied and the resulting seuquence can also be uniquely decomposed into summands $k \leftrightarrow \cdots \leftrightarrow k$ where every arrow is the identity, giving *zigzag persistent homology.*

Remark 3.2.6. A filtration could also be viewed as a functor $F : \{0, \ldots, m\} \to \mathrm{Top}$, where $F(i) = X_i$ and $F(i \to j)$, for $j \geqslant i$, is the composition of maps $X_i \to \cdots \to X_j$. Hence the degree-k persistent homology of X_i can be defined as the image of the maps $H_k F(i \to j)$, for all $j \geqslant i$, and the functor $H_k F : \{0, \ldots, m\} \to \mathrm{Vec}$ may be viewed as the kth persistence module. This is a *categorification* of persistent homology.

Remark 3.2.7. A space X can be filtered in several different ways. A *multifiltration* X_α, for α a multi-index, is a collection of filtrations such that fixing all but one of the indices in α gives a (one-dimensional) filtration of X. The *multidimensional persistence* of X_α is a $|\alpha|$-dimensional grid of homology groups, with the barcode generalizing to the *rank invariant*, a map on the grid.

Another generalization, viewing filtrations as quivers, will not be discussed here, but rather presented as a separate post later.

References: Edelsbrunner and Morozov (Persistent homology: theory and practice), Carlsson, de Silva, and Morozov (Zigzag persistent homology and real-valued functions), Bubenik and Scott (Categorification of persistent homology), Carlsson and Zomorodian (The theory of multidimensional persistence)

3.3 Distance and persistence diagrams

2017-04-09

Keywords: *persistent homology, extended persistence, persistence diagram, Wasserstein distance, bottleneck distance*

We assume we have a Morse-type function $f : X \to \mathbf{R}$, whose associated persistence diagram is $D(f) = \{f_1, \dots, f_n\}$, which we will think of as a collection of persistence birth-death pairs f_i in the extended real plane $(\mathbf{R}^*)^2$. If the topological space X was filtered without such a function, define one by $x \mapsto i$ where i is the smallest index such that $x \in X_i$.

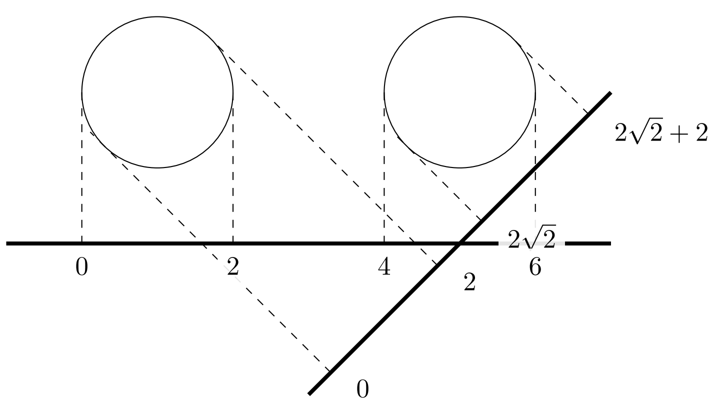
Definition 3.3.1. Let $f, g : X \to \mathbf{R}$ be two Morse-type functions with associated persistence diagrams $D(f)$, $D(g)$. The *(Wasserstein) q-distance* between f and g is defined as

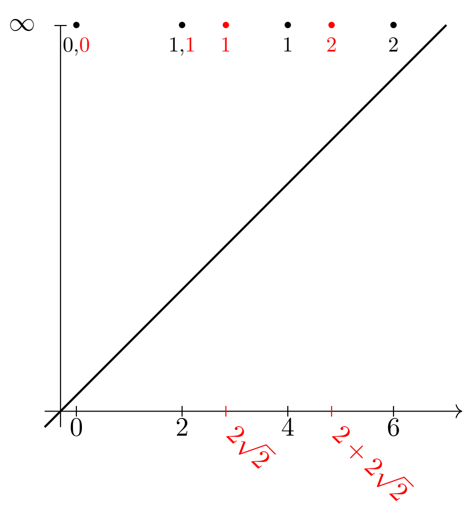
$$W_q(f,g) := \inf_{\sigma \in S_n} \left(\sum_{i=1}^n ||f_i - g_{\sigma(i)}||_\infty^q \right)^{1/q}.$$

The *bottleneck distance* between f and g is

$$\begin{aligned} W_\infty(f,g) &:= \lim_{q\to\infty} \{W_q(f,g)\} && \text{(limit of } q\text{-distances)} \\ &= \max_i \left\{ ||f_i - g_{\sigma(i)}||_\infty \ : \ \sigma = \arg W_q(f,g) \right\}. && \text{(length of longest edge in best matching)} \end{aligned}$$

Example 3.3.2. Consider the torus of inner and outer radius 1 embedded in the natural way. Left $f, g : T^2 \to \mathbf{R}$ be height functions of the torus, but projecting to the planes $z = -2$ and $z = x - 4$, respectively. Note all critical points occur on the plane $y = 0$. Below, the slice at this plane is given (distances along planes from the first critical point are shown), as well as $D(f), D(g)$ on the same diagram (degrees of homology classes are shown).

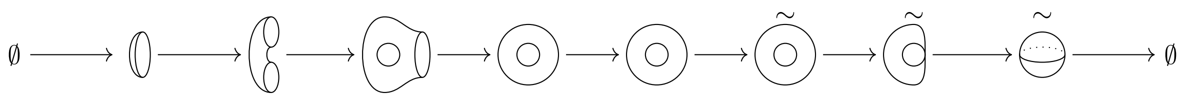




For $D(f) = \{(0,\infty),(2,\infty),(4,\infty),(6,\infty)\}$ and $D(g) = \{(0,\infty),(2,\infty),(2\sqrt{2},\infty),(2+2\sqrt{2},\infty)\}$, it is clear that $\sigma = \text{id}$ will be the best matching. The q-distance between f and g is then given by

$$W_q(f,g) = \Big(||(4,\infty) - (2\sqrt{2},\infty)||_\infty^q + ||(6,\infty) - (2+2\sqrt{2},\infty)||_\infty^q \Big)^{1/q} = 2^{1/q}(4-2\sqrt{2}),$$

with bottleneck distance $4-2\sqrt{2}$. However, we would like to say that these two functions are the same in some way, as no critical points are switched, and extended persistence allows us to do that. The decomposed extended persistence module is given below.



$$H_k(X_0) \longrightarrow H_k(X_1) \longrightarrow H_k(X_2) \longrightarrow H_k(X_3) \longrightarrow H_k(X_4) \longrightarrow H_k(X_4) \to \widetilde{H}_k(X,X^3) \to \widetilde{H}_k(X,X^2) \to \widetilde{H}_k(X,X^1) \longrightarrow H_k(\emptyset)$$

0-class: $\mathbf{Z} \longrightarrow \mathbf{Z}$

1-class: $\mathbf{Z} \longrightarrow \mathbf{Z}$

1–class: $\mathbf{Z} \longrightarrow \mathbf{Z}$

2-class: $\mathbf{Z} \longrightarrow \mathbf{Z}$

The extended persistence classes have length 3 ((1,4) for the 0-class, (4,1) for the 2-class) and 1 ((2,3) and (3,2) for the 1-classes), no matter if we use f or g to define the X_i and X^j.

Remark 3.3.3. An interesting question to ask is how long does it take for an essential homology class to be built? Some things to keep in mind while resolving this question:
- The 0-class case should be treated spearately because of reduced homology
- A class may be encountered several times (like the first 1-class in the example above)
- What does it mean for a class to be "begin being built" (this is probably the key)
- A class is certainly "done being built" (the first time) when it first appears in the persistence module

It seems that the extended persistence pair gives the length between when the class is "done being built" the first time f encounters it fully and when it "begins to be built" the last time f encounters it.

The bottleneck distance satisfies a nice stability condition for *tame* functions $f : X \to \mathbf{R}$, which have finite dimensional homology groups $H_k(f^{-1}(-\infty,a])$ for all $a \in \mathbf{R}$.

Theorem 3.3.4. [Cohen-Steiner, Edelsbrunner, Harer 2007]
Let $f,g : X \to \mathbf{R}$ be tame. Then $W_\infty(f,g) \leqslant ||f-g||_\infty$.

This bound is reached when $g = f + c$ for some constant c, and the Wasserstein distance is 0 when $g(p_i) = f(p_i)$ for all critical values. Hence it seems without stronger assumptions about f and g, this bound is as good as we can get.

References: Edelsbrunner and Morozov (Persistent homology: theory and practice), Cohen-Steiner, Edelsbrunner and Harer (Stability of persistence diagrams)

3.4 Categories and the TDA pipeline

2017-05-21

Keywords: *persistent homology, TDA, categories, functor, filtration, frame, barcode*

This post contains topics and ideas from ACAT at HIM, April 2017, as presented by Professor Ulrich Bauer (see slide 11 of his presentation, online at `ulrich-bauer.org/persistence-bonn-talk.pdf`). The central theme is to assign categories and functors to analyze the process

$$\text{filtration} \longrightarrow \text{(co)homology} \longrightarrow \text{barcode}. \tag{pipe}$$

Remark 3.4.1. The categories we will use are below. For filtrations, we have the ordered reals (though any poset P would work) and topological spaces:

$$R \ : \mathrm{Obj}(R) = \mathbf{R}, \qquad \mathrm{Hom}(r,s) = \begin{cases} \{r \mapsto s\}, & \text{if } r \leqslant s, \\ \emptyset, & \text{else}, \end{cases}$$

$$\mathrm{Top} \ : \mathrm{Obj}(\mathrm{Top}) = \{\text{topological spaces}\}, \qquad \mathrm{Hom}(X,Y) = \{\text{functions } f : X \to Y\}.$$

For (co)homology groups, we have the category of (framed) vector spaces. We write V^n for $V^{\oplus n} = V \oplus V \oplus \cdots \oplus V$, and e_n for a *frame* of V^n (see below).

$$\begin{aligned} \mathrm{Vect} \ &: \mathrm{Obj}(\mathrm{Vect}) = \{V^{\oplus n} \ : \ 0 \leqslant n < \infty\}, \\ &\quad \mathrm{Hom}(V^n, V^m) = \{\text{homomorphisms } f : V^n \to V^m\}, \end{aligned}$$

$$\begin{aligned} \mathrm{Vect}^{fr} \ &: \mathrm{Obj}(\mathrm{Vect}^{fr}) = \{V^n \times e^n \ : \ 0 \leqslant n < \infty\}, \\ &\quad \mathrm{Hom}(V^n \times e^n, V^m \times e^m) = \{\text{hom. } f : V^n \to V^m,\ g : e^n \to e^m,\ g \in \mathrm{Mat}(n,m)\}. \end{aligned}$$

Finally for barcodes, we have Δ, the *category of finite ordered sets*, and its variants. A *partial injective function*, or *matching* $f : A \nrightarrow B$ is a bijection $A' \to B'$ for some $A' \subseteq A$, $B' \subseteq B$.

$$\begin{aligned} \Delta \ &: \mathrm{Obj}(\Delta) = \{[n] = (0,1,\ldots,n) \ : \ 0 \leqslant n < \infty\}, \\ &\quad \mathrm{Hom}([n],[m]) = \{\text{order-preserving functions } \ f : [n] \to [m]\}, \end{aligned}$$

$$\begin{aligned} \Delta' \ &: \mathrm{Obj}(\Delta') = \{a = (a_0 < a_1 < \cdots < a_n) \ : \ a_i \in \mathbf{Z}_{\geqslant 0}, 0 \leqslant n < \infty\}, \\ &\quad \mathrm{Hom}(a,b) = \{\text{order-preserving functions } f : a \to b\}, \end{aligned}$$

$$\begin{aligned} \Delta'' \ &: \mathrm{Obj}(\Delta'') = \{a = (a_0 < a_1 < \cdots < a_n) \ : \ a_i \in \mathbf{Z}_{\geqslant 0}, 0 \leqslant n < \infty\}, \\ &\quad \mathrm{Hom}(a,b) = \{\text{order-preserving partial injective functions } f : a \nrightarrow b\}. \end{aligned}$$

Definition 3.4.2. A *frame* e of a vector space V^n is equivalently:
- an ordered basis of V^n,
- a linear isomorphism $V^n \to V^n$, or
- an element in the fiber of the principal rank n frame bundle over a point.

Frames (of possibly different sizes) are related by full rank elements of $\mathrm{Mat}(n,m)$, which contains all $n \times m$ matrices over a given field.

Definition 3.4.3. Let $(P, \leqslant)$ be a poset. A (*indexed topological*) *filtration* is a functor $F : P \to \mathrm{Top}$, with

$$\mathrm{Hom}(F(r), F(s)) = \begin{cases} \{\iota : F(r) \hookrightarrow F(s)\}, & \text{if } r \leqslant s, \\ \emptyset, & \text{else}, \end{cases}$$

where ι is the inclusion map. That is, we require $F(r) \subseteq F(s)$ whenever $r \leqslant s$.

Definition 3.4.4. A *persistence module* is the composition of functors $M_i : P \xrightarrow{F} \mathrm{Top} \xrightarrow{H_i} \mathrm{Vect}$.

Homology will be taken over some field k. A *framed* persistence module is the same composition as above, but mapping into Vect^{fr} instead. The framing is chosen to describe how many different vector spaces have already been encountered in the filtration.

Definition 3.4.5. A *barcode* is a collection of intervals of $\mathbf{R}$. It may also be viewed as the composition of functors $B_i : P \xrightarrow{F} \mathrm{Top} \xrightarrow{H_i} \mathrm{Vect} \xrightarrow{\dim} \Delta$.

Similarly as above, we may talk about a *framed* barcode by instead mapping into Vect^{fr} and then to Δ'', keeping track of which vector spaces we have already encountered. This allows us to interpret the process (pipe) in two different ways. First we have the unframed approach

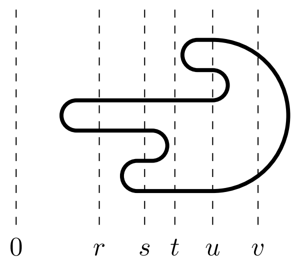
$$\begin{array}{ccccc} \mathrm{Top} & \to & \mathrm{Vect} & \to & \Delta, \\ X_t & \mapsto & H_i(X_t;k) & \mapsto & [\dim(H_i(X_t;k))]. \end{array}$$

The problem here is interpreting the inclusion $X_t \hookrightarrow X_s$ as a map in Δ, for instance, in the case when $H_i(X_t;k) \cong H_i(X_s;k)$, but $H_i(X_t \hookrightarrow X_s) \neq \text{id}$. To fix this, we have the framed interpretation of (pipe)

$$\begin{array}{rcccl} \text{Top} & \to & \text{Vect}^{fr} & \to & \Delta'', \\ X_t & \mapsto & H_i(X_t;k) \times e & \mapsto & [e]. \end{array}$$

The first map produces a frame e of size n, where n is the total number of different vector spaces encountered over all $t' \leqslant t$, by setting the first $\dim(H_i(X_t;k))$ coordinates to be the appropriate ones, and then the rest. This is done with the second map to Δ'' in mind, as the size of $[e]$ is $\dim(H_i(X_t;k))$, with only the first $\dim(H_i(X_t;k))$ basis vectors taken from e. As usual, these maps are best understood by example.

Example 3.4.6. Given the closed curve X in $\mathbf{R}^2$ below, let $\varphi : X \to \mathbf{R}$ be the height map from the line 0, with $X_i = \varphi^{-1}(-\infty, i]$, for $i = r, s, t, u, v$. Let e_i be the standard ith basis vector in $\mathbf{R}^N$.



$$\begin{array}{rcccl} X_r & \mapsto & k \times (e_1) & \mapsto & (1) \\ X_s & \mapsto & k^2 \times (e_1, e_2) & \mapsto & (1 < 2) \\ X_t & \mapsto & k \times (e_1, e_2) & \mapsto & (1) \\ X_u & \mapsto & k^2 \times (e_1, e_3, e_2) & \mapsto & (1 < 3) \\ X_v & \mapsto & k \times (e_1, e_2, e_3) & \mapsto & (1) \end{array}$$

Remark 3.4.7. This seems to make (pipe) functorial, as the maps $X_t \hookrightarrow X_{t'}$ may be naturally viewed as partial injective functions in Δ'', to account for the problem mentioned with the unframed interpretation. However, we have traded locality for functoriality, as the image of X_t in Δ'' can not be calculated without having calculated $X_{t'}$ for all $t' < t$.

References: Bauer (Algebraic perspectives of persistence), Bauer and Lesnick (Induced matchings and the algebraic stability of persistence barcodes)

4 The Ran space - stratifications

4.1 Constructible sheaves

2017-06-13

Keywords: *sheaf, constructible sheaf, derived category, Ran space, distance, filtration*

Let X be a topological space with an open cover $\mathcal{U} = \{U_i\}$, and category $Op(X)$ of open sets of X. The goal is to define *constructible sheaves* and consider some applications. Thanks to Joe Berner for helpful pointers in this area.

Definition 4.1.1. *Constructible subsets* of X are the smallest family F of subsets of X such that
- $Op(X) \subset F$,
- F is closed under finite intersections, and
- F is closed under complements.

This idea can be applied to sheaves. Recall that a *locally closed subset* of X is the intersection of an open set and a closed set.

Definition 4.1.2. A sheaf $\mathcal{F}$ over X is *constructible* if there exists, equivalently,
- a filtration $\emptyset = U_0 \subset \cdots \subset U_n = X$ of X by opens such that $\mathcal{F}|_{U_{i+1}\setminus U_i}$ is constant for all i, or
- a cover $\{V_i\}$ of locally closed subsets of X such that $\mathcal{F}|_{V_i}$ is constant for all i.

Since the category of abelian sheaves over a topological space has enough injectives, we may consider an injective resolution of a sheaf $\mathcal{F}$ rather than the sheaf itself. The resolution may be considered as living inside the *derived category* of sheaves on X.

Definition 4.1.3. Let A be an abelian category.
- $C(A)$ is the category of *cochain complexes* of A,
- $K(A) = C(A)$ modulo cochain homotopy, and
- $D(A) = K(A)$ modulo $F \in K(A)$ such that $H^n(F) = 0$ for all n, called the *derived category* of A.

Next we consider an example. Recall the *Ran space* $\text{Ran}(M) = \{X \subset M \ : \ 0 < |X| < \infty\}$ of non-empty finite subsets of a manifold M and the *Čech complex* of radius $t > 0$ of $P \in \text{Ran}(M)$, a simplicial complex with n-cells for every $P' \subset P$ of size $n+1$ such that $d(P_1', P_2') < t$ for all $P_1', P_2' \in P'$.

Example 4.1.4. Consider the subset $\text{Ran}^{\leqslant 2}(M) = \{X \subset M \ : \ 1 \leqslant |X| \leqslant 2\}$ of the Ran space. Decompose $X = \text{Ran}^{\leqslant 2}(M) \times \mathbf{R}_+$ into disjoint sets $U_\alpha \cup U_\beta$, where

$$U_\alpha = \underbrace{\left(\text{Ran}^1(M) \times \mathbf{R}_+\right)}_{U_{\alpha,1}} \cup \underbrace{\bigcup_{P \in \text{Ran}^2(M)} \{P\} \times (d_M(P_1, P_2), \infty)}_{U_{\alpha,2}}, \qquad U_\beta = \bigcup_{P \in \text{Ran}^2(M)} \{P\} \times (0, d_M(P_1, P_2)],$$

with d_M the distance on the manifold M. The idea is that for every $(P, t) \in U_\alpha$, the Čech complex of radius t on P has the homotopy type of a point, whereas on U_β has the homotopy type of two points. With this in mind, define a constructible sheaf $F \in \text{Shv}(\text{Ran}^{\leqslant 2}(M) \times \mathbf{R}_+)$ valued in simplicial complexes, with $F|_{U_\alpha}$ and $F|_{U_\beta}$ constant sheaves. Set

$$F_{(P,t)\in U_\alpha} = F(U_\alpha) = (0 \to \{*\} \to 0)\,, \qquad F_{(P,t)\in U_\beta} = F(U_\beta) = (0 \to \{*,*\} \to 0)\,.$$

Note that the chain complex $F(U_\alpha)$ is chain homotopic to $0 \to \{-\} \to \{*,*\} \to 0$, where $-$ is a single 1-cell with endpoints $*, *$. To show that this is a constructible sheaf, we need to filter $\text{Ran}^{\leqslant 2}(M) \times \mathbf{R}_+$ into an increasing sequence of opens. For this we use a distance on $\text{Ran}^{\leqslant 2}(M) \times \mathbf{R}_+$, given by $d((P,t),(P',t')) = d_{\text{Ran}(M)}(P,P') + d_{\mathbf{R}}(t,t')$, where $d_{\mathbf{R}}(t,t') = |t - t'|$ and

$$d_{\text{Ran}(M)}(P, P') = \max_{p \in P} \left\{ \min_{p' \in P'} \{d_M(p, p')\} \right\} + \max_{p' \in P'} \left\{ \min_{p \in P} \{d_M(p, p')\} \right\}.$$

Note that U_α is open. Indeed, for $(P, t) \in U_{\alpha,1}$, every other $P' \in \text{Ran}^1(M)$ close to P is also in $U_{\alpha,1}$, and if $P' \in \text{Ran}^2(M)$ is close to P, then the non-zero component $t \in \mathbf{R}_+$ still guarantees the same homotopy type. The set $U_{\alpha,2}$ is open as well, so U_α is open. The whole space is open, so a filtration $\emptyset \subset U_\alpha \subset X$ works for us.

References: Hartshorne (Algebraic geometry, Section II.3), Hartshorne (Residues and Duality, Chapter IV.1), Kashiwara and Schapira (Sheaves on manifolds, Chapters 2 and 8), Lurie (Higher algebra, Section 5.5.1)

4.2 A constructible sheaf over the Ran space

2017-06-24

Keywords: *constructible sheaf, Ran space, simplicial complex*

Let M be a manifold. The goal of this post is to show that the sheaf $\mathcal{F}_{(P,t)} = \text{Rips}(P,t)$ valued in simplicial complexes over $\text{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$ is constructible, a goal not quite achieved (see "A naive constructible sheaf," 2017-12-19 for a solution to the problems encountered here). This space will be described using filtered diagram of open sets, with the sheaf on consecutive differences of the diagram giving simplicial complexes of the same homotopy type.

Definition 4.2.1. Let $P = \{P_1, \dots, P_n\} \in \text{Ran}(M)$. For every collection of open neighborhoods $\{U_i \ni P_i\}_{i=1}^n$ of the P_i in M, there is an open neighborhood of P in $\text{Ran}(M)$ given by

$$\text{Ran}(\{U_i\}_{i=1}^n) = \left\{ Q \in \text{Ran}(M) \ : \ Q \subset \bigcup_{i=1}^n U_i, \ Q \cap U_i \neq \emptyset \right\}.$$

Moreover, these are a basis for any open neighborhood of P in $\text{Ran}(M)$.

Sets

We begin with a few facts about sets. Let X be a topological space.

Lemma 4.2.2. Let $A, B \subset X$. Then:
(a) If $A \subset B$ is open and $B \subset X$ is open, then $A \subset X$ is open.
(b) If $A \subset B$ is closed and $B \subset X$ is open, then $A \subset X$ is locally closed.
(c) If $A \subset B$ is open and $B \subset X$ is locally closed, then $A \subset X$ is locally closed.
(d) If $A \subset B$ is locally closed and $B \subset X$ is locally closed, then $A \subset X$ is locally closed.

Proof: For part (a), first recall that open sets in B are given by intersections of B with open sets of A. Hence there is some $W \subset X$ open such that $A = B \cap W$. Since both B and W are open in X, the set A is open in X.

For part (b), since $A \subset B$ is closed, there is some $Z \subset X$ closed such that $A = B \cap Z$. Since B is open in X, A is locally closed in X.

For parts (c) and (d), let $B = W_1 \cap W_2$, for $W_1 \subset X$ open and $W_2 \subset X$ closed. For part (c), again there is some $W \subset X$ open such that $A = B \cap W$. Then $A = (W_1 \cap W_2) \cap W = (W \cap W_1) \cap W_2$, and since $W \cap W_1$ is open in X, the set A is locally closed in X.

For part (d), let $A = Z_1 \cap Z_2$, where $Z_1 \subset B$ is open and $Z_2 \subset B$ is closed. Then there exists $Y_1 \subset X$ open such that $Z_1 = B \cap Y_1$ and $Y_2 \subset X$ closed such that $Z_2 = B \cap Y_2$. So $A = Z_1 \cap Z_2 = (B \cap Y_1) \cap (B \cap Y_2) = (B \cap Y_1) \cap Y_2$, where $(B \cap Y_1) \subset X$ is open and $Y_2 \subset X$ is closed. Hence $A \subset X$ is locally closed. ■

Lemma 4.2.3. Let $U \subset X$ be open and $f : X \to \mathbf{R}$ continuous. Then $\bigcup_{x \in U} \{x\} \times (f(x), \infty)$ is open in $X \times \mathbf{R}$.

Proof: Consider the function

$$\begin{array}{rcl} g \ : \ X \times \mathbf{R} & \to & X \times \mathbf{R}, \\ (x,t) & \mapsto & (x, t - f(x)). \end{array}$$

Since f is continuous and subtraction is continuous, g is continuous (in the product topology). Since $U \times (0, \infty)$ is open in $X \times \mathbf{R}$, the set $g^{-1}(U \times (0, \infty))$ is open in $X \times \mathbf{R}$. This is exactly the desired set. ■

Filtered diagrams

Definition 4.2.4. A *filtered diagram* is a directed graph such that
- for every pair of nodes u, v there is a node w such that there exist paths $u \to w$ and $v \to w$, and
- for every multi-edge $u \overset{1,2}{\to} v$, there is a node w such that $u \overset{1}{\to} v \to w$ is the same as $u \overset{2}{\to} v \to w$.

For our purposes, the nodes of a filtered diagram will be subsets of $\mathrm{Ran}^n(M) \times \mathbf{R}_{\geqslant 0}$ and a directed edge will be open inclusion of one set into another set (that is, the first is open inside the second). Although we require below that loops $u \to u$ be removed, we consider the first condition above to be satisfied if there exists a path $u \to v$ or a path $v \to u$.

Remark 4.2.5. In the context given,
- edge loops $U \to U$ and path loops $U \to \cdots \to U$ may be replaced by a single node U ($U \subseteq U$ is the identity),
- multi-edges $U \rightrightarrows V$ may be replaced by a single edge $U \to V$ (inclusions are unique), and
- multi-edges $U \rightleftarrows V$ may be replaced by a single node U (if $U \subseteq V$ and $V \subseteq U$, then $U = V$).

A diagram with all possible replacements of the types above is called a *reduced* diagram.

Lemma 4.2.6. In the context above, a reduced filtered diagram D of open sets of any topological space X gives an increasing sequence of open subsets of X, with the same number of nodes.

Proof: Order the nodes of D so that if $U \to V$ is a path, then U has a lower index than V (this is always possible in a reduced diagram). Let $U_1, U_2, \ldots, U_N$ be the order of nodes of D (we assume we have finitely many nodes). For every pair of indices i, j, set

$$\delta_{ij} = \begin{cases} \emptyset & \text{if } U_i \to U_j \text{ is a path in } D, \\ U_i & \text{if } U_i \to U_j \text{ is not a path in } D. \end{cases}$$

Then the following sequence is an increasing sequence of nested open subsets of X:

$$U_1 \to \delta_{12} \cup U_2 \to \delta_{13} \cup \delta_{23} \cup U_3 \to \cdots \to \underbrace{\left(\bigcup_{i=1}^{j-1} \delta_{ij} \right) \cup U_j}_{V_j} \to \cdots \to U_N.$$

Indeed, if $U_i \to U_j$ is a path in D, then U_i is open in V_j, as $U_i \subset V_j$. If $U_i \to U_j$ is not a path in D, then U_i is still open in V_j, as $U_i \subset V_j$. As unions of opens are open, and by Lemma 4.2.2(a), V_{j-1} is open in V_j for all $1 < j < N$. ■

Remark 4.2.7. Note that every consecutive difference $V_j \setminus V_{j-1}$ is a (not necessarily proper) subset of U_j.

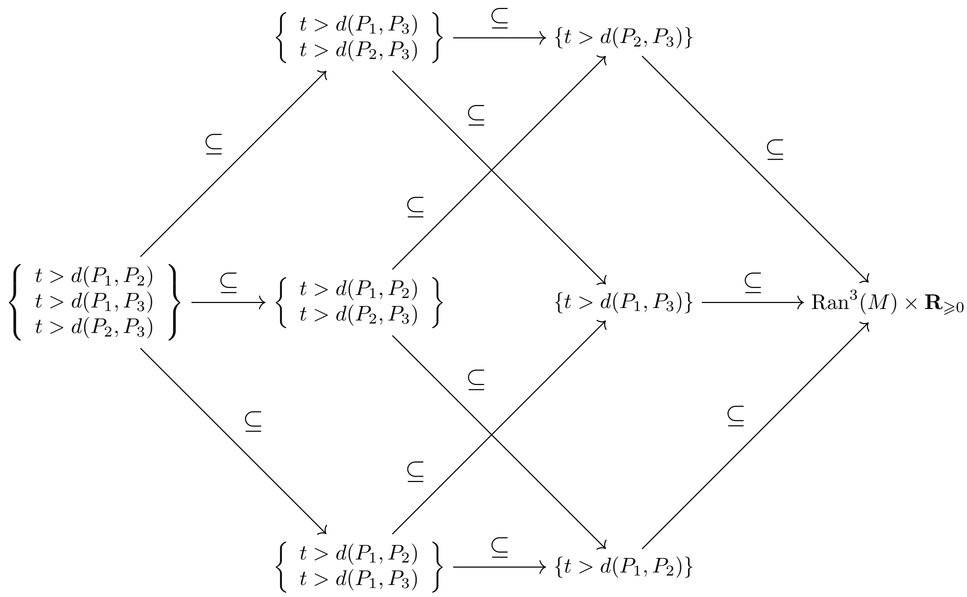
Definition 4.2.8. For $k \in \mathbf{Z}_{>0}$, define a filtered diagram D_k over $\mathrm{Ran}^k(M) \times \mathbf{R}_{\geqslant 0}$ by assigning a subset to every corner of the unit N-hypercube in the following way: for the ordered set $S = \{(i,j) \ : \ 1 \leqslant i < j \leqslant k\}$ (with $|S| = N = k(k-1)/2$), write $P = \{P_1, \ldots, P_k\} \in \mathrm{Ran}^k(M)$, and assign

$$(\delta_1, \ldots, \delta_k) \mapsto \left\{ (P,t) \in \mathrm{Ran}^k(M) \times \mathbf{R}_{\geqslant 0} \ : \ t > d(P_{(S_\ell)_1}, P_{(S_\ell)_2}) \text{ whenever } \delta_\ell = 0, \ \forall\, 1 \leqslant \ell \leqslant k \right\},$$

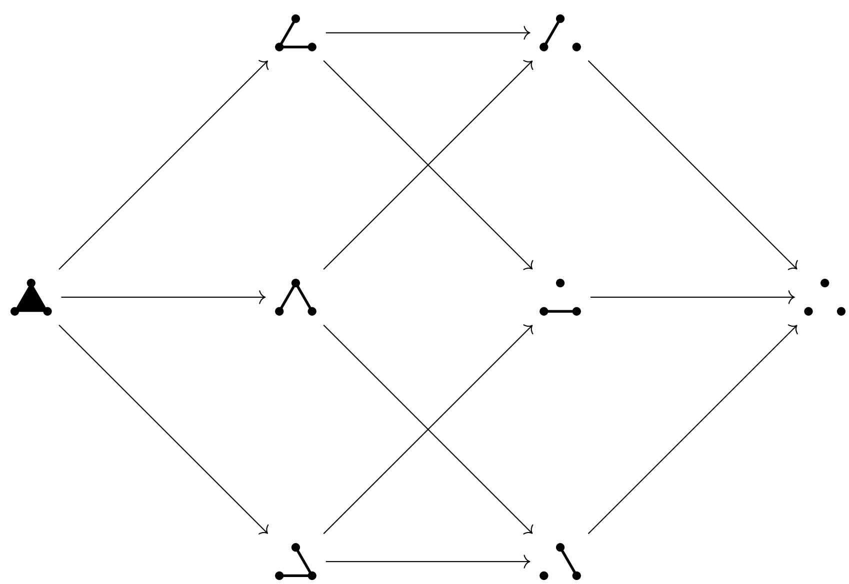
where $\delta_\ell \in \{0,1\}$ for all ℓ, and $d(x,y)$ is the distance on the manifold M between $x, y \in M$. The edges are directed from smaller to larger sets.

Remark 4.2.9. This diagram has $2^{k(k-1)/2}$ nodes, as $k(k-1)/2$ is the number of pairwise distances to consider. Moreover, the difference between the head and tail of every directed edge is elements (P,t) for which $\mathrm{Rips}(P,t)$ is constant.

Example 4.2.10. For example, if $k = 3$, then $2^{3 \cdot 2/2} = 8$, and D_3 is the diagram below. For ease of notation, we write $\{t > \cdots\}$ to mean $\{(P,t) \ : \ P = \{P_1, P_2, P_3\} \in \mathrm{Ran}^3(M), \ t > \cdots\}$.



The diagram of corresponding Vietoris–Rips complexes introduced at each node is below. Note that each node contains elements (P,t) whose Vietoris–Rips complex may be of type encountered in any paths leading to the node.



Lemma 4.2.11. In the filtered diagram D_k, every node is open inside every node following it.

Proof: The left-most node of D_k may be expressed as

$$\{(P,t) \ : \ P = \{P_1, \ldots, P_k\} \in \mathrm{Ran}^k(M), t > d(P_i,P_j) \ \forall \ P_i, P_j \in P\} = \bigcup_{P \in \mathrm{Ran}^k(M)} \{P\} \times \left(\max_{P_i, P_j \in P} \{d(P_i,P_j)\}, \infty \right).$$

Applying a slight variant of Lemma 4.2.3 (replacing $\mathbf{R}$ by an open ray that is bounded below), with the max function continuous, we get that the left-most node is open in the nodes one directed edge away from it. Repeating this argument, we get that every node is open inside every node following it. ■

The constructible sheaf

Recall that a constructible sheaf can be given in terms of a nested cover of opens or a cover of locally closed sets (see post "Constructible sheaves," 2017-06-13). The approach we take is more the latter, and illustrates the relation between the two. Let $n \in \mathbf{Z}_{>0}$ be fixed.

Definition 4.2.12. Define a sheaf $\mathcal{F}$ over $X = \text{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$ valued in simplicial complexes, where the stalk $\mathcal{F}_{(P,t)}$ is the Vietoris–Rips complex of radius t on the set P. For any subset $U \subset X$ such that $\mathcal{F}_{(P,t)}$ is constant for all $(P,t) \in U$, let $\mathcal{F}(U) = \mathcal{F}_{(P,t)}$.

Note that we have not described what $\mathcal{F}(U)$ is when U contains stalks with different homotopy types. Omitting this (admittedly large) detail, we have the following:

Theorem 4.2.13. The sheaf $\mathcal{F}$ is constructible.

Proof: First, by Remark 5.5.1.10 in Lurie, we have that $\text{Ran}^n(M)$ is open in $\text{Ran}^{\leqslant n}(M)$. Hence $\text{Ran}^{\leqslant n-1}(M)$ is closed in $\text{Ran}^{\leqslant n}(M)$. Similarly, $\text{Ran}^{\leqslant n-2}(M)$ is closed in $\text{Ran}^{\leqslant n-1}(M)$, and so closed in $\text{Ran}^{\leqslant n}(M)$, meaning that $\text{Ran}^{\leqslant k}(M)$ is closed in $\text{Ran}^{\leqslant n}(M)$ for all $1 \leqslant k \leqslant n$. This implies that $\text{Ran}^{\geqslant k}(M)$ is open in $\text{Ran}^{\leqslant n}(M)$ for all $1 \leqslant k \leqslant n$, meaning that $\text{Ran}^k(M)$ is locally closed in $\text{Ran}^{\leqslant n}(M)$, for all $1 \leqslant k \leqslant n$.

Next, for every $1 \leqslant k \leqslant n$, let $V_{k,1} \to \cdots \to V_{k,N_k}$ be a sequence of nested opens covering $\text{Ran}^k(M) \times \mathbf{R}_{\geqslant 0}$, as given in Definition 4.2.8 and flattened by Lemma 4.2.6. The sets are open by Lemma 4.2.11. This gives a cover $\mathcal{V}_k = \{V_{k,1}, V_{k,2}, \setminus V_{k,1}, \dots, V_{k,N_k} \setminus V_{k,N_k-1}\}$ of $\text{Ran}^k(M) \times \mathbf{R}_{\geqslant 0} = V_{k,N_k}$ by consecutive differences, with $V_{k,1}$ open in V_{k,N_k} and all other elements of $\mathcal{V}_k$ locally closed in V_{k,N_k}, by Lemma 4.2.2(b). By Lemma 4.2.2 parts (c) and (d), every element of $\mathcal{V}_k$ is locally closed in $\text{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$, and so $\mathcal{V} = \bigcup_{k=1}^n \mathcal{V}_k$ covers $\text{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$ by locally closed subsets.

Finally, by Remarks 4.2.7 and 4.2.9, over every $V \in \mathcal{V}$ the function $\text{Rips}(P,t)$ is constant. Hence $\mathcal{F}|_V$ is a locally constant sheaf, for every $V \in \mathcal{V}$. As the V are locally closed and cover X, $\mathcal{F}$ is constructible. ■

References: Lurie (Higher algebra, Section 5.5.1)

4.3 The Ran space and singularity sets

2017-08-11

Keywords: *Ran space, singularity, dense set, triangle inequality, base of topology*

Fix a manifold M along with an embedding of M into $\mathbf{R}^N$ and set $X = \text{Ran}(M) \times \mathbf{R}_{\geqslant 0}$. The goal of this post is to show that every $(P,t) \in X$ has an open neighborhood that contains no points of the type $(Q, d(Q_i, Q_j))$, for some $i \neq j$. The collection of all such elements of X is called the *singularity set* of X, as the Vietoris–Rips complex at Q with such a radius changes at such elements.

Following Lurie, given a collection of open sets $\{U_i\}_{i=1}^k$ in M, set

$$\text{Ran}(\{U_i\}_{i=1}^k) = \left\{ P \in \text{Ran}(M) \;:\; P \subset \bigcup_{i=1}^k U_i,\; P \cap U_i \neq \emptyset \;\forall\; i \right\}.$$

The topology on $\text{Ran}(M)$ is the smallest topology for which every $\text{Ran}(\{U_i\}_{i=1}^k)$ is open, for any $\{U_i\}_{i=1}^k$, for any k. The topology on the product X is the product topology.

Remark 4.3.1. Note that the Ran space $\text{Ran}(M)$ by itself can be split up into the pieces $\text{Ran}^k(M)$, with "singularities" viewed as when a point splits into two (or more) points, or two (or more) combine into one. Then every element of $\text{Ran}(M)$ is on the edge of the singularity set, as any neighborhood of a single point on the manifold contains two points on the manifold.

Fix $(P,t) \in X$ not in the singularity set of X, with $P = (P_1, \dots, P_k)$, for $1 \leqslant k \leqslant n$. Set

$$\mu = \min \left\{ t, \min_{1 \leqslant i < j \leqslant k} \{ |t - d(P_i, P_j)| \} \right\},$$

with distance d being Euclidean distance in $\mathbf{R}^N$. The quantity μ should be thought of as the upper bound on how "far" we may move from (P,t) without hitting the singularity set.

Proposition 4.3.2. Let (P,t) be as above and $t,\alpha,\beta > 0$ such that $\alpha+\beta=\mu$. Then

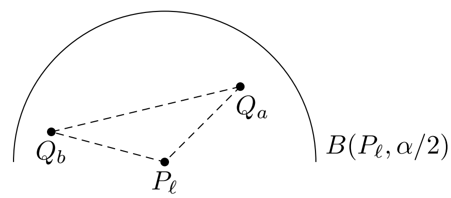
$$U = \text{Ran}\left(\{B(P_i,\alpha/2)\}_{i=1}^k\right)\times(t-\beta,t+\beta)$$

is an open neighborhood of (P,t) in X and does not contain any points of the singularity set of X.

If $t=0$, then having $[0,\beta)$ as the second component of U, with $\alpha+\beta=\min_{i,j} d(P_i,P_j)$ works as the open neighborhood of (P,t). The balls $B(x,r)$ are N-dimensional in $\mathbf{R}^N$. The proof is mostly applications of the triangle inequality.

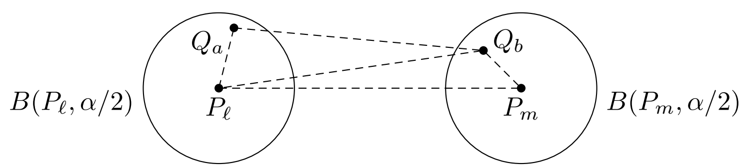
Proof: By construction we have that U is open in X and that it contains (P,t). For $(Q,s)\in U$ any other element, we have three cases. We will show that the distance between any two $Q_a,Q_b\in Q$ is never s. Fix distinct indices $\ell,m\in\{1,\dots,k\}$.

Case 1: $Q_a,Q_b\in B(P_\ell,\alpha/2)$. The situation looks as in the diagram below.



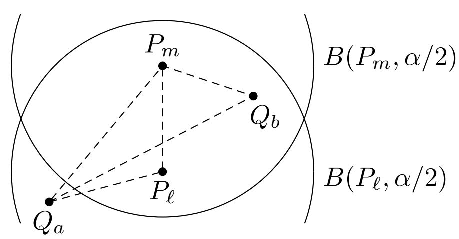
Observe that $d(Q_a,Q_b)\leqslant d((Q_a,P_\ell)+d(Q_b,P_\ell)<\alpha=\mu-\beta\leqslant t-\beta$. Hence $d(Q_a,Q_b)<s$.

Case 2: $Q_a\in B(P_\ell,\alpha/2), Q_b\in B(P_m,\alpha/2), d(P_\ell,P_m)>t$. The situation looks as in the diagram below.



Observe that $d(P_\ell,P_m)\leqslant d(P_\ell,Q_b)+d(P_m,Q_b)\leqslant d(P_\ell,Q_a)+d(Q_a,Q_b)+d(P_m,Q_b)<\alpha+d(Q_a,Q_b)$. Since $d(P_\ell,P_m)>t$, the definition of μ gives us that $\mu\leqslant d(P_\ell,P_m)-t$, so combining this with the previous inequality, we get $d(Q_a,Q_b)>d(P_\ell,P_m)-\alpha\geqslant\mu+t-(\mu-\beta)=t+\beta$. Hence $d(Q_a,Q_b)>s$.

Case 3: $Q_a\in B(P_\ell,\alpha/2), Q_b\in B(P_m,\alpha/2), d(P_\ell,P_m)<t$. The situation looks as in the diagram below.



Observe that $d(Q_a,Q_b)\leqslant d(P_m,Q_b)+d(P_m,Q_a)\leqslant d(P_\ell,Q_a)+d(P_\ell,P_m)+d(P_m,Q_a)<\alpha+d(P_\ell,P_m)$. Since $d(P_\ell,P_m)<t$, the definition of μ gives us that $\mu\leqslant t-d(P_\ell,P_m)$, so combining this with the previous inequality, we get $d(Q_a,Q_b)<\mu-\beta+t-\mu=t-\beta$. Hence $d(Q_a,Q_b)<s$. ■

As an extension, it would be nice to show that the Vietoris–Rips complex of every element in U is homotopy equivalent. This seems to be intuitively true, but a similar case analysis as above seems daunting.

References: Lurie (Higher Algebra, Section 5.5.1)

4.4 Exit paths, part 1

2017-08-31

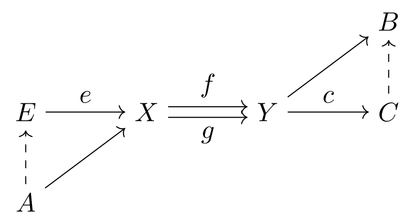
Keywords: *equalizer, fibration, simplicial set, nerve, horn, Kan complex, Kan fibration, Kan extension, infinity category, upset, stratification, exit path*

This post is meant to set up all the necessary ideas to define the category of exit paths.

Preliminaries

Let X be a topological space and C a category. Recall the following terms:

- Δ*:* The category whose objects are finite ordered sets $[n] = (1, \dots, n)$ and whose morphisms are non-decreasing maps. It has several full subcategories, including
  - Δ_s, comprising the same objects of Δ and only injective morphisms, and
  - $\Delta_{\leqslant n}$, comprising only the objects $[0], \dots, [n]$ with the same morphisms.
- *equalizer:* An object E and a universal map $e : E \to X$, with respect to two maps $f, g : X \to Y$. It is universal in the sense that all maps into X whose compositions with f, g are equal factor through e. Equalizers and coequalizers are described by the diagram below, with universality given by existence of the dotted maps.



- *fibered product* or *pullback:* The universal object $X \times_Z Y$ with maps to X and Y, with respect to maps $X \to Z$ and $Y \to Z$.
- *fully faithful:* A functor F whose morphism restriction $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(X), F(Y))$ is surjective (full) and injective (faithful).
- *locally constant sheaf:* A sheaf $\mathcal{F}$ over X for which every $x \in X$ has a neighborhood U such that $\mathcal{F}|_U$ is a constant sheaf. For example, constructible sheaves are locally constant on every stratum.
- *simplicial object:* A contravariant functor from Δ to any other category. When the target category is Set, it is called a *simplicial set*. They may also be viewed as a collection $S = \{S_n\}_{\geqslant 0}$ for $S_n = S([n])$ the value of the functor on each $[n]$. Simplicial sets come with two natural maps:
  - *face maps* $d_i : S_n \to S_{n-1}$ induced by the map $[n-1] \to [n]$ which skips the ith piece, and
  - *degeneracy maps* $s_i : S_n \to S_{n+1}$ induced by the map $[n+1] \to [n]$ which repeats the ith piece.
- *stratification:* A property of a cover $\{U_i\}$ of X for which consecutive differences $U_{i+1} \setminus U_i$ have "nicer" properties than all of X. For example, $E_i \to U_{i+1} \setminus U_i$ is a rank i vector bundle, but there is no vector bundle $E \to X$ that restricts to every E_i.

Now we get into new territory.

Definition 4.4.1. The *nerve* of a category C is the collection $N(C) = \{N(C)_n = Fun([n], C)\}_{n \geqslant 0}$, where $[n]$ is considered as a category with objects $0, \dots, n$ and a single morphism in $\mathrm{Hom}_{[n]}(s, t)$ iff $s \leqslant t$.

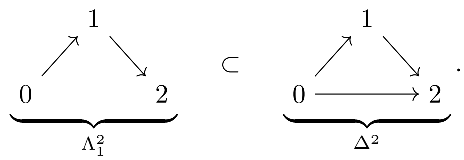
Note that the nerve of C is a simplicial set, as it is a functor from $\Delta^{op} \to Fun(\Delta, C)$. Moreover, the pieces $N(C)_0$ are the objects of C and $N(C)_1$ are the morphisms of C, so all the information about C is contained in its nerve. There is more in the higher pieces $N(C)_n$, so the nerve (and simplicial sets in general) may be viewed as a generalization of a category.

Kan structures

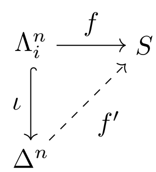
Let sSet be the category of simplicial sets. We may consider $\Delta^n = \mathrm{Hom}_\Delta(-, [n])$ as a contravariant functor $\Delta \to \mathrm{Set}$, so it is an object of sSet.

Definition 4.4.2. Fix $n \geqslant 0$ and choose $0 \leqslant i \leqslant n$. Then the *ith n-horn* of a simplicial set is the functor $\Lambda^n_i \subset \Delta^n$ generated by all the faces $\Delta^n(d_j)$, for $j \neq i$.

We purposefully do not describe what "⊂" or "generated by" mean for functors, hoping that intuition fills in the gaps. In some sense the horn feels like a partially defined functor (though it is a true simplicial set), well described by diagrams, for instance with $n = 2$ and $i = 1$ we have



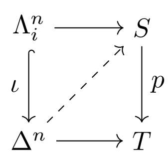
Definition 4.4.3. A simplicial set S is a *Kan complex* whenever every map $f : \Lambda^n_i \to S$ factors through Δ^n. That is, when there exists a map $f' : \Delta^n \to S$ such that the diagram below commutes.



The map ι is the inclusion. Moreover, S is an ∞-*category*, or *quasi-category*, if the extending map f' is unique.

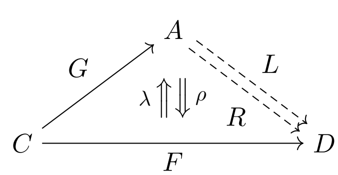
Example 4.4.4. Some basic examples of ∞-categories, for X a topological space, are
- $Sing(X)$, made up of pieces $Sing(X)_n = \mathrm{Hom}(\Delta^n, X)$, and
- $LCS(X)$, the category of locally constant sheaves over X. Here $LCS(X)_n$ over an object A, whose objects are $B \to A$ and morphisms are the appropriate commutative diagrams

Definition 4.4.5. A morphism $p \in \mathrm{Hom}_{\mathrm{sSet}}(S, T)$ is a *Kan fibration* if for every commutative diagram (of solid arrows)



the dotted arrow exists, making the new diagram commute.

Definition 4.4.6. Let C, D, A be categories with functors $F : C \to D$ and $G : C \to A$.
- The *left Kan extension* of F along G is a functor $A \xrightarrow{L} D$ and a universal natural transformation $F \overset{\lambda}{\rightsquigarrow} L \circ G$.
- The *right Kan extension* of F along G is a functor $A \xrightarrow{R} D$ and a universal natural transformation $R \circ G \overset{\rho}{\rightsquigarrow} F$.



Exit paths

The setting for this section is constructible sheaves over a topological space X. We begin with a slightly more technical definition of a stratification.

Definition 4.4.7. Let $(A, \leqslant)$ be a partially ordered set with the *upset topology*. That is, if $x \in U$ is open and $x \leqslant y$, then $y \in A$. An A-*stratification* of X is a continuous function $f : X \to A$.

We now begin with a Treumann's definition of an exit path, combined with Lurie's stratified setting.

Definition 4.4.8. An *exit path* in an A-stratified space X is a continuous map $\gamma : [0, 1] \to X$ for which there exists a pair of chains $a_1 \leqslant \cdots \leqslant a_n$ in A and $0 = t_0 \leqslant \cdots \leqslant t_n = 1$ in $[0, 1]$ such that $f(\gamma(t)) = a_i$ whenever $t \in (t_{i-1}, t_i]$.

This really is a path, and so gives good intuition for what is happening. Recall that the *geometric realization* of the functor Δ^n is $|\Delta^n| = \{(t_0, \dots, t_n) \in \mathbf{R}^{n+1} \ : \ t_0 + \cdots + t_n = 1\}$. Oserving that $[0,1] \cong |\Delta^1|$, Lurie's definition of an exit path is more general by instead considering maps from $|\Delta^n|$.

Definition 4.4.9. The category of *exit paths* in an A-stratified space X is the simplicial subset $Sing^A(X) \subset Sing(X)$ consisting of those simplices $\gamma : |\Delta^n| \to X$ for which there exists a chain $a_0 \leqslant \cdots \leqslant a_n$ in A such that $f(\gamma(t_0, \dots, t_i, 0, \dots, 0)) = a_i$ for $t_i \neq 0$.

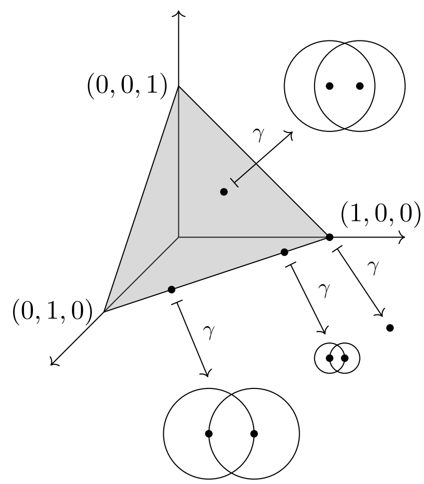
Example 4.4.10. As with all new ideas, it is useful to have an example. Consider the space $X = \mathrm{Ran}^{\leqslant 2}(M) \times \mathbf{R}_{\geqslant 0}$ of a closed manifold M (see post "A constructible sheaf over the Ran space" 2017-06-24 for more). With the poset $(A, \leqslant)$ being $(a \leqslant b \leqslant c)$ and stratifying map

$$\begin{aligned} f \ : \ X & \to A, \\ (P,t) & \mapsto \begin{cases} a & \text{if } P \in \mathrm{Ran}^1(M), \\ b & \text{if } P \in \mathrm{Ran}^2(M), t \leqslant d(P_1, P_2), \\ c & \text{else,} \end{cases} \end{aligned}$$

we can make a continuous map $\gamma : \Delta^3 \to X$ by

$$\begin{aligned} (1,0,0) & \mapsto (P \in \mathrm{Ran}^1(M), 0), \\ (t_0, t_1 \neq 0, 0) & \mapsto (P \in \mathrm{Ran}^2(M), d(P_1, P_2)), \\ (t_0, t_1, t_2 \neq 0) & \mapsto (P \in \mathrm{Ran}^2(M), t > d(P_1, P_2)). \end{aligned}$$

Then $f(\gamma(t_0 \neq 0, 0, 0)) = a$, and $f(\gamma(t_0, t_1 \neq 0, 0)) = b$, and $f(\gamma(t_0, t_1, t_2 \neq 0)) = c$, as desired. The embedding of such a simplex γ is described by the diagram below.



Both the image of $(1,0,0)$ and the 1-simplex from $(1,0,0)$ to $(0,1,0)$ lie in the singularity set of $\mathrm{Ran}^{\leqslant 2}(M) \times \mathbf{R}_{\geqslant 0}$, which is pairs (P,t) where $t = d(P_i, P_j)$ for some i, j. The idea that the simplex "exits" a stratum is hopefully made clear by this image.

References: Lurie (Higher algebra, Appendix A), Lurie (What is... an ∞-category?), Groth (A short course on ∞-categories, Section 1), Joyal (Quasi-categories and Kan complexes), Goerss and Jardine (Simplicial homotopy theory, Chapter 1), Treumann (Exit paths and constructible stacks)

4.5 Stratifying correctly

2017-09-17

Keywords: *stratification, upset, poset, group action, continuity*

In a previous blog post ("A constructible sheaf over the Ran space," 2017-06-24) it was claimed that there was a particular constructible sheaf over $\mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$. However, the proof actually uses finite ordered subsets of M to make the stratification, rather than finite unordered subsets. This means that the sheaf is actually over $M^{\times n} \times \mathbf{R}_{\geqslant 0}$, and in this post we try to fix that problem.

Let Δ_n be the "fat diagonal" of $M^{\times n}$, that is, the collection of $P \in M^{\times n}$ for which at least two coordinates are the same. For every $k > 0$, there is an S_k action on $M^{\times k} \setminus \Delta_k$, quotienting by which we get a map

$$M^{\times k} \setminus \Delta_k \xrightarrow{q_k} \mathrm{Ran}^k(M)$$

to the Ran space of degree k. The stratification of $M^{\times k} \times \mathbf{R}_{\geqslant 0}$ given in the previous post will be pushed forward to a stratification of $\mathrm{Ran}^k(M) \times \mathbf{R}_{\geqslant 0}$, for all $0 < k \leqslant n$. A large part of the work already has been done, it remains to put everything in the right order and check openness. The process is given as follows:

1. Stratify $\mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$ into n pieces, each being $\mathrm{Ran}^k(M) \times \mathbf{R}_{\geqslant 0}$.
2. Stratify $(M^{\times k} \setminus \Delta_k) \times \mathbf{R}_{\geqslant 0}$ as in the previous post.
3. Quotient by S_k-action to get stratification of $\mathrm{Ran}^k(M) \times \mathbf{R}_{\geqslant 0}$.

Step 1

As stated in the proof of Theorem 4.2.13, $\mathrm{Ran}^{\geqslant k}(M) \times \mathbf{R}_{\geqslant 0}$ is open inside $\mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$, allowing us to make a stratification $f : \mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0} \to A$, where A is the poset

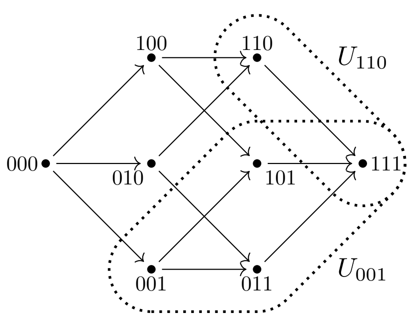
$$\underset{a_1}{\bullet} \longrightarrow \underset{a_2}{\bullet} \longrightarrow \underset{a_3}{\bullet} \to \quad \cdots \quad \to \underset{a_{n-1}}{\bullet} \longrightarrow \underset{a_n}{\bullet} \quad ,$$

where the tail of an arrow is ordered lower than the head. The map f sends $\mathrm{Ran}^k(M) \times \mathbf{R}_{\geqslant 0}$ to a_k, which is a continuous map in the upset topology on A.

Step 2

As stated in Definition 4.2.8, we have a stratification $g_k : (M^{\times k} \setminus \Delta_k) \times \mathbf{R}_{\geqslant 0} \to B_k$, where B_k may be viewed as a directed graph $B_k = (V_k, E_k)$. The vertex set is $V_k = \{0,1\}^{k(k+1)/2}$, whose elements are strings of 1 and 0, and the edge set E_k contains $v \to v'$ iff $d_H(v, v') = 1$ and $d_H(v, 0) < d_H(v', 0)$, for d_H the Hamming distance. Let $U_v \subset B_k$ denote the upset based at v, that is, all elements $v' \in B_k$ with $v \leqslant v'$.

Order all distinct pairs $(i, j) \in \{1, \ldots, k\}^2$, of which there are $k(k+1)/2$. Under the stratifying map g_k, each upset U_v based at the vertex $v \in \{0,1\}^{k(k+1)/2}$ receives elements $(P, t) \in (M^{\times k} \setminus \Delta_k) \times \mathbf{R}_{\geqslant 0}$ satisfying $t > d(P_i, P_j)$ whenever the position representing (i, j) in v is 1. For example, when $k = 3$,



$$\begin{aligned} \{(P,t) \ : \ d(P_1,P_2) > t, d(P_1,P_3) > t\} &\mapsto U_{110}, \\ \{(P,t) \ : \ d(P_2,P_3) > t\} &\mapsto U_{001}. \end{aligned}$$

To check that g_k is continuous in the upset topology, we restate Lemma 4.2.3 in a clearer way.

Lemma 4.5.1. Let $U \subset X$ be open and $\varphi : X \to A \subset \mathbf{R}_{\geqslant 0}$ continuous, with $|A| < \infty$. Then

$$\bigcup_{x \in U} \{x\} \times (\varphi(x), \infty) \ \subseteq \ X \times (z', \infty)$$

is open, for any $z' \leqslant z := \min_{x \in U}\{\varphi(x)\}$.

Proof: Consider the function

$$\begin{array}{rcccl} \psi & : & X \times (z', \infty) & \to & X \times (-\infty, z), \\ & & (x,t) & \mapsto & (x, \varphi(x) - t). \end{array}$$

Since φ is continuous and subtraction is continuous, ψ is continuous (in the product topology). Since $U \times (-\infty, 0)$ is open in $X \times (-\infty, z)$, the set $\psi^{-1}(U \times (-\infty, 0))$ is open in $X \times (z', \infty)$. For any $x \in U$ and $t = \varphi(x)$, we have $\varphi(x) - t = 0$. For any $x \in U$ and $t \to \infty$, we have $\varphi(x) - t \to -\infty$. It is immediate that all other $t \in (\varphi(x), \infty)$ give $\varphi(x) - t \in (-\infty, 0)$. Hence $\psi^{-1}(U \times (-\infty, 0))$ is the collection of points (x, t) with $t \in (\varphi(x), \infty)$, which is then open in $X \times [0, z')$. ■

Applying Lemma 4.5.1 to $U = X = M^{\times k} \setminus \Delta_k$ and $\varphi(P) = \max_{i \neq j}\{d(P_i, P_j)\}$, which is continuous, gives that $g_k^{-1}(U_{11\cdots 1}) \subseteq M^{\times k} \setminus \Delta_k$ is open. This also works to show that $g_k^{-1}(U_v) \subseteq g_k^{-1}(U_{v'})$ is open, for any $v' \leqslant v$, by limiting the pairs of indices iterated over by the function φ. Hence g_k is continuous.

Step 3

The symmetric group S_k acts on $(M^{\times k} \setminus \Delta_k) \times \mathbf{R}_{\geqslant 0}$ by permuting the order of elements in the first factor. That is, for $\sigma \in S_k$, we have

$$\sigma(P = \{P_1, \ldots, P_k\}, t) = (\{P_{\sigma(1)}, \ldots, P_{\sigma(k)}\}, t).$$

Note that $((M^{\times k} \setminus \Delta_k) \times \mathbf{R}_{\geqslant 0})/S_k = \mathrm{Ran}^k(M) \times \mathbf{R}_{\geqslant 0}$.

Remark 4.5.2. Graph isomorphism for two graphs with k vertices may also be viewed as the equivalence relation induced by S_k acting on $\Gamma_k = \{$simple vertex-labeled graphs with k vertices$\}$. First, let G_v be the (unique) graph first introduced at element $v \in B_k$ by g_k. That is, we have $G_v = VR(P, t)_1$ (the ordered 1-skeleton of the Vietoris–Rips complex on the set P with radius t) whenever $g_k((P, t)) \in U_v$ and $g_k((P, t)) \notin U_{v'}$ for any $v' \leqslant v$, $v' \neq v$. Then the elements of B_k are in bijection with the elements of Γ_k (given by $v \leftrightarrow G_v$), so we have $B_k/S_k = B'_k$. Recall that $v \leqslant v'$ in B_k iff adding an edge to G_v gives $G_{v'}$. In $B_{k'}$, this becomes a partial order on equivalence classes $[w] = \{v \in B_k \ : \ \sigma G_v = G_w$ for some $\sigma \in S_k\}$. We write $[w] \leqslant [w']$ iff there is a collection of pairs $\{(v_1, v'_1), \ldots, (v_\ell, v'_\ell)\}$ such that $v_i \leqslant v'_i$ for all i, and $\{v_1, \ldots, v_\ell\} = [w]$ and $\{v'_1, \ldots, v'_\ell\} = [w']$ (there may be repetition among the v_i or v'_i).

By the universal property of the quotient, there is a unique map $h_k : \mathrm{Ran}^k(M) \times \mathbf{R}_{\geqslant 0} \to B'_k$ that makes the following diagram commute.

$$\begin{array}{ccccc} (M^{\times k} \setminus \Delta_k) \times \mathbf{R}_{\geqslant 0} & \xrightarrow{g_k} & B_k & \xrightarrow{\pi} & B'_k \\ \downarrow S_k \curvearrowright & & & \nearrow h_k & \\ \mathrm{Ran}^k(M) \times \mathbf{R}_{\geqslant 0} & & & & \end{array}$$

This will be our stratifying map. To check that h_k is continuous take $U \subseteq B'_k$ open. As π is the projection under a group action, it is an open map, so $\pi^{-1}(U) \subseteq B_k$ is open. Since g_k is continuous in the upset topology, $g_k^{-1}(\pi^{-1}(U))$ is open. Again, $S_k \curvearrowright$ is the projection under a group action, so $(S_k \curvearrowright)(g_k^{-1}(\pi^{-1}(U)))$ is open, giving continuity of h_k.

4.6 Ordering simplicial complexes

2017-09-26

Keywords: *informal, simplicial complex, simple graph, graph, Ran space, ordering*

In the context of trying to make a constructible sheaf over the Ran space, we have made several attempts to stratify $X = \mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$ correctly, the hope being for each stratum to have a unique simplicial complex (the Vietoris–Rips complex of the elements of X). In this post we make some observations and examine what it means to move around in X.

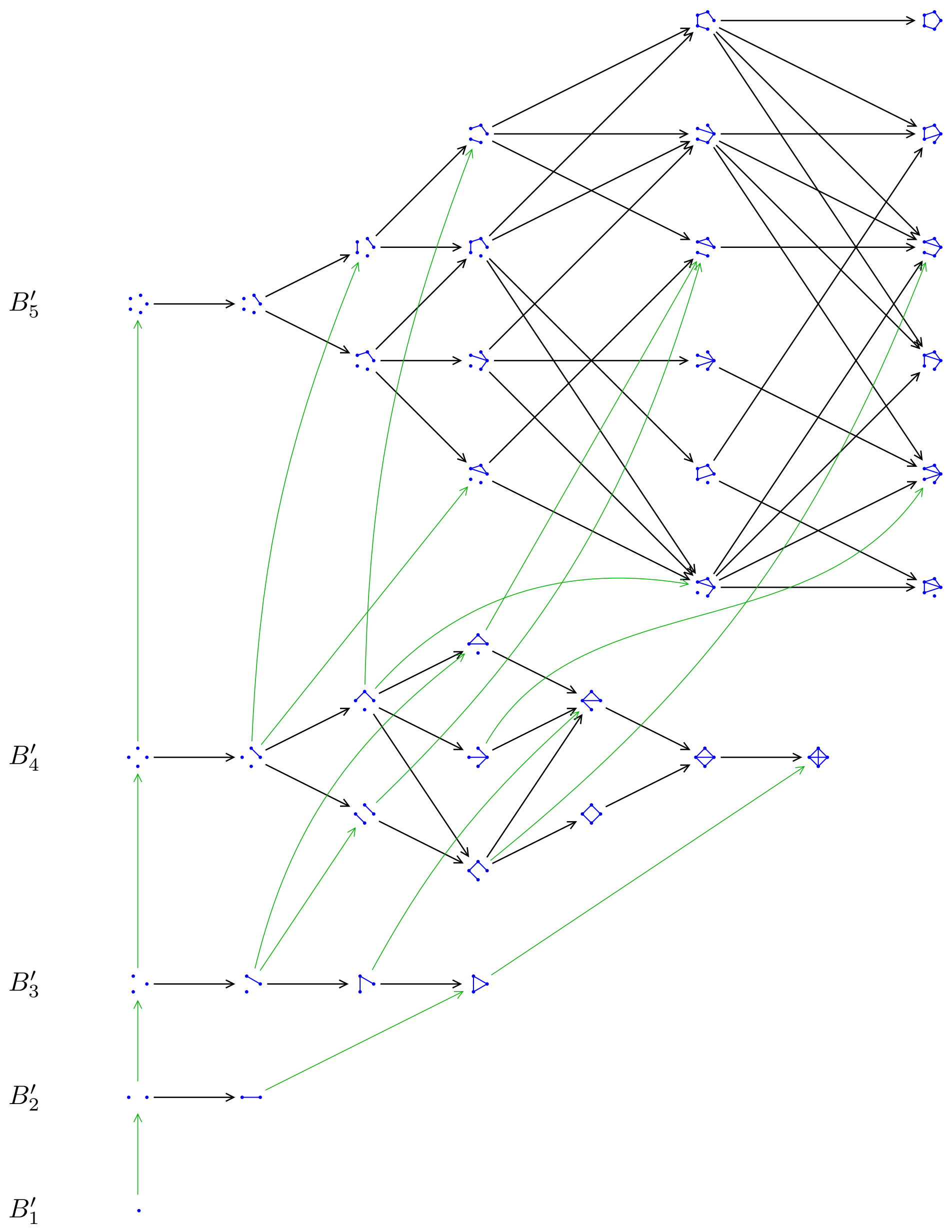
We use the convention that a Vietoris–Rips complex $VR(P,t)$ of an element $(P,t) \in X$ contains an edge (P_i, P_j) iff $d(P_i, P_j) > t$ (as opposed to $d(P_i, P_j) \geqslant t$).

Observation 1: The VR complex $VR(P,t)$ is completely described by its 1-skeleton $sk_1(VR(P,t))$, as having a complete subgraph $K_\ell \subseteq sk_1(VR(P,t))$ is equivalent to $VR(P,t)$ having an $(\ell-1)$-cell spanning that subgraph. The 1-skeleton is a simple graph $G = (V, E)$ on k vertices, so if we can order simple graphs with $\leqslant n$ vertices, we can order VR complexes of $\leqslant n$ vertices.

Let Γ_k be the collection of simple gaphs on k vertices. From now on we talk about an element $(P = \{P_1, \ldots, P_k\}, t) \in X$, a k-vertex VR complex $S = VR(P,t)$, and its 1-skeleton $G = sk_1(VR(P,t)) \in \Gamma_k$ interchangeably. Consider the following informal defintion of how the stratification of X should work.

Definition 4.6.1. A VR complex S is *ordered lower* than another VR complex T if there is a path from the stratum of type S to the stratum of type T that does not pass through strata of type R with $|V(R)| < |V(S)|$ or $|E(R)| < |E(S)|$. If S is *ordered lower* than T and we can move from the stratum of type S to the stratum of type T without passing through another stratum, then we say that S is *directly below* T.

To gain intuition of what this ordering means, consider the ordering on the posets B'_k, as defined in a previous post ("Stratifying correctly," 2017-09-17) and the 1-skeleta of the VR-complexes mapped to their elements. A complete description for $k = 1, 2, 3, 4$ and partially for $k = 5$ is given below, with arrows $S \to T$ indicating the minimal number of *directly below* relationships. That is, if $S \to R$ but also $S \to T$ and $T \to R$, then $S \to R$ is not drawn.



The orderings on each B'_k are clear and can be found in an algorithmic manner. However, it is more difficult to see which S at level k are *directly below* which T at level $k+1$. The green arrows follow no clear pattern.

Observation 2: If $G \in \Gamma_k$ has an isolated vertex and $t > 0$, then it can be *directly below* $H \in \Gamma_{k+1}$ only if $|E(H)| = |E(G)| + 1$. In general, if the smallest degree of a vertex of $G \in \Gamma_k$ is d and $t > 0$, then G can be *directly below* $H \in \Gamma_{k+1}$ only if $|E(H)| = |E(G)| + d + 1$.

Recall the posets B'_k are made by quotienting the nodes of the hypercube $B_k = \{0,1\}^{k(k-1)/2}$ by the action of S_k, where an element of B_k is viewed as a graph $G \in \Gamma_k$ having an edge (i,j) if the coordinate corresponding to the edge (i,j) is 1 (there are $k(k-1)/2$ pairs (i,j) of a k-element set).

Observation 3: It is not clear that G not being *ordered lower* than H in the hypercube context (order increases when increasing in any coordinate) implies that the VR complex of G is not *ordered lower* than the VR complex of H in X. No counterexample exists in the example given above, but this does not seem to exclude the possibility.

If any conclusion can be made from this, it is that this may not be the best approach to take when stratifying X.

4.7 Refining stratifications

2018-03-11

Keywords: *stratification, conical stratification, partial order, ordering*

The goal of this post is to describe a natural stratification associated to any stratification, with hopes of it being conical. Let X be a topological space, $(A, \leqslant_A)$ a finite partially ordered set, and $f : X \to A$ a stratifying map. For every $x \in X$, write $A_{>f(x)} = \{a \in A \ : \ a > f(x)\} \subseteq A$, and analogously for $A_{\geqslant f(x)}$. For every $a \in A$, write $X_a = \{x \in X \ : \ f(x) = a\}$.

Definition 4.7.1. For any other stratified space $g\colon Y \to B$, a *stratified map* $\varphi\colon (X \to A) \to (Y \to B)$ is a pair of maps $\varphi_{XY} \in \mathrm{Hom}_{\mathrm{Top}}(X, Y)$ and $\varphi_{AB} \in \mathrm{Hom}_{\mathrm{Set}}(A, B)$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi_{XY}} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{\varphi_{AB}} & B \end{array}$$

commutes. A stratified map φ is an *open embedding* if both φ_{XY} and $\varphi_{XY}|_{X_a}\colon X_a \to Y_{\varphi_{AB}(a)}$ are open embeddings.

Recall the cone $C(Y)$ of a space Y is defined as $Y \times [0,1)/Y \times \{0\}$.

Definition 4.7.2. A stratification $f\colon X \to A$ is *conical at* $x \in X$ if there exist

- a stratified space $f_x\colon Y \to A_{>f(x)}$,
- a topological space Z, and
- an open embedding $Z \times C(Y) \hookrightarrow X$ of stratified spaces whose image contains x.

The cone $C(Y)$ has a natural stratification $f'_x\colon C(Y) \to A_{\geqslant f(x)}$, as does the product $Z \times C(Y)$. The space X itself is *conically stratified* if it is conically stratfied at every $x \in X$.

The image to have in mind is that Z is a neighborhood of x in its stratum $X_{f(x)}$, and $C(Y)$ is an upwards-directed neighborhood of $f(x)$ in A. Now we describe how to refine the stratification of an arbitrary stratified space to make it conical.

Definition 4.7.3. Let $\leqslant_{\mathbf{P}(A)}$ be the partial order on $\mathbf{P}(A)$ defined in the following way:

- For every $x, y \in A$, set $x \leqslant_{\mathbf{P}(A)} y$ whenever $x \leqslant_A y$, and
- for every $C \in \mathbf{P}(A)$, set $C \leqslant_{\mathbf{P}(A)} C'$ for all $C' \in \mathbf{P}(C)$.

Note that $(A, \leqslant_A)$ is open in $(\mathbf{P}(A), \leqslant_{\mathbf{P}(A)})$ in the upset topology. Hence for $i : A \hookrightarrow \mathbf{P}(A)$ the inclusion map, $i \circ f : X \to A \hookrightarrow \mathbf{P}(A)$ is also a stratifying map for X. We now define another $\mathbf{P}(A)$-stratification for X.

Definition 4.7.4. Let $f_{\mathbf{P}}\colon X \to \mathbf{P}(A)$ be defined by $f_{\mathbf{P}}(x) = \min\limits_{(\mathbf{P}(A), \leqslant_{\mathbf{P}(A)})} \big\{C \ : \ x \in \mathrm{cl}(f^{-1}(C')) \ \forall \ C' \in C\big\}$.

This map is well defined because for each $x \in X$ there are finitely many strata $f^{-1}(a)$ which contain x in their closure. The element $C \in \mathbf{P}(A)$ containing all such a is the C to which x gets mapped. We now claim this is a stratifying map for X.

Proposition 4.7.5. The map $f_{\mathbf{P}}\colon X \to \mathbf{P}(A)$ is continuous.

Proof: Let $C \in \mathbf{P}(A)$. We will show that the preimage via $f_\mathbf{P}$ of the open set $U_C = \mathbf{P}(C) \subseteq \mathbf{P}(A)$ is open in X (and such sets U_C are a basis of topology for $\mathbf{P}(A)$). By definition of the map $f_\mathbf{P}$, we have

$$f_\mathbf{P}^{-1}(U_C) = f^{-1}(U_{\min\{C' \in C\}}) \setminus \left(\bigcup_{(D,E)\in A\times(A\setminus C)} \mathrm{cl}(f^{-1}(D)) \cap \mathrm{cl}(f^{-1}(E)) \right).$$

By continuity of f, the set $f^{-1}(U_{\min\{C'\in C\}})$ is open in X, and the sets we are subtracting from this open set are all closed. Hence $f_\mathbf{P}^{-1}(U_C)$ is open in X. ■

Unfortunately, this stratification is difficult to work with. Recall the space $\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_+$ for a very nice (smooth, compact, connected, embedded) manifold M, along with the map

$$\begin{aligned} f\colon \mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0} &\to SC, \\ (P,t) &\mapsto VR(P,t), \end{aligned}$$

for VR the Vietoris–Rips complex on P with radius t. To put a partial order on SC, we first say that $S \leqslant T$ in SC whenever there is a path $\gamma : I \to X$ satisfying

- $\widetilde{f}(\gamma(0)) = S$ and $\widetilde{f}(\gamma(1)) = T$,
- $\widetilde{f}(\gamma(t)) = \widetilde{f}(\gamma(1))$ for all $t > 1$.

Let $(SC, \leqslant_p)$ denote the partial order on SC generated by all relations of this type. We would like to prove some results about $f_\mathbf{P}$ induced by this f, and by any stratifying f in general, but the results seem difficult to prove. We give a list, in order of (percieved) increasing difficulty.

- The stratification $f_\mathbf{P}\colon \mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_+ \to \mathbf{P}(SC)$ is conical.
- The stratification $f_\mathbf{P}\colon X \to \mathbf{P}(A)$ is conical for any stratified space $f\colon X \to A$.
- If $f\colon X \to A$ is already conical, the map $j\colon A \to \mathbf{P}(A)$ given by $j(a) = \{b \in A \ :\ f^{-1}(a) \subseteq \mathrm{cl}(f^{-1}(b))\}$ is an isomorphism onto its image, and $f_\mathbf{P} = j \circ f$.

References: Ayala, Francis, Tanaka (Local structure on stratified spaces)

4.8 Conical stratifications via semialgebraic sets

2018-04-16

Keywords: *stratification, conical stratification, partial order, simplicial complex, semialgebraic, triangulation, compatible, piecewise linear*

The goal of this post is to describe a conical stratification of $\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$ that refines the stratification previously seen (in “Exit paths, part 2,” 2017-09-28, and “Refining stratifiations,” 2018-03-11). Thanks to Shmuel Weinberger for the key observation that the strata under consideration are nothing more than semialgebraic sets, which are triangulable, and so admit a conical stratification via this triangulation.

Remark 4.8.1. Fix $n \in \mathbf{Z}_{>0}$, let M be a smooth, compact, connected, embedded submanifold in $\mathbf{R}^N$, and let M^n have the Hausdorff topology. We will be interested in $M^n \times \mathbf{R}_{>0}$, though this will be viewed as the compact set $M^n \times [0,K] \subseteq \mathbf{R}^{nN+1}$ for some K large enough (for instance, larger than the diameter of M) when necessary. The point 0 is added for compactness.

Stratification of the Ran space by semialgebraic sets

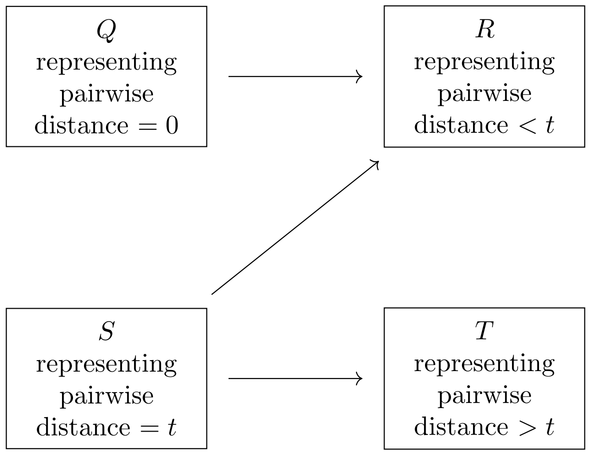
We begin by stratifying $M^n \times \mathbf{R}_{>0}$ by a poset A, creating strata based on the pairwise distance between points in each M component. Then we take that to a stratification of the quotient $\mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{>0}$ via the action of the symmetric group S_n and overcounting of points.

Definition 4.8.2. Define a partial order $\leqslant$ on the set $A = \{\text{partitions of } (\{1,\ldots,n\}^2 \setminus \Delta)/S_2 \text{ into 4 parts}\}$ of ordered 4-tuples of sets by

$$(Q,R,S,T) \leqslant (Q \setminus Q',\ R \cup Q' \cup S',\ S \setminus (S' \cup S''),\ T \cup S''),$$

for all $Q' \subseteq Q$ and $S', S'' \subseteq S$, with $S' \cap S'' = \emptyset$.

The diagram to keep in mind is the one below, with arrows pointing from lower-ordered elements to higher-ordered elements. Once we pass to valuing the 4-tuple in simplicial complexes, moving between Q and R will not change the simplicial complex type (this comes from the definition of the Vietoris–Rips complex).



Lemma 4.8.3. The map $f\colon M^n \times \mathbf{R}_{>0} \to (A, \leqslant)$ defined by

$$(\{P_1,\ldots,P_n\},t) \mapsto \Big(\{(i,j>i)\ :\ P_i = P_j\},\ \{(i,j>i)\ :\ d_M(P_i,P_j) < t\},\\ \{(i,j>i)\ :\ d_M(P_i,P_j) = t\},\ \{(i,j>i)\ :\ d_M(P_i,P_j) > t\}\Big)$$

is continuous in the upset topology on $(A, \leqslant)$.

Proof: Choose $(Q,R,S,T) \in A$ and consider the open set $U = U_{(Q,R,S,T)}$ based at (Q,R,S,T). Take $(P,t) \in f^{-1}(U)$, which we claim has a small neighborhood still contained within $f^{-1}(U)$. If we move a point P_i slightly that was exactly distance t away from P_j, then the pair (i,j) was in S, but is now in either R or T, and both $(Q, R\cup\{(i,j)\}, S\setminus\{(i,j)\}, T)$ and $(Q, R, S\setminus\{(i,j)\}, T\cup\{(i,j)\})$ are ordered higher than (Q,R,S,T), so the perturbed point is still in $f^{-1}(U)$. If $P_i = P_j$ in P and we move them apart slightly, since $t \in \mathbf{R}_{>0}$, the pair (i,j) will move from Q to R, and $(Q,R,S,T) \leqslant (Q \setminus \{(i,j)\}, R \cup \{(i,j)\}, S, T)$, so the perturbed point is still in $f^{-1}(U)$. For all pairs (i,j) in R or T, the distances can be changed slightly so that the pair still stays in R or T, respectively. Hence f is continuous. ■

This shows that $M^n \times \mathbf{R}_{>0}$ is stratified by $(A, \leqslant)$, using Lurie's definition of a (poset) stratification, which just needs a continuous map to a poset. Our goal is to work with the Ran space of M, instead of the n-fold product of M, which are related by the natural projection map $\pi\colon M^n \to \mathrm{Ran}^{\leqslant n}(M)$, taking $P = \{P_1,\ldots,P_n\}$ to the unordered set of distinct elements in P. We also would like to stratify $\mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{>0}$ by simplicial complex type, so we need the following map.

Definition 4.8.4. Let $g\colon (A,\leqslant) \to SC$ be the map into simplicial complexes that takes (Q,R,S,T) to the clique complex of the simple graph C on $n-k$ vertices, for $|Q| = k(k+1)/2$, defined as follows:

- $V(C) = \{[i]\ :\ i = 1,\ldots,n,\ [j] = [i] \text{ iff } (i,j) \in Q\}$,
- $E(C) = \{([i],[j])\ :\ (i,j) \in R \cup S\}$.

We require C to be simple, so if $(i,j) \in Q$ and $(i,\ell),(j,\ell) \in R \cup S$, we only add one edge $([i],[\ell]) = ([j],[\ell])$ to C.

The map g induces a partial order $\leqslant$ on SC from the partial order on A, with $C \leqslant C'$ in SC whenever there is $(Q, R, S, T) \in g^{-1}(C)$ and $(Q', R', S', T') \in g^{-1}(C')$ such that $(Q, R, S, T) \leqslant (Q', R', S', T')$ in A. Note that if $C \in SC$ is not in the image of g, then it is not related to any other element of SC. By the universal property of the quotient and continuity of f and g (as A and SC are discrete), there is a continuous map $h : \mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{>0} \to (SC, \leqslant)$ such that the diagram

$$\begin{array}{ccccc} M^n \times \mathbf{R}_{>0} & \xrightarrow{\;f\;} & (A, \leqslant) & \xrightarrow{\;g\;} & (SC, \leqslant) \\ \Big\downarrow{\scriptstyle \pi \times \mathrm{id}} & & & \nearrow{\scriptstyle h} & \\ \mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{>0} & & & & \end{array} \tag{8}$$

commutes. Hence $\mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{>0}$ is stratified by $(SC, \leqslant)$.

Remark 4.8.5. The map π can be thought of as a quotient by the action of the symmetric group S_n, followed by the quotient of the equivalence relation

$$\{P_1^1, \ldots, P_1^{\ell_1}, P_2^1, \ldots, P_2^{\ell_2}, P_3^1, \ldots, P_k^{\ell_k}\} \quad \sim \quad \{P_1^1, \ldots, P_1^{\ell_1 - 1}, P_2^1, \ldots, P_2^{\ell_2 + 1}, P_3^1, \ldots, P_k^{\ell_k}\}$$

on M^n, for all possible combinations $\ell_1 + \cdots + \ell_k = n$ and $1 \leqslant k \leqslant n-1$, where $P_m^i = P_m^j$ for all $1 \leqslant i < j \leqslant \ell_m$.

Semialgebraic geometry

Next we move into the world of semialgebraic sets and triangulations, following Shiota. Here we come across a more restrictive notion of *stratification* of a manifold X, which requires a partition of X into submanifolds $\{X_i\}$. If Lurie's stratification $f : X \to A$ gives back submanifolds $\{f^{-1}(a)\}_{a \in A}$, then we have Shiota's stratification. Conversely, the poset $(\{X_i\}, \leqslant)$, for $X_i \leqslant X_j$ iff $X_i \subseteq \mathrm{cl}(X_j)$ is always a stratification in the sense of Lurie.

Definition 4.8.6. A *semialgebraic set* in $\mathbf{R}^N$ is a set of the form

$$\bigcup_{\text{finite}} \{x \in \mathbf{R}^N \ : \ f_1(x) = 0, f_2(x) > 0, \ldots, f_m(x) > 0\},$$

for polynomial functions $f_1, \ldots, f_m$ on $\mathbf{R}^N$. A *semialgebraic stratification* of a space $X \subseteq \mathbf{R}^N$ is a partition $\{X_i\}$ of X into submanifolds that are semialgebraic sets.

Next we observe that the strata of $M^n \times \mathbf{R}_{>0}$ are semialgebraic sets, with the preimage theorem and I.2.9.1 of Shiota, which says that the intersection of semialgebraic sets is semialgebraic. Take $(Q, R, S, T) \in A$ and note that

$$f^{-1}(Q, R, S, T) = \left\{ (\{P_1, \ldots, P_n\}, t) \in M^n \times \mathbf{R}_{>0} \ : \ \begin{array}{rl} d(P_i, P_j) = 0 & \forall (i,j) \in Q, \\ t - d(P_i, P_j) = 0 & \forall (i,j) \in S, \\ t - d(P_i, P_j) > 0 & \forall (i,j) \in R, \\ d(P_i, P_j) - t > 0 & \forall (i,j) \in T. \end{array} \right\}$$

Here d means distance on the manifold, and we assume the metric to be analytic. Alternatively, d could be Euclidean distance between points on the embedding of $M^n \times \mathbf{R}_{>0}$, induced by the assumed embedding of M.

For his main Theorem II.4.2, Shiota uses cells, but we opt for simplices instead, and for cell complexes we use simplicial complexes. Every cell and cell complex admits a decomposition into simplicial complexes, even without introducing new 0-cells (by Lemma I.3.12), so we do not lose any generality.

Definition 4.8.7. Let X, Y be semialgebraic sets.

- A map $f \colon X \to Y$ is *semialgebraic* if the graph of f is semialgebraic.
- A *semialgebraic cell triangulation* of a semialgebraic set X is a pair (C, π), where C is a simplicial complex and $\pi \colon |C| \to X$ is a semialgebraic homeomorphism for which $\pi|_{\mathrm{int}(\sigma)}$ is a diffeomorphism onto its image.
- A semialgebraic cell triangulation (C, π) is *compatible* with a family $\{X_i\}$ of semialgebraic sets if $\pi(\mathrm{int}(\sigma)) \subseteq X_i$ or $\pi(\mathrm{int}(\sigma)) \cap X_i = \emptyset$ for all $\sigma \in C$ and all X_i.

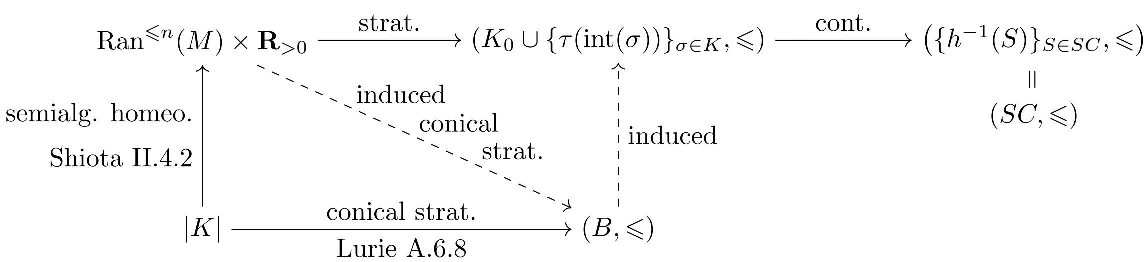
A semialgebraic cell triangulation (C, π) of X induces a stratification $X \to (C_0 \cup \{\pi(\mathrm{int}(\sigma))\}, \leqslant)$, where the order is the one mentioned just before Definition 4.8.6. We use the induced stratification and the cell triangulation interchangeably, specifically in Proposition 4.8.8.

A compatible conical stratification

Finally we put everything together to get a conical stratification of $\mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{>0}$. Unfortunately we have to restrict ourselves to piecewise linear manifolds, or PL manifolds, which are homeomorphic images of geometric realizations of simplicial complexes, as otherwise we cannot claim M is a semialgebraic set. We can also just let $M = \mathbf{R}^k$, as the point samples we are given could be coming from an unknown space.

Proposition 4.8.8. Let M be a PL manifold embedded in $\mathbf{R}^N$. There is a conical stratification $\widetilde{h}\colon \mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{>0} \to (B, \leqslant)$ compatible with the stratification $h\colon \mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{>0} \to (SC, \leqslant)$.

Proof: (Sketch) The main lifting is done by Theorem II.4.2 of Shiota. Since M is PL, it is semialgebraic, and so $M^n \times \mathbf{R}_{>0} \subseteq \mathbf{R}^{nN+1}$ is semialgebraic, by I.2.9.1 of Shiota. Since the quotient π of diagram (8) is semialgebraic, the space $\mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{>0}$ is semialgebraic, by Scheiderer. Similarly, $\{f^{-1}(a)\}_{a \in A}$ is a family of semialgebraic sets, where f is the map from Lemma 4.8.3. Theorem II.4.2 gives that $\mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{>0}$ admits a cell triangulation (K, τ) compatible with $\{h^{-1}(S)\}_{S \in SC}$. By the comment after Definition 4.8.7, this means we have a stratification $\mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{>0} \to (K_0 \cup \{\tau(\mathrm{int}(\sigma))\}_{\sigma \in K}, \leqslant)$. Further, by Proposition A.6.8 of Lurie, we have a conical stratification $|K| \to (B, \leqslant)$. This is all described by the solid arrow diagram below.



The vertical induced map comes as the poset B has the exact same structure as the abstract suimplicial complex K. The diagonal induced map comes as the map $|K| \to \mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{>0}$ is a homeomorphism, and so has a continuous inverse. Composing the inverse with the conical sratification of Lurie, we get a conical stratification of $\mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{>0}$. Composing the vertical induced arrow and the maps to $(SC, \leqslant)$ show that there is a conical stratification of $\mathrm{Ran}^{\leqslant n} \times \mathbf{R}_{>0}$ compatible with its simplicial complex stratification from diagram 8. ∎

Shiota actually requires that the space that admits a triangulation be closed semialgebraic, and having $\mathbf{R}_{>0}$ violates that condition. Replacing this piece with $\mathbf{R}_{\geqslant 0}$, then applying Shiota, and afterwards removing the $t = 0$ piece we get the same result.

Remark 4.8.9. Every (sufficiently nice) manifold admits a triangulation, so it may be possible to extend this result to a larger class of manifolds, but it seems more sophisticated technology is needed.

References: Shiota (Geometry of subanalytic and semialgebraic sets, Chapters I.2, I.3, II.4), Scheiderer (Quotients of semi-algebraic spaces), Lurie (Higher algebra, Appendix A.6)

4.9 Visualizing paths in configuration space

2018-11-25

Keywords: *configuration space, persistent homology, simplicial complex, visualization, code*

The goal of this post is to visualize how point configurations induce persistent homology, and how paths between point samples induce changes in the simplicial complexes producing the homology. We use the Čech simplicial complex construction of a finite subset of $\mathbf{R}^N$.

Definition 4.9.1. For M a Riemannian mandifold, $\mathrm{Conf}_n(M) := \{P \subseteq M : |P| = n\}$ is the *configuration space* of n points on M.

The space $\mathrm{Conf}_n(M)$ is itself a topological space, with topology induced by the Hausdorff distance of subsets. Let SC be the set of abstract simplicial complexes (V, S), where V is a set and $S \subseteq P(V)$ closed under subsets. Let uSC be the set of unlabeled abstract simplicial complexes, with the natural projection map $\mathsf{SC} \to \mathsf{uSC}$.

Definition 4.9.2. The *Čech map* is the function $\check{C}\colon \mathrm{Conf}_n(M) \times \mathbf{R}_{\geqslant 0} \to \mathsf{SC}$ given by $V(\check{C}(P,r)) = P$ and $P' \in S(\check{C}(P,r))$ whenever $\bigcap_{p\in P'} B(p,r) \neq \emptyset$, for every $P' \subseteq P$. The *unlabeled Čech map* is the composition of $\check{C}$ with the projection to uSC.

We will consider the case $M = \mathbf{R}^2$ and $n = 4$. To describe an implementation of the Čech map, we only need to consider double and triple intersections. Finding if $B(P_1,r) \cap B(P_2,r)$ is empty or not is easy, but to determine if $B(P_1,r)\cap B(P_2,r)\cap B(P_3,r)$ is empty or not requires more care. Below is an implementation in *Mathematica*.

```
(* CechPt : Finds the coordinate where balls of the same radii around three points a,b,c will
    first intersect *)
(* Input : 3 coordinates {x, y}. Output : 1 coordinate {x, y} *)
CechPt[a_,b_,c_] := Module[{
    cenx = Det[{{Norm[a]^2, a[[2]], 1}, {Norm[b]^2, b[[2]], 1}, {Norm[c]^2, c[[2]], 1}}],
    ceny = Det[{{a[[1]], Norm[a]^2, 1}, {b[[1]], Norm[b]^2, 1}, {c[[1]], Norm[c]^2, 1}}],
    scal = 2*Det[{{a[[1]], a[[2]], 1}, {b[[1]], b[[2]], 1}, {c[[1]], c[[2]], 1}}]},
   cen = {cenx/scal, ceny/scal};
   If[Max[ArcCos[(b-a).(c-a)/(Norm[b-a]*Norm[c-a])],
          ArcCos[(a-b).(c-b)/(Norm[a-b]*Norm[c-b])],
          ArcCos[(a-c).(b-c)/(Norm[a-c]*Norm[b-c])]] < Pi/2, cen,
     If[Norm[cen-(a+b)/2] < Norm[cen-(a+c)/2],
       If[Norm[cen-(a+b)/2] < Norm[cen-(b+c)/2], (a+b)/2, (b+c)/2],
       If[Norm[cen-(a+c)/2] < Norm[cen-(b+c)/2], (a+c)/2, (b+c)/2]]]];
```

Here `cen` is the circumcenter of the input points, which corresponds to our desired point only if it lies within the convex hull of the points. Now $B(P_1,r)\cap B(P_2,r)\cap B(P_3,r)$ is non-empty if and only if the distance from each of P_1, P_2, P_3 to `CechPt[`P_1`,`P_2`,`P_3`]` is less than or equal to r.

Let $\gamma\colon I \to \mathrm{Conf}_4(\mathbf{R}^2)$ be a path, and $\gamma(0) = \{P_1,P_2,P_3,P_4\}$. At each $t \in I$ and for every pair and triple $P' \subseteq \gamma(t)$, we can find the smallest r such that $\bigcap_{p\in P'} B(P,r) \neq \emptyset$. This gives 6 curves for the pairs P', and 4 curves for the triples P', which we can plot all together in *Mathematica*.

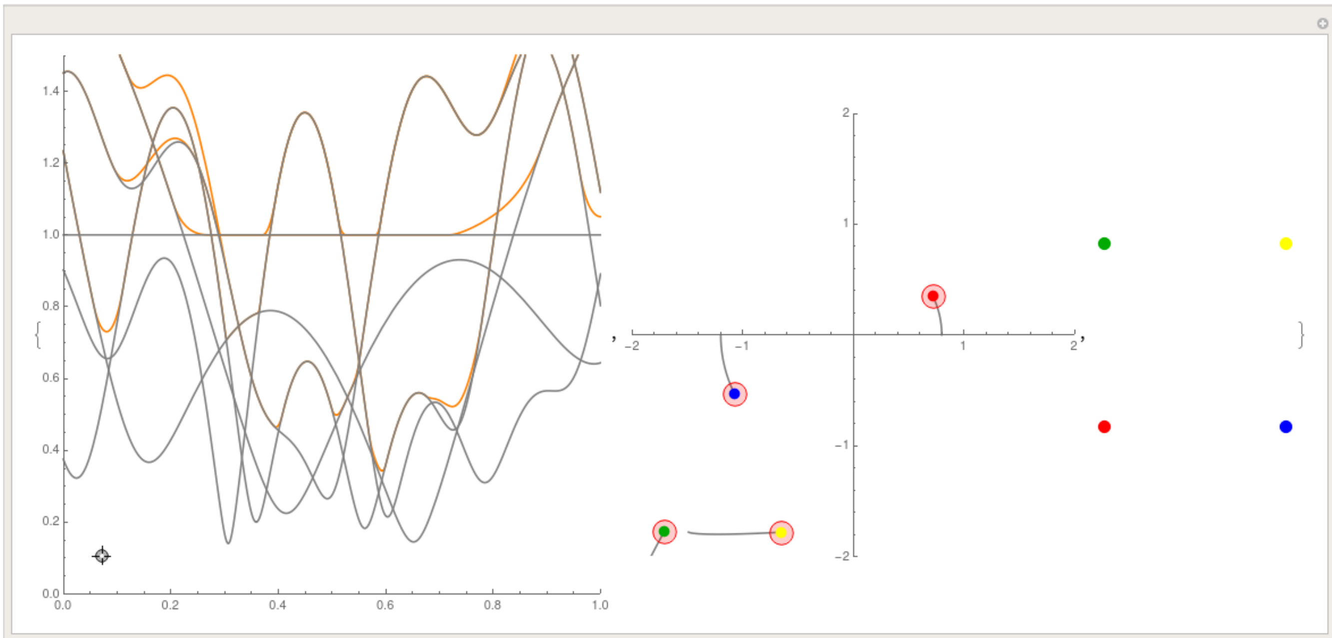
```
PList[t_] := {P1[t],P2[t],P3[t],P4[t]};
(* Graphs of pairwise distances *)
DistGraph1 = Plot[Table[Norm[pair[[1]]-pair[[2]]]/2, {pair,Subsets[PList[t],{2}]}], {t, 0, 1},
    PlotRange -> {{0,1},{0,1.5}}, PlotStyle -> {Gray}, AspectRatio -> 1];
(* Graphs of minimum distance from every triple to its CechPt*)
DistGraph2 = Plot[Table[Max[Table[Norm[triple[[k]]-CechPt@@triple],{k,1,3}]],
    {triple,Subsets[PList[t],{3}]}], {t, 0, 1}, PlotRange -> {{0,1},{0,1.5}}, PlotStyle ->
    {Orange}, AspectRatio -> 1];
```

The code is given so that it may be easily generalized to more than 4 points. Next, use the `Manipulate` command to add interactivity to the graphs.

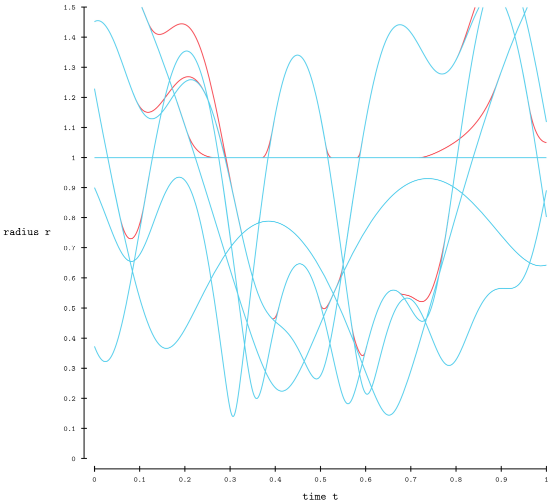
```
Manipulate[{
  Show[DistGraph2, DistGraph1],
  Show[
    ParametricPlot[PList[t],{t,0,X[[1]]},PlotRange -> {{-2,2},{-2,2}},PlotStyle -> {Black}],
    Graphics[Join[
      {Opacity[.2],Red}, Table[Disk[point,X[[2]]],{point,PList[X[[1]]]}],
      {Opacity[1],Red}, Table[Circle[point,X[[2]]],{point,PList[X[[1]]]}],
      {Red,Disk[P1[X[[1]]],.05]},
      {Blue,Disk[P2[X[[1]]],.05]},
      {Darker[Green],Disk[P3[X[[1]]],.05]},
      {Yellow,Disk[P4[X[[1]]],.05]}]]],
  Graphics[Join[
    {Black, Thick},
```

```
  Flatten[Table[{{Opacity[0], Opacity[.3]}[[Boole[X[[2]] >= Norm[pair[[1]][[1]][X[[1]]] -
      pair[[2]][[1]][X[[1]]]]/2] + 1]], Line[#[[2]]&/@pair]},
           {pair,Subsets[{{P1,{0,0}},{P2,{2,0}},{P3,{0,2}},{P4,{2,2}}},{2}]}]],
  Flatten[Table[{{Opacity[0], Opacity[.3]}[[Boole[X[[2]] >=
      Max[Table[Norm[triple[[k]][[1]][X[[1]]] - CechPt@@(#[[1]][X[[1]]]&/@triple)],
      {k,1,3}]]] + 1]], Polygon[#[[2]]&/@triple]},
           {triple,Subsets[{{P1,{0,0}},{P2,{2,0}},{P3,{0,2}},{P4,{2,2}}},{3}]}]],
  {Opacity[1], Red, Disk[{0,0},.07], Blue, Disk[{2,0},.07], Darker[Green], Disk[{0,2},.07],
      Yellow, Disk[{2,2},.07]}]]
}, {{X, {.1, .1}}, Locator}]
```

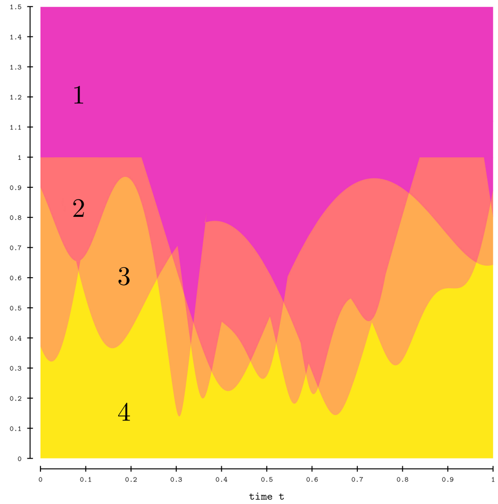
This produces the interactive visualization below, allowing the user to drag the crosshairs on the graph on the left (graphs of when double and triple intersections are reached). The paths of the individual points P_1, P_2, P_3, P_4 are in the middle and the image of the unlabeled Čech map is on the right.



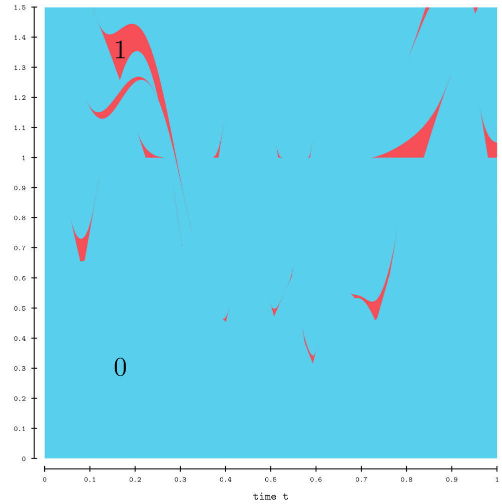
The graphs on the left stratify the strip $I \times \mathbf{R}_{\geqslant 0}$, so that the unlabeled Čech map is constant on each stratum. Computing the Betti numbers of each simplicial complex gives the *CROCKER plot* (see TZH) of the stratified space. We use the Čech instead of the Rips complex, so perhaps this should be called the *CROCKEČ plot*. The stratified space, 0-dimensional, and 1-dimensional plots are given below.



stratified space $I \times \mathbf{R}_{\geqslant 0}$



Betti numbers b_0



Betti numbers b_1

Here the Betti numbers were computed by inspection, since the complexes are so small. An extension would be to make this computation automatic once the input path γ is given.

The *Mathematica* code for this post is available online.

References: Topaz, Ziegelmeier, Halverson (Topological Data Analysis of Biological Aggregation Models)

5 The Ran space - constructibility

5.1 Exit paths, part 2

2017-09-28

Keywords: *exit path, universality, stratification, conical stratification, constructible sheaf*

In this post we continue on a previous topic ("Exit paths, part 1," 2017-08-31) and try to define a constructible sheaf via universality. Let X be an A-stratified space, that is, a topological space X and a poset $(A, \leqslant)$ with a continuous map $f : X \to A$, where A is given the upset topology relative to its ordering $\leqslant$. Recall the full subcategory $\mathrm{Sing}^A(X) \subseteq \mathrm{Sing}(X)$ of exit paths on an A-stratified space X.

Proposition 5.1.1. If $X \to A$ is conically stratified, $\mathrm{Sing}^A(X)$ is an ∞-category.

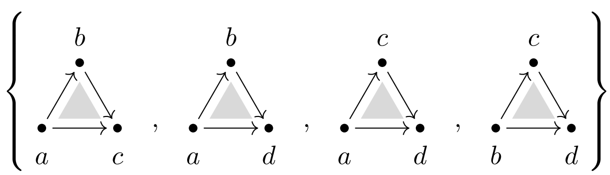
Briefly, a stratification $f : X \to A$ is *conical* if for every stratum there exists a particular embedding from a stratified cone into X (see Lurie for "conical stratification" and Ayala, Francis, Tanaka for "conically smooth stratified space," which seem to be the same). We will leave confirming the described stratification as conical to a later post.

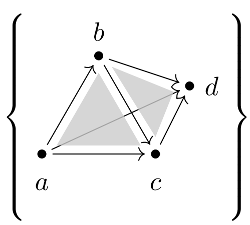
This proposition, given as part of Theorem A.6.4 in Lurie, has a very long proof, so is not repeated here. Lurie actually proves that the natural functor $\mathrm{Sing}^A(X) \to N(A)$ described below is a (inner) fibration, which implies the unique lifting property of $\mathrm{Sing}^A(X)$ via the unique lifting property of $N(A)$ (and we already know nerves are ∞-categories).

Example 5.1.2. The nerve of a poset is an ∞-category. Being a nerve, it is already immediate, but it is worthwhile to consider the actual construction. For example, if $A = \{a \leqslant b \leqslant c \leqslant d\}$ is the poset with the ordering $\leqslant$, then the pieces $N(A)_i$ are as below.

$$N(A)_0 = \left\{ \underset{a}{\bullet}\ ,\ \underset{b}{\bullet}\ ,\ \underset{c}{\bullet}\ ,\ \underset{d}{\bullet} \right\}$$

$$N(A)_1 = \left\{ \underset{a}{\bullet} \longrightarrow \underset{b}{\bullet}\ ,\ \underset{a}{\bullet} \longrightarrow \underset{c}{\bullet}\ ,\ \underset{a}{\bullet} \longrightarrow \underset{d}{\bullet}\ ,\ \underset{b}{\bullet} \longrightarrow \underset{c}{\bullet}\ ,\ \underset{b}{\bullet} \longrightarrow \underset{d}{\bullet}\ ,\ \underset{c}{\bullet} \longrightarrow \underset{d}{\bullet} \right\}$$

$N(A)_2 =$ 

$N(A)_3 =$ 

It is immediate that every 3-horn can only be filled in one unique way (as there is only one element of $N(A)_3$), as well as that every 2-horn can be filled in one unique way (as every sequence of two composable morphisms appears as a horn of exactly one element of $N(A)_2$).

In Appendix A.9 of Higher Algebra, Lurie says that there is an equivalence of categories

$$(A\text{-constructible sheaves on } X) \quad \cong \quad [(A\text{-exit paths on } X), \mathcal{S}]\,,$$

given that X is conically stratified, and for $\mathcal{S}$ the ∞-category of spaces (equivalently $N(Kan)$, the nerve of all the simplicial sets that are Kan complexes). So, instead of trying to define a particular constructible sheaf on

$X = \mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$, (as in previous posts "Stratifying correctly," 2017-09-17 and "A constructible sheaf over the Ran space," 2017-06-24) we will try to make a functor that takes an exit path of X and gives back a space.

Fix $n \in \mathbf{Z}_{>0}$ and set $X = \mathrm{Ran}^{\leqslant n} \times \mathbf{R}_{\geqslant 0}$. Let SC be the category of simplicial complexes and simplicial maps, with SC_n the full subcategory of simplicial complexes with at most n vertices. There is a map

$$\begin{array}{rcl} g \ : \ X & \to & SC_n \\ (P,t) & \mapsto & VR(P,t), \end{array}$$

allowing us to say

$$X = \bigcup_{S \in SC_n} g^{-1}(S).$$

Here we consider that two elements $P_i, P_j \in P$ give an edge of $VR(P,t)$ whenever $t > d(P_i, P_j)$ (this is chosen instead of $t \geqslant d(P_i,P_j)$ so that the boundaries of the strata "facing downward," with respect to the poset ordering, are open). Now we define a stratifying poset A for X.

Definition 5.1.3. Let $A = \{a_S \ : \ S \in SC_n\}$ and define a relation $\leqslant$ on A by

$$(a_S \leqslant a_T) \quad \Longleftarrow \quad \left(\begin{array}{c} \exists \ \sigma \in \mathrm{Sing}(X)_1 \text{ such that} \\ g(\sigma(0)) = S, \ g(\sigma(t>0)) = T. \end{array} \right)$$

Let $(A, \leqslant)$ be the poset generated by relations of the type given above.

We claim that $f : X \to A$ given by $f(P,t) = a_{g(P,t)}$ is a stratifying map, that is, continuous in the upset topology on A. To see this, take the open set $U_S = \{a_T \in A \ : \ a_S \leqslant a_T\}$ in the basis of the upset topology of A, for any $S \in SC_n$, and consider $x \in f^{-1}(U_S)$. If for all $\epsilon > 0$ we have $B_X(x,\epsilon) \cap f^{-1}(U_S)^C \neq \emptyset$, then there exists $T_\epsilon \in SC_n$ with $B_X(x,\epsilon) \cap f^{-1}(a_{T_\epsilon}) \neq \emptyset$, for $S \not\leqslant T_\epsilon$ (as $T_\epsilon \notin U_S$). This means there exists $\sigma \in \mathrm{Sing}(X)_1$ with $\sigma(0) = x$ and $\sigma(t>0) \in f^{-1}(a_{T_\epsilon})$, which in turn implies $S \leqslant T_\epsilon$, a contradiction. Hence f is continuous, so $f : X \to A$ is a stratification.

As all morphisms in $\mathrm{Sing}(X)$ are compsitions of the face maps s_i and degeneracy maps d_i, so are all morphisms in $\mathrm{Sing}^A(X)$. There is a natural functor $F : \mathrm{Sing}^A(X) \to N(A)$ defined in the following way:

$$\text{objects} \qquad \left(\begin{array}{c} \sigma : |\Delta^k| \to X \\ a_0 \leqslant \cdots \leqslant a_k \subseteq A \\ f(\sigma(t_0, \dots, t_i \neq 0, 0, \dots, 0)) = a_i \end{array} \right) \quad \mapsto \quad (a_0 \to \cdots \to a_k \in N(A)_k)$$

$$\text{face maps} \qquad \left(\begin{array}{c} \left(\begin{array}{c} \sigma : |\Delta^k| \to X \\ a_0 \leqslant \cdots \leqslant a_k \subseteq A \end{array} \right) \\ \downarrow \\ \left(\begin{array}{c} \tau : |\Delta^{k+1}| \to X \\ a_0 \leqslant \cdots \leqslant a_i \leqslant a_i \leqslant \cdots a_k \subseteq A \end{array} \right) \end{array} \right) \quad \mapsto \quad \left(\begin{array}{c} (a_0 \to \cdots \to a_k) \\ \downarrow \\ \left(a_0 \to \cdots \to a_i \xrightarrow{\mathrm{id}} a_i \to \cdots \to a_k \right) \end{array} \right)$$

$$\text{degeneracy maps} \qquad \left(\begin{array}{c} \left(\begin{array}{c} \sigma : |\Delta^k| \to X \\ a_0 \leqslant \cdots \leqslant a_k \subseteq A \end{array} \right) \\ \downarrow \\ \left(\begin{array}{c} \tau : |\Delta^{k-1}| \to X \\ a_0 \leqslant \cdots \leqslant a_{i-1} \leqslant a_{i+1} \leqslant \cdots a_k \subseteq A \end{array} \right) \end{array} \right) \quad \mapsto \quad \left(\begin{array}{c} (a_0 \to \cdots \to a_k) \\ \downarrow \\ \left(a_0 \to \cdots \to a_{i-1} \xrightarrow{\circ} a_{i+1} \to \cdots \to a_k \right) \end{array} \right)$$

As all maps in $\mathrm{Sing}^A(X)$ are generated by compositions of face and degeneracy maps, this completely defines F. Naturality of F follows precisely because of this.

A poset (which can be viewed as a directed simple graph) may be naturally viewed as a 1-dimensional simplicial set, moreover an ∞-category (by virtue of being a *simple* graph, with no multi-edges or loops). Hence there is a natural map, the inclusion, that takes $N(A)$ into $N(\mathcal{K}an) = \mathcal{S}$. Finally, Construction A.9.2 of Lurie describes a map that takes a functor from A-exit paths into spaces and gives back an A-constructible sheaf over X, which Theorem A.9.3 shows to be an equivalence, given the following conditions:

- X is paracompact,
- X is locally of singular shape,
- the A-stratification of X is conical, and
- A satisfies the ascending chain condition.

The first condition is satisfied as both $\text{Ran}^{\leqslant n}(M)$ and $\mathbf{R}_{\geqslant 0}$ are locally compact and second countable. The last condition is satisfied because A is a finite poset. We already mentioned that the conical property will be checked later, as will the singular shape property. Unfortunately, Lurie gives a definition of singular shape only for ∞-topoi, so some work must be done to translate this into our simpler setting. However, in the introduction to Appendix A, Lurie says that if X is "sufficiently nice" and we assume some "mild assumptions" about A, then the described categorical equivalence follows, so it seems there is hope that everything will work out well in the end.

References: Lurie (Higher algebra, Appendix A), Ayala, Francis and Tanaka (Local structures on stratified spaces, Sections 2 and 3)

5.2 The Ran space is locally conical

2017-10-22

Keywords: *cone, Ran space, ordering, stratification*

In this post we show that every point in the Ran space $\text{Ran}^{\leqslant n}(M)$, for M a compact, smooth embedded manifold, is the base of a cone in $\text{Ran}^{\leqslant n}(M)$. Let $\dim(M) = m$ and let $P = \{P_1, \dots, P_k\} \in \text{Ran}^k(M) \subseteq \text{Ran}^{\leqslant n}(M)$. We write $d(x, y)$ for distance in Euclidean space $\mathbf{R}^N$ where M is embedded, and $d_M(x, y)$ for distance on the embedded manifold M (note $d \leqslant d_M$). Define the following objects:

$$\begin{aligned} N_\epsilon(x) &= \{z \in M \ : \ d_M(x,z) < \epsilon\}, \\ E_n &= \{\text{distinct partitions of an unlabeled set of } n \text{ elements}\}, \\ T(e) &= \{\text{distinct total orderings of } e \in E_n\}. \end{aligned}$$

We write $\tau = (\tau_1 < \cdots < \tau_{|\tau|})$ for an element $\tau \in T(e)$.

Example 5.2.1. Let $n = 4$, so then

$$E_4 = \Big\{\{\{\ast\},\{\ast\},\{\ast\},\{\ast\}\}, \quad \{\{\ast,\ast\},\{\ast\},\{\ast\}\}, \quad \{\{\ast,\ast\},\{\ast,\ast\}\}, \quad \{\{\ast,\ast,\ast\},\{\ast\}\}, \quad \{\{\ast,\ast,\ast,\ast\}\}\Big\}.$$

By stacking the $\ast$ on top of one another to indicate containment in a single set, and for order increasing from left to right, we have the following distinct total orderings for every element of E_4.

$$\begin{aligned} T(\{\{\ast\},\{\ast\},\{\ast\},\{\ast\}\}) &= \ast\ast\ast\ast & T(\{\{\ast,\ast\},\{\ast\},\{\ast\}\}) &= \substack{\ast\\\ast}\ast\ast \ , \ \ast\substack{\ast\\\ast}\ast \ , \ \ast\ast\substack{\ast\\\ast} \\ T(\{\{\ast,\ast\},\{\ast,\ast\}\}) &= \substack{\ast\\\ast}\substack{\ast\\\ast} & T(\{\{\ast,\ast,\ast\},\{\ast\}\}) &= \substack{\ast\\\ast\\\ast}\ast \ , \ \ast\substack{\ast\\\ast\\\ast} \\ T(\{\{\ast,\ast,\ast,\ast\}\}) &= \substack{\ast\\\ast\\\ast\\\ast} \end{aligned}$$

Set $\epsilon = \min_{1\leqslant i<j\leqslant k}\{d(P_i, P_j)\}$, $t_0 \in (0, \epsilon/2)$, and $t_{j>0} \in (0, t_{j-1})$. By construction, the object

$$\begin{aligned} C_P &= \{P\} \cup \coprod_{\substack{\sum \ell_i = n-k \\ \ell_i \in \mathbf{Z}_{>0}}} \prod_{i=1}^k \coprod_{\substack{\tau\in T(e)\\ e\in E_{\ell_i}}} \prod_{j=1}^{|\tau|} \text{Ran}^{|\tau_j|}\left(\partial N_{t_j}(P_i)\right) \times (0, t_{j-1}) \\ &= \coprod_{\substack{\sum \ell_i = n-k \\ \ell_i \in \mathbf{Z}_{>0}}} \prod_{i=1}^k \coprod_{\substack{\tau\in T(e)\\ e\in E_{\ell_i}}} \left(\text{Ran}^{|\tau_1|}(\partial N_{t_0}(P_i)) \times \prod_{j=2}^{|\tau|} \text{Ran}^{|\tau_j|}\left(\partial N_{t_j}(P_i)\right) \times (0, t_{j-1}) \right) \times [0, \epsilon/2) \Big/ \sim \end{aligned}$$

is an open cone based at P sitting inside $\mathrm{Ran}^{\leqslant n}(M)$. Here $\sim$ is the equivalence relation of all elements with $t_0 = 0$, with $[0, \epsilon/2) \ni t_0$ representing the unit interval in the usual cone construction. Moreover, given the point-counting stratification $f : \mathrm{Ran}^{\leqslant n}(M) \to A$, there is a natural stratification $g : C_p \to A_{\geqslant f(P)}$, with $P \in C_P$ the only element mapping to $f(P)$ under g.

The next step is to show that P has an open neighborhood in $\mathrm{Ran}^{\leqslant n}(M)$ that is the image of an open embedding $Z \times C_P$, for some topological space Z. The obvious choice $Z = \prod_{i=1}^k N_{\epsilon/2}(P_i)$ does not work, because we double count points in higher strata, so we do not have an embedding.

5.3 Attempts at proving conical stratification

2017-10-27

Keywords: *cone, Ran space, stratification, conical stratification, informal*

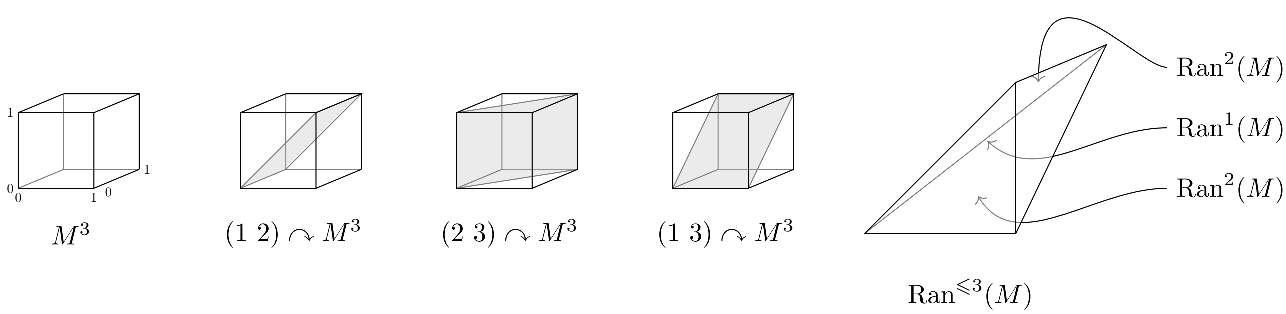
This post chronicles several attempts and failures to show that $X = \mathrm{Ran}^{\leqslant n}(M)$ is conically stratified. Here M will be a smooth, compact manifold of dimension m, embedded in $\mathbf{R}^N$ for $N \gg 0$. Recall that a stratified space $f : X \to A$ is *conically stratitifed at $x \in X$* if there exist:
- a stratified space $g : Y \to A_{>f(x)}$,
- a topological space Z, and
- an open embedding $Z \times C(Y) \hookrightarrow X$ of stratified spaces whose image contains x.

The cone $C(Y)$ has a natural stratification $g' : C(Y) \to A_{\geqslant f(x)}$, as does the product $Z \times C(Y)$. The space X itself is *conically stratified* if it is conically stratfied at every $x \in X$.

Let $P = \{P_1, \dots, P_k\} \in \mathrm{Ran}^k(M) \subseteq \mathrm{Ran}^{\leqslant n}(M) = X$, and $2\epsilon = \min_{1 \leqslant i < j \leqslant k}\{d(P_i, P_j)\}$.

Observations

Observation 1: When $M = I = (0,1)$, the interval, we can visualize what $\mathrm{Ran}^{\leqslant 3}(M)$ looks like via the construction $\mathrm{Ran}^{\leqslant 3}(M) = (M^3 \setminus \Delta_3)/S_3$, to gain some intuition about what the Ran space looks like in general.



A drawback is that $\dim(M) = 1$, which masks the problems in higher dimensions.

Observation 2: An open neighborhood of $P \in X$ looks like

$$\coprod_{\substack{\sum \ell_i = n \\ \ell_i \in \mathbf{Z}_{>0}}} \prod_{i=1}^k \mathrm{Ran}^{\leqslant \ell_i}(B_\epsilon^M(P_i)) = B_{\epsilon/2}^X(P) \times \coprod_{\substack{\sum \ell_i = n \\ \ell_i \in \mathbf{Z}_{>0}}} \prod_{i=1}^k \mathrm{Ran}^{\leqslant \ell_i}(B_{\epsilon/2}^M(P_i)), \tag{9}$$

for $B_\epsilon^M(x) = \{y \in M \ : \ d_M(x,y) < \epsilon\}$ the open ball of radius ϵ around $x \in M$, and similarly for $P \in X$. Most attempts to prove conical stratification are based around expressing these as $Z \times C(Y)$, usually for $Z = B_{\epsilon/2}^X(P)$.

Observation 3: When $k < n$, the “steepest” direction from P_i into the highest stratum of X is given by P_i splitting into $n-k+1$ points uniformly distributed on $\partial B_t^M(P_i)$. Hence the $[0,1)$ part of the cone (recall $C(Y) = Y \times [0,1)/\sim$) should be along $t \in [0,1)$.

Attempts

Attempt 1: *Use more resrictive (but better described) AFT definition.*
Ayala–Francis–Tanaka describe C^0 *stratified spaces*, a special type of stratified space. Any space that has a cover by topological manifolds is a C^0 stratified space, however it seems that X **cannot be covered by topological manifolds**. Even further, each element in the cover must have the trivial stratification, and since we must have overlaps, $f : X \to A$ will have $A = \{*\}$, which is not what we want.

Attempt 2: *Stratify* $\mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$ *instead.*
This is more difficult, but was the original impetus, with strata defined by collecting the Vietoris–Rips complexes $VR(P, t)$ of the same type. The problem is that this space **has strata next to each other of the same dimension**, which does not conform to a standard definition of stratification, and so doesn't admit a conical stratification. Dimension counting and requiring an open embedding $Z \times C(Y) \hookrightarrow X$ shows this is impossible at the boundary point between two such strata.

Weinberger gives some standard stratifed space types, among them a *manifold stratified space*, a *manifold stratified space with boundary*, and a *PL stratified space*, but $X \times \mathbf{R}_{\geqslant 0}$ is none of these.

Attempt 3: *Naively describe the neighborhood of P as a cone.*
This is the most direct attempt to write (9) as $Z \times C(Y)$. If we say

$$C(Y) = \underbrace{\coprod_{\substack{\sum \ell_i = n \\ \ell_i \in \mathbf{Z}_{>0}}} \prod_{i=1}^{k} \mathrm{Ran}^{\leqslant \ell_i}(\partial B_t^M(P_i))}_{Y} \times [0, \epsilon/2) \Big/ \sim,$$

then we **miss points splitting off at different "speeds"**. That is, in this presentation P_i can only split into points that are all the same distance away from it. Between such a collection of points and P_i are points that are some closer, some the same distance away, and those are not accounted for.

Moreover, using $Z = B^X_{\epsilon/2}(P)$, leads to **overcounting**, and the map into X would not be injective.

Attempt 4: *Iterate over different number of points at common radius.*
This came out of an attempt to fix the previous attempt. As in a previous post ("The Ran space is locally conical," 2017-10-22), let E_ℓ be the collection of distinct partitions of ℓ elements, and for $e \in E_\ell$, let $T(e)$ be the collection of distinct total orderings of e. A candidate for $Z \times C(Y)$ would then be

$$\underbrace{\prod_{i=1}^{k} B^M_\epsilon(P_i)}_{Z} \times \overbrace{\{P\} \cup \coprod_{\substack{\sum \ell_i \leqslant n-k \\ \ell_i \in \mathbf{Z}_{>0}}} \prod_{i=1}^{k} \coprod_{\substack{\tau_i \in T(e_i) \\ e_i \in E_{\ell_i}}} \prod_{j=1}^{|\tau_i|} \mathrm{Ran}^{|\tau_i^j|}\left(\partial B^M_{t_{i,j}}(P_i)\right) \times (0, t_{i,j-1})}^{C(Y)},$$

- $\coprod_{\sum \ell_i \leqslant n-k}$ → choose a partition of $n-k$
- $\prod_{i=1}^{k}$ → take all points $P_i \in P$
- $\coprod_{\tau_i \in T(e_i)}$ → choose a total ordering of a partition of ℓ_i
- $\prod_{j=1}^{|\tau_i|}$ → take all Ran spaces for each element in total ordering

with $t_{i,0} = \epsilon$ and $t_{i,j>0}$ the chosen element of $(0, t_{i,j-1})$. The open embedding $Z \times C(Y) \to X$ would be the inclusion on the $C(Y)$ component, and would scale every factor in the Z component to a neighborhood of P_i of radius $t_{i,|\tau_i|}$. However, **this embedding is not continuous**, because a point in $\mathrm{Ran}^k(M)$ is next to a point in $\mathrm{Ran}^n(M)$, where P_i has split off into $n-k$ points, but the radius of $B^M_\epsilon(P_i)$ in $\mathrm{Ran}^k(M)$ is ϵ, while in $\mathrm{Ran}^n(M)$ it is the shortest distance from one of the new points to P_i.

Attempt 5: *Iterate over common radii, but only "antipodal" points.*
This was an attempt to fix the previous attempt and combine it with the naive description. In fact, this approach works when $k = 1$ and $n = 2$. Then $P = \{P_1\}$, and

$$B^M_\epsilon(P_1) \times \left(\mathbf{P}\partial B^M_t(P_1) \times [0,1)\right)/\sim$$

maps into $B^X_\epsilon(P_1)$ by first scaling $[0,1)$ down to $[0,\epsilon - d_M(P,P_1))$, where $P \in B^M_\epsilon(P_1)$ is the chosen point. The object $\mathbf{P}\partial B^M_t(P_1)$ is the projectivization of the boundary of the open $\dim(M)$-ball of radius t around P_1 on M. That is, every element in it is a pair of antipodal points on the boundary of this ball that are exactly $t \in [0, \epsilon - d_M(P,P_1))$ away from P_1.

This works because every pair of points in a contractible neighborhood of P_1 is described uniquely by a pair (P, v), for P the midpoint of the two points and v the $\dim(M)$-vector giving the direction of the points from P (this may rely on working in charts, which is fine, as M is a manifold). However, trying to generalize to more than two points fails because $\ell > 2$ points in general **are not equally distributed on a sphere**. If instead of using the "antipodal" property we take a point from which all ℓ points are equidistant, this point may not be in the ϵ-neighborhood of P_1.

Possible solutions

Solution 1: Instead of a smooth manifold, let M be a simplicial complex. Then $\text{Ran}^{\leqslant n}(M)$ should also be a simplicial complex. Then it may be possible to apply a general theorem to find appropriate cones.

Solution 2: Extend the only partially successful attempt, Attempt 5. Extend by describing a point splitting off into ℓ pieces as a sequence of points splitting into 2 pieces. Or, extend by using the *centroid* of ℓ points instead of the midpoint.

Solution 3: Weaken definition of "conically stratifed" to exclude either open embedding condition or $A_{>f(x)}$ stratification of Y, though this would involve following out Lurie's proof to see what can not be concluded.

References: Lurie (Higher algebra, Appendix A), Ayala, Francis and Tanaka (Local structures on stratified spaces, Sections 2 and 3), Weinberger (The classification of topologically stratified spaces)

5.4 Splitting points in two

2017-11-02

Keywords: *stratification, cone, conical stratification, shape, singular shape, locally singular shape*

The goal of this post is to expand upon some final ideas in a previous post ("Atempts at proving conical stratification," 2017-10-27). Let M be a compact smooth m-manifold embedded in $\mathbf{R}^N$, and fix $n \in \mathbf{Z}_{>0}$. Let $X = \text{Ran}^{\leqslant n}(M)$ and $f : X \to A = \{1, \dots, n\}$ the usual point-counting stratification. Let

$$\begin{aligned} B^X_\epsilon(P) &= \left\{ Q \in X \ : \ 2d_M(P,Q) = \sup_{p\in P}\inf_{q\in Q} d_M(p,q) + \sup_{q\in Q}\inf_{p\in P} d_M(p,q) < 2\epsilon \right\}, \\ B^M_\epsilon(p) &= \{q \in M \ : \ d_M(p,q) < \epsilon\}, \\ B^{\mathbf{R}^m}_\epsilon(0) &= \{x \in \mathbf{R}^m \ : \ d(0,x) < \epsilon\} \end{aligned}$$

be open balls in their respective spaces. We use d_M for distance on M and d for distance in $\mathbf{R}^N$. Since M is an m-manifold, we will work in charts in $\mathbf{R}^m$ when necessary.

Proposition 5.4.1. The stratification $f : X \to A$ is conical in the top two strata $\text{Ran}^n(M)$ and $\text{Ran}^{n-1}(M)$.

Proof: Let $P = \{P_1, \dots, P_n\} \in \text{Ran}^n(M)$ and $2\epsilon = \min_{1\leqslant i<j\leqslant n} d(P_i,P_j)$. Let $Y = \emptyset$ which has a natural $(A_{>n} = \emptyset)$-stratification with $C(Y) = \{*\}$ having a natural $(A_{\geqslant n} = \{n\})$-stratification. Let $Z = B^X_\epsilon(P) = \prod_{i=1}^n B^M_\epsilon(P_i)$, for which the identity map $Z \times \{*\} \cong Z \hookrightarrow X$ is an open embedding. Hence X is stratified at every $P \in \text{Ran}^n(M)$.

Let $P = \{P_1, \ldots, P_{n-1}\} \in \text{Ran}^{n-1}(M)$ and $2\epsilon = \min_{1 \leqslant i < j \leqslant n-1} d(P_i, P_j)$. Let

$$Y = \coprod_{i=1}^{n-1} \mathbf{P}\partial B_{\epsilon/2}^{\mathbf{R}^m}(0), \qquad Z = B_{\epsilon/2}^{\mathbf{R}^m}(0),$$

where $\mathbf{P}\partial B$ is the projectivization of the sphere, so may be viewed as a collection of unique pairs $\{\vec{v}, -\vec{v}\}$. Then the cone $C(Y)$ may be viewed as a collection of pairs $\{\vec{v}, t > 0\}$ along with the singleton $\{0\}$, with the usual cone topology. Define a map

$$\begin{aligned} \varphi \ : \ Z \times C(Y) &\to X, \\ (x, \vec{v}, t) &\mapsto \{x + t\vec{v}, x - t\vec{v}\}, \\ (x, 0) &\mapsto \{x\}. \end{aligned}$$

Note that $B^X_{\epsilon/2}(P) \subseteq \text{im}(\varphi) \subseteq B^X_\epsilon(P)$. This map is injective as every pair of points on M within an $\epsilon/2$-radius of P_i is uniquely defined by their midpoint (the element of Z), a direction from that midpoint (the element of Y) and a distance from that midpoint (the cone component $t \in [0, 1)$). By construction φ is continuous and an embedding. The map takes open sets to open sets, so we have an open embedding into X. Hence X is conically stratified at every $P \in \text{Ran}^{n-1}(M)$. ■

The problem with generalizing this to $P \in \text{Ran}^k(M)$ for all other k is that an $(n - k + 1)$-tuple of points has no unique midpoint. It does have a unique centroid, but it is not clear what the $[0, 1)$ component of the cone should then be.

Proposition 5.4.2. The space X is of locally singular shape.

Proof: First note that every $P \in X$ has an open neighborhood that is homemorphic to an open ball of dimension mn (see Equation (9) of previous post "Attempts at proving conical stratification," 2017-10-27). Hence we may cover X by contractible sets. By Remark A.4.16 of Lurie, X will be of locally singular shape if every element of the cover is of singular shape. Since all elements of the cover are contractible, by Remark A.4.11 of Lurie we only need to check if the topological space $*$ is of singular shape. Finally, Example A.4.12 of Lurie gives that $*$ has singular shape. ■

References: Lurie (Higher algebra, Appendix A)

5.5 The point-counting stratification of the Ran space is conical

2017-11-06

Keywords: *stratification, cone, conical stratification, centroid, Ran space*

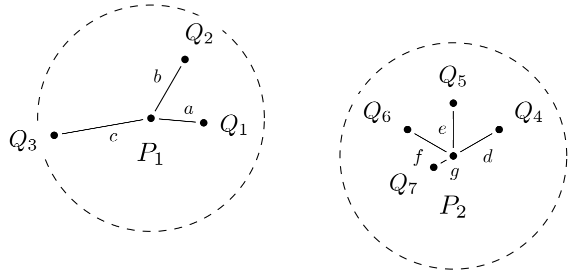
This post completes the effort of several previous posts to show that $f : \text{Ran}^{\leqslant n}(M) \to A = \{1, \ldots, n\}$ is a conically stratified space, where f is the point-counting map, for M a compact smooth m-manifold embedded in $\mathbf{R}^N$.

Remark 5.5.1. Since M is a manifold, we will work on M or through charts in $\mathbf{R}^m$, as necessary, without explicitly mentioning the charts or domains. Balls $B^M_\lambda, B^{\mathbf{R}^m}_\lambda$ of radius λ will be closed and $\mathcal{B}^{\mathbf{R}^m}_\lambda, \mathcal{B}^X_\lambda$ will be open. We write d for distance between points of M (or $\mathbf{R}^m$) and $\mathbf{d}$ for distance between finite subsets of $\mathbf{R}^m$. This is essentially the definition given by Remark 5.5.1.5 of Lurie:

$$\mathbf{d}(P, Q) = \frac{1}{2} \left(\sup_{p \in P} \inf_{q \in Q} d(p, q) + \sup_{q \in Q} \inf_{p \in P} d(p, q) \right).$$

We add the $\frac{1}{2}$ so that $\mathbf{d}(\{p\}, \{q\}) = d(p, q)$. Note also sup, inf may be replaced by max, min in the finite case.

Remark 5.5.2. In our context, given $P \in X$, $\mathbf{d}$ may be thought of as how far away have new points split off from the P_i. That is, if $Q \in X$ is close to P representing the P_i splitting up, then $\mathbf{d}(P, Q)$ is (half) the sum of the distance to the farthest point splitting off from the P_i and to the farthest point among every P_i's closest point. The diagram below gives the idea.



Then the distance between P and Q is given by

$$\begin{aligned}\mathbf{d}(P,Q) &= \frac{1}{2}\left(\sup_{P_i}\left\{\inf_{Q_j}\{d(P_i,Q_j)\}\right\} + \sup_{Q_j}\left\{\inf_{P_i}\{d(P_i,Q_j)\}\right\}\right)\\ &= \frac{1}{2}\left(\sup\{\inf\{a,b,c\},\inf\{d,e,f,g\}\} + \sup\{a,b,c,d,e,f,g\}\right)\\ &= \frac{1}{2}\left(\sup\{a,g\}+c\right)\\ &= \frac{1}{2}(a+c).\end{aligned}$$

Now we move on to the main result.

Proposition 5.5.3. The point-counting stratification $f : X \to A$ is conical.

Proof: Fix $P = \{P_1,\dots,P_k\} \in \mathrm{Ran}^k(M) \subseteq \mathrm{Ran}^{\leqslant n}(M)$ and set $2\epsilon = \min_{i<j} d(P_i,P_j)$. Set

$$Z = \prod_{i=1}^k \mathcal{B}_\epsilon^{\mathbf{R}^m}(0), \qquad Y = \coprod_{\substack{\sum \ell_i = n\\ \sum t_i=\epsilon}} \prod_{i=1}^k \left\{Q \in \mathrm{Ran}^{\ell_i}(B_{t_i}^{\mathbf{R}^m}(0)) \ :\ \mathbf{d}(0,Q)=t_i,\ \sum Q_j = 0\right\},$$

both of which are topological spaces. The first condition on elements of Y is the *cone condition*, which ensures the right topology at the cone point in $C(Y)$. The second condition on Y is the *centroid condition*, which ensures that the point to which 0 maps to (under φ) is the centroid of points splitting off it, so that we don't overcount when multiplying by Z. For $C(Y) = (Y\times[0,1))/(Y\times\{0\})$ the cone of Y, define a map

$$\begin{aligned}\varphi\ :\ C(Y)\times Z &\to X,\\ \left(\mathrm{Ran}^{\ell_i}(B_{t_i}^{\mathbf{R}^m}(0)),t,R\right) &\mapsto \mathrm{Ran}^{\ell_i}(B_{tt_i}^M(R_i)),\end{aligned}$$

where $t\in[0,1)$ is the cone component and $R = \{R_1,\dots,R_k\} \in Z$ is an element of $\mathrm{Ran}^k(M)$ near P. It is sufficient to describe where the Ran^{ℓ_i} map to, as all the Q in a fixed Ran^{ℓ_i} map in the same way into X.

The map φ is continuous by construction, injective by the centroid condition, and a homeomorphism onto its image by the cone condition. Hence φ is an embedding, and since the image is open, it is an open embedding. Note that we are taking "open embedding" to mean an embedding whose image is open. Hence every $P \in X$ satisfies Definition A.5.5 of Lurie, so $f : X \to A$ is conically stratified. ■

Remark 5.5.4. Observe that $\mathcal{B}^X_{\epsilon/k}(P) \subseteq \mathrm{im}(\varphi) \subseteq \mathcal{B}^X_\epsilon(P)$, both inclusions coming from the $\sum t_i = \epsilon$ condition.

Combined with Proposition 5.4.2 of a previous post ("Splitting points in two," 2017-11-02) and Theorem A.9.3 of Lurie, it follows that A-constructible sheaves on X are equivalent to functors of A-exit paths on X to the category $\mathcal{S}$ of spaces. A previously given construction (in "Exit paths, part 2," 2017-09-28) gives such a functor, indicating that there exists an A-constructible sheaf on X.

Next steps may involve applying this approach to the space $\mathrm{Ran}^{\leqslant n}(M)\times\mathbf{R}_{\geqslant 0}$, which was the motivator for all this, or continuing with Lurie's work to see how far this can be taken.

References: Lurie (Higher Algebra, Appendix A), nLab (article "Embedding of topological spaces")

5.6 Towards a sheaf of simplicial complexes

2017-11-26

Keywords: *stratification, simplicial complex, poset*

The goal of this post is to describe a new stratification of $\text{Ran}^n(M)\times\mathbf{R}_{\geqslant 0}$ that builds on the ideas from a previous post (see " The point-counting stratification of the Ran space is conical (really though) ," 2017-11-15) and some newer ones.

Let SC_n be the set of simplicial complexes on n ordered vertices. There is a natural partial order on SC_n given by inclusion of sets, viewing every simplex as a subset of the power set $\mathbf{P}(\{1,\dots,n\})$. The symmetric group S_n has a natural action on SC_n and SC_n/S_n has an induced partial order as well. Hence we have a map

$$\begin{array}{rcl} f \ : \ \text{Ran}^n(M)\times\mathbf{R}_{\geqslant 0} & \to & SC_n/S_n, \\ (P,t) & \mapsto & VR(P,t), \end{array}$$

where $VR(P,t)$ is the Vietoris–Rips complex on P with radius t. We include a k-cell in $VR(P,t)$ at the vertices $\{P_0,\dots,P_k\}\subset P$ if $d(P_i,P_j)<t$ for all $0\leqslant i<j\leqslant k$. Because we have strict inequality, the map is continuous in the upwards-directed, or Alexandrov topology on SC_n/S_n. Indeed, taking the preimage of an open set U_S in SC_n/S_n based at some simplicial complex S (such U_S form the basis of topology on SC_n/S_n), there is an open ball of radius $\min_{i<j} d(P_i,P_j)/2$ in the $\text{Ran}^n(M)$ component and $\min_{(P_i,P_j)\subset f(P,t)} |t-d(P_i,P_j)|$ in the $\mathbf{R}_{\geqslant 0}$ component around any $(P,t)\in f^{-1}(U_S)$.

Remark 5.6.1. The above shows that $\text{Ran}^n(M)\times\mathbf{R}_{\geqslant 0}$ is *poset-stratified* by SC_n/S_n, in the sense of Definition A.5.1 of Lurie. However, the strata are all of the same dimension, so there is no chance of this being a *conical* stratification, in the sense of Definition A.5.5 of Lurie. We hope to fix that with a different stratification.

Definition 5.6.2. Construct a poset $(A,\leqslant_A)$ in the following way:

- $SC_n/S_n\subset A$, with $S\leqslant_A T$ whenever $S\leqslant_{SC_n/S_n} T$,
- for every $S\neq T\in SC_n/S_n$, let $a_{ST}\in A$ with $a_{ST}\leqslant_A S$ and $a_{ST}\leqslant_A T$,
- for every $\{S_1,\dots,S_{k>2}\}\subset SC_n/S_n$, let $a_{S_1\cdots S_k}\in A$ with $a_{S_1\cdots S_k}\leqslant_A a_{S_1\cdots\widehat{S_i}\cdots S_k}$ for all $1\leqslant i\leqslant k$.

Define a map into $(A,\leqslant_A)$ in the following way:

$$\begin{array}{rcl} g \ : \ \text{Ran}^n(M)\times\mathbf{R}_{\geqslant 0} & \to & A, \\ (P,t) & \mapsto & \begin{cases} S, & \text{if } (P,t)\in\text{int}(f^{-1}(S)) \text{ for some } S\in SC_n/S_n, \\ a_{S_1\cdots S_k}, & \text{if } (P,t)\in\text{cl}(f^{-1}(T)) \iff T\in\{S_1,\dots,S_k\}. \end{cases} \end{array}$$

We now claim that g is a stratifying map.

Proposition 5.6.3. The map g is continuous.

Proof: Since $\text{int}(f^{-1}(S))\cap\text{int}(f^{-1}(T))=\emptyset$ for all $S\neq T\in SC_n/S_n$, the open sets $U_S\subseteq A$ based at S all have open preimage $g^{-1}(U_S)\subseteq X$. Now take $(P,t)\in g^{-1}(U_{a_{S_1\cdots S_k}})$, for $k\geqslant 2$. If every open ball around $(P,t)\in X$ intersects $X_{a_{\mathbf{T}}}$, for some $\mathbf{T}\subseteq SC_n/S_n$, then (P,t) must be in the closure of $f^{-1}(T)$, for every $T\in\mathbf{T}$. Hence the only possible such $\mathbf{T}$ are $\mathbf{T}\subseteq\{S_1,\dots,S_k\}$, so $g^{-1}(U_{a_{S_1\cdots S_k}})$ is open in X. ■

The next step would be to show that this stratification is conical, though it is not clear yet if it is.

References: Lurie (Higher Algebra, Appendix A)

5.7 Perspectives on the Ran space

2017-11-29

Keywords: *Ran space, mapping space, compact-open, topology, stratification, coincidence, colimit*

This post combines the finite subset approach with the mapping space approach of the Ran space, in the context of stratifications. The goal is to understand the colimit construction of the Ran space, as that leads to more powerful results.

Topology

Let X, Y be topological spaces.

Definition 5.7.1. The *mapping space* of X with respect to Y is the topological space $X^Y = \{f : Y \to X \text{ continuous}\}$. The topology on X^Y is the *compact-open* topology which has as basis finite intersections of sets

$$\{f \in X^Y \ : \ f(K) \subseteq U\}, \tag{10}$$

for all $K \subseteq Y$ compact and all $U \subseteq X$ open.

Now fix a positive integer n.

Definition 5.7.2. The *Ran space* of X is the space $\text{Ran}^{\leqslant n}(X) = \{P \subseteq X \ : \ 0 < |P| \leqslant n\}$. The topology on $\text{Ran}^{\leqslant n}(X)$ is the coarsest which contains

$$\left\{P \in \text{Ran}^{\leqslant n}(X) \ : \ P \subseteq \bigcup_{i=1}^k U_i, \ P \cap U_i \neq \emptyset \ \forall \ i\right\} \tag{11}$$

as open sets, for all nonempty finite collection of parwise disjoint open sets $\{U_i\}_{i=1}^k$ in X.

From now on, we let I be a set of size n and M be a compact, smooth, connected m-manifold. There is a natural map

$$\begin{array}{rcl} \varphi \ : \ M^I & \to & \text{Ran}^{\leqslant n}(M), \\ (f : I \to M) & \mapsto & f(I). \end{array}$$

This map is surjective, and for $n > 1$, is not injective.

Proposition 5.7.3. The map φ is continuous and an open map.

Proof: For continuity, take an open set $U \subseteq \text{Ran}^{\leqslant n}(M)$ as in (11) and consider $\varphi^{-1}(U)$. We use the fact that $\{*\} \subset I$ is a compact (in fact open and closed) subset of I and that all the U_i are open, as is their union. Observe that

$$\begin{aligned} \varphi^{-1}(U) &= \left\{f \in M^I \ : \ f(I) \subset \bigcup_{i=1}^k U_i, \ f(I) \cap U_i \neq \emptyset \ \forall \ i\right\} \\ &= \left\{f \in M^I \ : \ f(I) \subset \bigcup_{i=1}^k U_i\right\} \cap \bigcap_{i=1}^k \left\{f \in M^I \ : \ f(* \in I) \in U_i\right\}, \end{aligned}$$

which is a finite intersection of sets of the type (10), and so $\varphi^{-1}(U)$ is open in M^I.

For openness, take an open set V as in (10), so $V = \bigcap_{i=1}^k \{f \in M^I \ : \ f(K) \subseteq U_i\}$ for different subsets $K \subseteq I$. By Lemma 5.7.4, we may assume that the U_i are pairwise disjoint. For each U_i, let $\{U_{i,j}\}_{j=1}^\infty$ be a sequence of increasing open sets in U_i such that $U_{i,j} \subseteq U_{i,j+1}$ and $U_{i,j} \xrightarrow{j\to\infty} U_i$. Then

$$\varphi(V) = \underbrace{\left\{P \in M \ : \ P \subset \bigcup_{i=1}^k U_i, \ P \cap U_i \neq \emptyset \ \forall \ i\right\}}_{f \in M^I \text{ with image completely in the } U_i} \cup \underbrace{\bigcap_{i=1}^k \bigcup_{j=1}^\infty \left\{P \in M \ : \ P \subset U_{i,j} \cup \left(\overline{U_{i,j}}\right)^c, \ P \cap U_{i,j} \neq \emptyset, \ P \subset \left(\overline{U_{i,j}}\right)^c \neq \emptyset\right\}}_{f \in M^I \text{ with image partially in the } U_i}.$$

Note that $U_{i,j}$ and $\left(\overline{U_{i,j}}\right)^c$, the complement of the closure of $U_{i,j}$ are both open and disjoint in M. Since infinite unions and finite intersections of elements in the topology are also open, we have that $\varphi(V)$ is open in $\text{Ran}^{\leqslant n}(M)$. ■

The above proposition says that we may talk equivalently about the compact-open topology on M^I and the Ran space topology on $\text{Ran}^{\leqslant n}(M)$. Viewing the Ran space as a function space allows for more general terminology to be applied.

Lemma 5.7.4. Let $U_i \subseteq M$ be open, for $i = 1, \ldots, k$. Then $\bigcap_{i=1}^k \{f \in M^I \ : \ f(K) \subseteq U_i\}$ may be written as a union of intersections $\bigcap_{j=1}^\ell \{f \in M^I \ : \ f(K) \subseteq V_j\}$ with the V_j open, pairwise disjoint, and $\ell \leqslant k$.

Proof: It suffices to prove this in the case $k = 2$. Let $U, V \subseteq M$ open and suppose than $U \cap V \neq \emptyset$. Note that $U \setminus V$ and $V \setminus U$ are separated (that is, $(U \setminus V) \cap \overline{V \setminus U} = \emptyset$ and $(V \setminus U) \cap \overline{U \setminus V} = \emptyset$), and since $\mathbf{R}^N$ is a completely normal space (equivalently, satisfies the $T5$ axiom), there exist disjoint open sets A, B with $U \setminus V \subseteq A$ and $U \setminus V \subseteq B$. So for $A' = A \cap (U \cup V)$ and $B' = B \cap (U \cup V)$, we have

$$\begin{aligned}\{f \in M^I \ : \ f(K) \subseteq U\} &\cap \{f \in M^I \ : \ f(K) \subseteq V\} \\ &= \big(\{f \in M^I \ : \ f(K) \subseteq U \setminus V\} \cap \{f \in M^I \ : \ f(K) \subseteq V \setminus U\}\big) \cup \{f \in M^I \ : \ f(K) \subseteq U \cap V\} \\ &= \big(\{f \in M^I \ : \ f(K) \subseteq A'\} \cap \{f \in M^I \ : \ f(K) \subseteq B'\}\big) \cup \{f \in M^I \ : \ f(K) \subseteq U \cap V\},\end{aligned}$$

for $A', B', U \cap V$ open, and $A' \cap B' = \emptyset$. ■

Note that in the last calculation of the proof, the intersection of sets in the second line is smaller than the intersection of sets in the last line (as $U \setminus V \subsetneq A$ and $V \setminus U \subsetneq B$). However, all the extra ones in the third line appear in the set $\{f \in M^I \ : \ f(K) \subseteq U \cap V\}$.

Stratifications

Now we compare stratifications on M^I and $\mathrm{Ran}^{\leqslant n}(M)$. As before, I is a set of size n.

Corollary 5.7.5. An image-constant A-stratification on M^I is equivalent to an A-stratification on $\mathrm{Ran}^{\leqslant n}(M)$.

This follows from Proposition 5.7.3. By *image-constant* we mean if $\alpha, \beta \in M^I$ have the same image (that is, $\alpha(I) = \beta(I)$), then α, β are sent to the same element of A.

Proof: If we start with a continuous map $f : M^I \to A$, setting $g(P) = f(I \to M)$ whenever $(I \to M) \in \varphi^{-1}(P)$ is continuous, as $\varphi(f^{-1}(U))$ is open, by continuity of f and openness of φ. The assignment $g(P) = f(I \to M)$ whenever $(I \to M) \in \varphi^{-1}(P)$ is well defined, as the stratification is image-constant, so any continuous map from M^I must send every element of $\varphi^{-1}(P)$ to the same place.

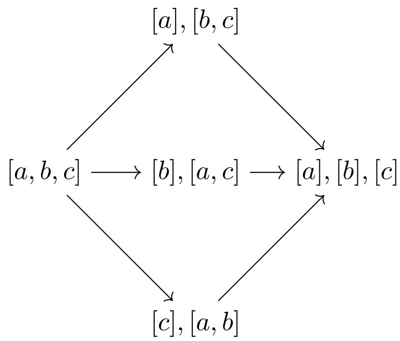
Conversely, if we start with a continuous map $g : \mathrm{Ran}^{\leqslant n}(M) \to A$, setting $f(I \to M) = g(\varphi(I \to M))$ is continuous, as $\varphi^{-1}(g^{-1}(U))$ is open, by continuity of g and continuity of φ. This map is image-constant, as $\varphi(\alpha : I \to M) = \alpha(I)$. ■

Next we consider a particular stratification of M^I, adapted from Example 3.5.17 of Ayala–Francis–Tanaka, simplified with $P = \{*\}$. That is, the example begins with a stratified space $M \to P$ and proceeds to construct another stratification $M^I \to P'$, but we only consider the trivial stratification $M \to \{*\}$.

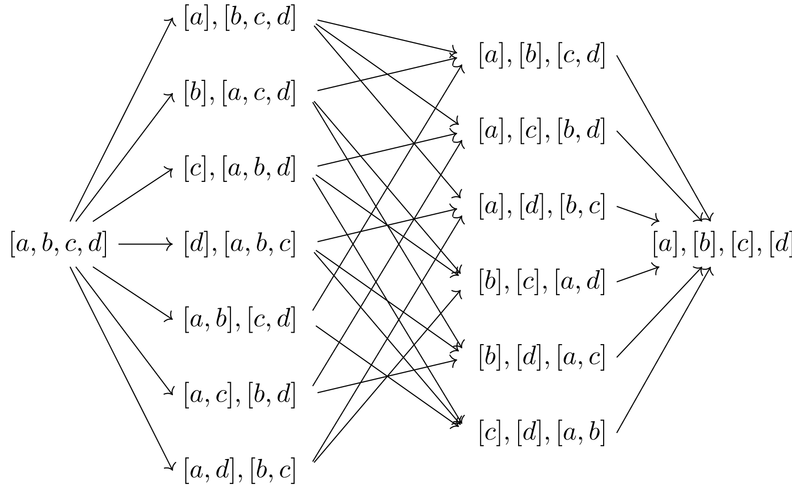
Definition 5.7.6. Given M and I, let the poset $\mathcal{P}(I)$ of *coincidences* on I be the set of equivalence relations on I, ordered by reverse set inclusion. Let $f_I : M^I \to \mathcal{P}(I)$ be the natural stratification that takes a map $\alpha : I \to M$ to the equivalence relation on I describing which elements of I coincide in the image of α.

Example 5.7.7. An element of $\mathcal{P}(I)$ is a subset of $I \times I$ always containing (a, a) for every $a \in I$ (reflexivity), and satisfying the symmetry and transitivity conditions. For example, if $|I| = 3$ or 4, then $\mathcal{P}(I)$ is ordered as in the diagrams below, with order increasing from left to right. We simplify things by writing $[x_1, \ldots, x_k]$ for the collection

(x_i, x_j) of all $i \neq j$ (the equivalence class).



$\mathcal{P}(\{a,b,c\})$



$\mathcal{P}(\{a,b,c,d\})$

To check that the map $f_I : M^I \to \mathcal{P}(I)$ is continuous, we first note that an element $U_{[x_1],\dots,[x_k]}$ in the basis of the upwards-directed topology on $\mathcal{P}(I)$ contains images of $\alpha \in M^I$ whose images have at most the elements of each equivalence class $[x_i]$ coinciding. Hence

$$f_I^{-1}(U_{[x_1],\dots,[x_k]}) = \bigcup_{\substack{U_1,\dots,U_k \subseteq M \\ \text{open, disjoint}}} \bigcap_{i=1}^{k} \left\{\alpha \in M^I \ : \ \alpha(K = \{x \in [x_i]\}) \subseteq U_i\right\},$$

which is an open set in the compact-open topolgy on M^I.

The Ran space as a colimit

Beilinson–Drinfeld (Section 3.4) and Ayala–Francis–Tanaka (Section 3.7) describe the Ran space as a colimit, the former of a functor into topological spaces, the latter of a functor into stratified spaces. See Mac Lane for a full treatment of colimits. Both BD and AFT use the category $\text{Fin}^{surj,\leqslant n}$ of finite sets and surjections, that is,

$$\text{Obj}(\text{Fin}^{surj,\leqslant n}) = \{I \in \text{Obj}(\text{Set}) \ : \ 0 < |I| \leqslant n\},$$
$$\text{Hom}_{\text{Fin}^{surj,\leqslant n}}(I,J) = \begin{cases} \emptyset, & \text{if } |I| < |J|, \\ \{\text{surjections } I \to J\}, & \text{if } |I| \geqslant |J|. \end{cases}$$

AFT uses more involved terminology, with "conically smooth" stratified spaces instead of just poset-stratified. They use a category Strat, which for our purposes we may define as

$$\text{Obj}(\text{Strat}) = \{\text{poset-stratified topological spaces } X \xrightarrow{f} A\},$$
$$\text{Hom}_{\text{Strat}}(X \xrightarrow{f} A, Y \xrightarrow{g} B) = \{(\mu \in \text{Hom}_{\text{Top}}(X,Y), \nu \in \text{Hom}_{\text{Set}}(A,B) \ : \ g \circ \mu = \nu \circ f\}.$$

Remark 5.7.8. There is a natural functor $\mathcal{F}_M : (\text{Fin}^{surj,\leqslant n})^{op} \to \text{Top}$, given by $I \mapsto M^I$. A surjection $s : I \to J$ induces a map $M^J \to M^I$, with $(f : J \to M) \mapsto (f \circ s : I \to M)$. BD use this to declare that $\text{Ran}^{\leqslant n}(M) = \text{colim}(\mathcal{F}_M)$.

Remark 5.7.9. There is also a natural functor $\mathcal{G}_M : (\text{Fin}^{surj,\leqslant n})^{op} \to \text{Strat}$, given by $I \mapsto (M^I \to \mathcal{P}(I))$. AFT use this to declare that $(\text{Ran}^{\leqslant n}(M) \to \{1,\dots,n\}) = \text{colim}(\mathcal{G}_M)$.

The construction of AFT is even more general, as they consider the Ran space of an already stratified space. Here we use their result for $M \to \{*\}$ trivially stratified.

References: Ayala, Francis, and Tanaka (Local structures on stratified spaces, Sections 3.5 and 3.7), Beilinson and Drinfeld (Chiral algebras, Section 3.4), Mac Lane (Categories for the working mathematician, Chapter III.3)

6 The Ran space - sheaves

6.1 A naive constructible sheaf

2017-12-19

Keywords: *sheaf, constructible sheaf, Ran space, direct image, simplicial complex*

In this post we describe a constructible sheaf over $X = \text{Ran}^{\leqslant n}(M) \times \mathbf{R}_{>0}$ valued in simplicial complexes, for a compact, smooth, connected manifold M. We note however that it does not capture all the information about the underlying space. Thanks to Joe Berner for helpful ideas.

Recall the category SC of simplicial complexes and simplicial maps, as well as the full subcategories SC_n of simplicial complexes with n vertices (the vertices are unordered). Let $A = \bigcup_{k=1}^n SC_n$ with the ordering $\leqslant_A$ as in a previous post ("Ordering simplicial complexes with unlabeled vertices," 2017-12-03), and $f : X \to A$ the stratifying map. Let $\{A_k\}_{k=1}^N$ be a cover of X by nested open sets of the type $f^{-1}(U_S) = f^{-1}(\{T \in A \ : \ S \leqslant_A T\})$, whose existence is guaranteed as A is finite. Note that $f(A_1)$ is a singleton containg the complete simplex on n vertices.

Remark 6.1.1. For every simplicial complex $S \in A$, there is a locally constant sheaf over $f^{-1}(S) \subseteq X$. Given the cover $\{A_k\}$ of X, denote this sheaf by $\mathcal{F}_k \in \text{Shv}(A_k \setminus A_{k-1})$ and its value by $S_k \in SC$.

Let $i^1 : A_1 \hookrightarrow A_2$ and $j^2 : A_2 \setminus A_1 \hookrightarrow A_2$ be the natural inclusion maps . Note that A_1 is open and $A_2 \setminus A_1$ is closed in A_2. The maps i^1, j^2 induce direct image functors on the sheaf categories

$$i^1_* : \text{Shv}(A_1) \to \text{Shv}(A_2), \qquad j^2_* : \text{Shv}(A_2 \setminus A_1) \to \text{Shv}(A_2).$$

The induced sheaves in $\text{Shv}(A_2)$ are extended by 0 on the complement of the domain from where they come. Note that since $A_2 \setminus A_1 \subseteq A_2$ is closed, j^2_* is the same as $j^2_!$, the direct image with compact support. We then have the direct sum sheaf $i^1_*\mathcal{F}_1 \oplus j^2_*\mathcal{F}_2 \in \text{Shv}(A_2)$, which we interpret as the disjoint union in SC. Then

$$\left(i^1_*\mathcal{F}_1 \oplus j_2^*\mathcal{F}_2\right)(U) = \begin{cases} S_1 & \text{if } U \subseteq A_1, \\ S_2 & \text{if } U \subseteq A_2 \setminus A_1, \\ S_1 \sqcup S_2 & \text{else,} \end{cases} \qquad \left(i^1_*\mathcal{F}_1 \oplus j_2^*\mathcal{F}_2\right)_{(P,t)} = \begin{cases} S_1 & \text{if } (P,t) \in A_1, \\ S_2 & \text{if } (P,t) \in \text{int}(A_2 \setminus A_1), \\ S_1 \sqcup S_2 & \text{else,} \end{cases}$$

for $U \subseteq A_2$ open and $(P,t) \in A_2$. Generalizing this process, we get a sheaf on X. The diagram

$$\begin{array}{ccccccccccc} A_1 & \xhookrightarrow{i^1} & A_2 & \xhookrightarrow{i^2} & A_3 & \xhookrightarrow{i^3} & A_4 & \xhookrightarrow{i^4} & \cdots & \xhookrightarrow{i^{N-1}} & A_N \\ & \nearrow_{j^2} & & \nearrow_{j^3} & & \nearrow_{j^4} & & & & \nearrow_{j^N} & \\ A_2 \setminus A_1 & & A_3 \setminus A_2 & & A_4 \setminus A_3 & & & & A_N \setminus A_{N-1} & & \end{array}$$

may be helpful to keep in mind. We use the fact that direct sums commute with colimits (used in the definition of the direct image sheaf) to simplify notation. We then get sheaves

$$\begin{aligned} \mathcal{F}^1 &\in \text{Shv}(A_1), \\ i^1_*\mathcal{F}^1 \oplus j^2_*\mathcal{F}^2 &\in \text{Shv}(A_2), \\ i^2_*i^1_*\mathcal{F}^1 \oplus i^2_*j^2_*\mathcal{F}^2 \oplus j^3_*\mathcal{F}^3 &\in \text{Shv}(A_3), \\ i^3_*i^2_*i^1_*\mathcal{F}^1 \oplus i^3_*i^2_*j^2_*\mathcal{F}^2 \oplus i^3_*j^3_*\mathcal{F}^3 \oplus j^4_*\mathcal{F}^4 &\in \text{Shv}(A_4), \end{aligned}$$

and finally

$$i_*^{N-1\cdots 1}\mathcal{F}^1 \oplus \left(\bigoplus_{k=2}^{N-1} i_*^{N-1\cdots k} j^k_*\mathcal{F}^k\right) \oplus j^N_*\mathcal{F}^N \in \text{Shv}(A_N = X),$$

where $i_*^{N-1\cdots k}$ is the composition $i_*^{N-1} \circ i_*^{N-2} \circ \cdots \circ i_*^k$ of direct image functors. Call this last sheaf simply $\mathcal{F} \in \text{Shv}(X)$. Each i^k_* extends the sheaf by 0 on an ever larger domain, so every summand in $\mathcal{F}$ is non-zero on exactly one stratum as defined by $f : X \to A$. We now have a functor $\mathcal{F} : Op(X) \to SC$ defined by

$$\mathcal{F}(U) = \bigsqcup_{k=1}^{N} S_k \delta_{U, A_K \setminus A_{k-1}}, \qquad \mathcal{F}_{(P,t)} = \bigsqcup_{k=1}^{N} S_k \delta_{(P,t), \mathrm{cl}(, A_K \setminus A_{k-1})},$$

where $\delta_{U,V}$ is the Kronecker delta that evaluates to the identity if $U \cap V \neq \emptyset$ and zero otherwise.

Remark 6.1.2. The sheaf $\mathcal{F}$ is A-constructible, as $\mathcal{F}|_{f^{-1}(S)}$ is a constant sheaf evaluating to the simplicial complex $S \in A$. However, if we want the cohomology groups to capture how the simplicial complexes change between strata, then we must use a different approach - all groups die when leaving a stratum because of the extension by zero construction.

References: nLab (article "Simplicial complexes")

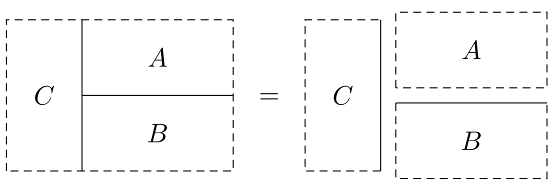
6.2 Artin gluing a sheaf 1: a small example

2018-01-21

Keywords: *Artin gluing, constructible sheaf, direct image, inverse image, pullback, simplicial complex*

The goal of this post is to describe a sheaf on a particular stratified space using locally constant sheaves defined on the strata. Thanks to Joe Berner for helpful discussions.

Recall the direct image and inverse image sheaves from a previous post ("Sheaves, derived and perverse," 2017-12-05). Let M be a smooth, compact, connected manifold, and $X = \mathrm{Ran}^{\leqslant 2}(M) \times \mathbf{R}_{\geqslant 0}$. Let SC be the category of abstract simplicial complexes and simplicial maps. All sheaves will be functors $\mathrm{Op}(-)^{op} \to SC$. The space X looks like the diagram below.



$$\begin{aligned} A &= \{(P,t) \in \mathrm{Ran}^2(M) \times \mathbf{R}_{\geqslant 0} \ : \ t > d(P_1, P_2)\}, \\ B &= \{(P,t) \in \mathrm{Ran}^2(M) \times \mathbf{R}_{\geqslant 0} \ : \ t \leqslant d(P_1, P_2)\}, \\ C &= \mathrm{Ran}^1(M) \times \mathbf{R}_{\geqslant 0}. \end{aligned}$$

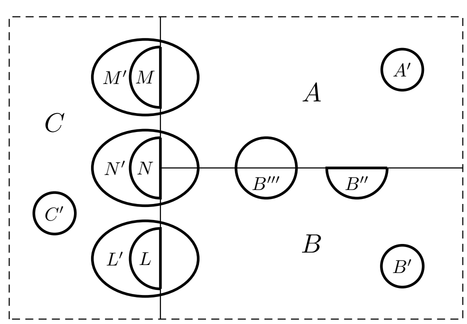
Let $Y = A \cup B$. Note that $A \subseteq Y$ is open, $B \subseteq Y$ is closed, $Y \subseteq X$ is open, and $C \subseteq X$ is closed. There is a natural stratified map $f : X \to \{1,2,3\}$, with $\{1,2,3\}$ given the natural ordering. The map f is described by $f^{-1}(3) = A$, $f^{-1}(2) = B$, and $f^{-1}(1) = C$. Define the inclusion maps

$$\begin{aligned} i &: A \hookrightarrow Y, & k &: Y \hookrightarrow X, \\ j &: B \hookrightarrow Y, & \ell &: C \hookrightarrow X. \end{aligned}$$

Define the following constant sheaves on A, B, C, respectively:

$$\mathcal{F}(U \subseteq A) = \bullet\!\!-\!\!\!-\!\!\bullet, \qquad \mathcal{G}(U \subseteq B) = \bullet \quad \bullet, \qquad \mathcal{H}(U \subseteq C) = \bullet.$$

If $U = \emptyset$, all three give back the simplicial complex on a single vertex $\bullet$. We will now attempt to define a sheaf on all of X by gluing sheaves on the strata. Choose some subsets of X as below on which to test the sheaves.



Step 1: Extend $\mathcal{F}$ and $\mathcal{G}$ to a sheaf on Y.

The direct image of $\mathcal{F}$ via i, as a sheaf on Y, is

$$i_*\mathcal{F}(U) = \mathcal{F}(i^{-1}(U)) = \mathcal{F}(U \cap A) = \begin{cases} \bullet\!\!-\!\!\!-\!\!\bullet & \text{if } U \cap A \neq \emptyset \\ \bullet & \text{else,} \end{cases}$$

for any $U \subseteq Y$. The inverse image of $i_*\mathcal{F}$ via j, as a sheaf on B, is

$$j^*i_*\mathcal{F}(U) = \operatorname*{colim}_{V \supseteq j(U)} [i_*\mathcal{F}(V)] = \operatorname*{colim}_{V \supseteq j(U)} [\mathcal{F}(V \cap A)] = \begin{cases} \bullet\!\!-\!\!\!-\!\!\bullet & \text{if } U \cap \text{cl}(A) \neq \emptyset \\ \bullet & \text{else,} \end{cases}$$

for any $U \subseteq B$. Note $j^*i_*\mathcal{F}(B') = \bullet$ and $j^*i_*\mathcal{F}(B'') = \bullet\!\!-\!\!\!-\!\!\bullet$. The inverse image sheaf is actually defined as the sheafification of the presheaf obtained by taking the colimit, but the sheaf axioms are easily seen to be satisfied here, as the support is on a closed subset.

Following the MathOverflow question, we need to define a map $\mathcal{G} \to j^*i_*\mathcal{F}$ of sheaves on B. Since the support of $j^*i_*\mathcal{F}$ is only $\text{cl}(A) \cap B$, it suffices to define the map here, and we can do it on stalks. There is a natural simplicial map

$$\bullet \quad \bullet \xrightarrow{\varphi} \bullet\!\!-\!\!\!-\!\!\bullet$$

which we use as the sheaf map. It seems we should now have a sheaf on all of Y now, but the result is not immediate. Following the proof of Theorem 3.10 in Chapter 2 of Milne, we need to take the fiber product, or pullback, of $i_*\mathcal{F}$ and $j_*\mathcal{G}$ over $j_*j^*i_*\mathcal{F}$, call it $\mathcal{K}$. Consider the pullback diagram on sets like B''':

$$\begin{array}{ccc} \mathcal{K}(B''') & \longrightarrow & i_*\mathcal{F}(B''') \\ \big\downarrow & & \big\downarrow \text{id} \\ j_*\mathcal{G}(B''') & \xrightarrow{j_*\varphi} & j_*j^*i_*\mathcal{F}(B''') \end{array} \quad = \quad \begin{array}{ccc} \mathcal{K}(B''') & \longrightarrow & \bullet\!\!-\!\!\!-\!\!\bullet \\ \big\downarrow & & \big\downarrow \text{id} \\ \bullet \quad \bullet & \xrightarrow{j_*\varphi} & \bullet\!\!-\!\!\!-\!\!\bullet \end{array}$$

Hence it makes sense to set $\mathcal{K}(B''') = \bullet \quad \bullet$. We now have a sheaf $\mathcal{K}$ on Y given by

$$\mathcal{K}(U \subseteq Y) = \begin{cases} \bullet\!\!-\!\!\!-\!\!\bullet & \text{if } U \subseteq \text{cl}(A), \\ \bullet \quad \bullet & \text{else,} \end{cases} \qquad \mathcal{K}_{x \in Y} = \begin{cases} \bullet\!\!-\!\!\!-\!\!\bullet & \text{if } x \in \text{cl}(A), \\ \bullet \quad \bullet & \text{else.} \end{cases}$$

Step 2: Extend $\mathcal{K}$ and $\mathcal{H}$ to a sheaf on X.

The direct image of $\mathcal{K}$ via k, as a sheaf on X, is

$$k_*\mathcal{K}(U) = \mathcal{K}(k^{-1}(U)) = \mathcal{K}(U \cap Y) = \begin{cases} \bullet\!\!-\!\!\!-\!\!\bullet & \text{if } U \cap Y \subseteq \text{cl}(A) \\ \bullet \quad \bullet & \text{else if } U \cap Y \neq \emptyset, \\ \bullet & \text{else,} \end{cases}$$

for any $U \subseteq X$. The inverse image of $k_*\mathcal{K}$ via ℓ, as a sheaf on C, is

$$\ell^*k_*\mathcal{K}(U) = \operatorname*{colim}_{V \supseteq \ell(U)} [k_*\mathcal{K}(V)] = \operatorname*{colim}_{V \supseteq \ell(U)} [\mathcal{K}(V \cap Y)] = \begin{cases} \bullet \quad \bullet & \text{if } U \cap \text{cl}(B) \neq \emptyset \\ \bullet\!\!-\!\!\!-\!\!\bullet & \text{else if } U \cap \text{cl}(A) \neq \emptyset, \\ \bullet & \text{else,} \end{cases}$$

for any $U \subseteq C$. We need to again define a map $\mathcal{H} \to \ell^*k_*\mathcal{K}$ of sheaves on C. On stalks we naturally have maps

$$\bullet \xrightarrow{\varphi} \bullet \quad \bullet, \qquad \text{and} \qquad \bullet \xrightarrow{\psi} \bullet\!\!-\!\!\!-\!\!\bullet,$$

due to the fact that both complexes are symmetric, so sending to one or the other vertex is the same. Let $\mathcal{L}$ be the sheaf we should now have defined over all of X, by taking the fiber product of $\ell_*\mathcal{H}$ and $k_*\mathcal{K}$ over $\ell_*\ell^*k_*\mathcal{K}$. Let us consider its pullback diagrams for the sets L', M', N'.

$$
\begin{array}{ccc}
\mathcal{L}(L') & \longrightarrow & \bullet \;\; \bullet \\
\downarrow & & \downarrow \text{id} \\
\bullet & \xrightarrow{\ell_*\varphi} & \bullet \;\; \bullet
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{L}(M') & \longrightarrow & \bullet\!\!-\!\!\bullet \\
\downarrow & & \downarrow \text{id} \\
\bullet & \xrightarrow{\ell_*\psi} & \bullet\!\!-\!\!\bullet
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{L}(N') & \longrightarrow & \bullet \;\; \bullet \\
\downarrow & & \downarrow \text{id} \\
\bullet & \xrightarrow{\ell_*\varphi} & \bullet \;\; \bullet
\end{array}
$$

It seems that we should set $\mathcal{L}(L') = \mathcal{L}(M') = \mathcal{L}(N') = \bullet$. We now have a sheaf $\mathcal{L}$ on X given by

$$
\mathcal{L}(U \subseteq X) = \begin{cases} \bullet\!\!-\!\!\bullet & \text{if } U \subseteq \text{cl}(A), \\ \bullet \;\; \bullet & \text{else if } U \subseteq \text{cl}(Y), \\ \bullet & \text{else,} \end{cases} \qquad \mathcal{L}_{x\in X} = \begin{cases} \bullet\!\!-\!\!\bullet & \text{if } x \in \text{cl}(A), \\ \bullet \;\; \bullet & \text{else if } x \in \text{cl}(B), \\ \bullet & \text{else.} \end{cases}
$$

The next goal is to extend this approach to $\text{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$. An immediate difficulty seems to be finding canonical simplicial maps like φ and ψ, but hopefully a choice of increasing nested open cover of the startifying set of X will solve this problem.

References: MathOverflow (Question 54037), Milne (Étale cohomology, Chapter 2.3)

6.3 Artin gluing a sheaf 2: simplicial sets and configuration spaces

2018-01-31

Keywords: *constructible sheaf, simplicial set, pullback, fiber product, Artin gluing, direct image, inverse image, configuration space*

The goal of this post is to extend the previous stratifying map to simplicial sets, and to generalize the sheaf construction to $X = \text{Conf}_n(M) \times \mathbf{R}_{\geqslant 0}$ for arbitrary integers n, where M is a smooth, compact, connected manifold. We work with $\text{Conf}_n(M)$ instead of $\text{Ran}^{\leqslant n}(M)$ because Lemma 6.3.1 and Proposition 6.3.4 have no chance of extending to $\text{Ran}^{\leqslant n}(M)$ without major modifications (see Remark 6.3.5 at the end of this post).

Recall SC is the category of simplicial complexes and simplicial maps, with SC_n the full subcategory of simplicial complexes on n vertices. Our main function is

$$
\begin{array}{rcccl}
f \;:\; X & \xrightarrow{f_1} & SC & \xrightarrow{f_2} & \text{sSet}, \\
(P, a) & \mapsto & VR(P, a) & \mapsto & \text{Hom}_{\text{Set}}(\Delta^\bullet, VR(P, a)).
\end{array}
$$

On $\text{Conf}_n(M)$ we have a natural metric, the Hausdorff distance $d_H(P, Q) = \max_{p\in P} \min_{q\in Q} d(p, q) + \max_{q\in Q} \min_{p\in P} d(p, q)$. This induces the 1-product metric on X, as

$$
d_X((P, a), (Q, b)) = d_H(P, Q) + d(a, b),
$$

where d without a subscript is Euclidean distance. We could have chosen any other p-product metric, but $p = 1$ makes computations easier. For a given $(P, t) \in X$, write $P = \{P_1, \ldots, P_n\}$ and define its *maximal neighborhood* to be the ball $B_X(\min\{\delta_1, \delta_2, t\}, P)$, where

$$
\delta_1 = \min_{i<j}\{d(P_i, P_j)\}, \qquad \delta_2 = \min_{i<j}\{|d(P_i, P_j) - t| \;:\; d(P_i, P_j) \neq t\}.
$$

Lemma 6.3.1. Any path $\gamma : I \to X$ induces a unique morphism $f(\gamma(0)) \to f(\gamma(1))$ of simplicial sets.

Proof: Write $\gamma(0) = \{P_1, \ldots, P_n\}$ and $\gamma(1) = \{Q_1, \ldots, Q_n\}$. The map γ induces n paths $\gamma_i : I \to M$ for $i = 1, \ldots, n$, with γ_i the path based at P_i. Let $s : \gamma(0) \to \gamma(1)$ be the map on simplicial complexes defined by $P_i \mapsto \gamma_i(1)$. Since we are in the configuration space, where points cannot collide (as opposed to the Ran space), this is a well-defined map. Then $f_2(s)$ is a morphism of simplicial complexes. ■

Note the morphism of simplicial sets induced by any path in a maximal neighborhood of $x \in X$ is the identity morphism. We now move to describing a sheaf over all of X.

Definition 6.3.2. Let X be any topological space and $\mathcal{C}$ a category with pullbacks. Let $A \subseteq X$ open and $B = X \setminus A \subseteq X$ closed, with $i : A \hookrightarrow X$ and $j : B \hookrightarrow X$ the inclusion maps. Let $\mathcal{F}$ be a $\mathcal{C}$-valued sheaf on A and $\mathcal{G}$ a $\mathcal{C}$-valued sheaf on B. Then the *Artin gluing* of $\mathcal{F}$ and $\mathcal{G}$ is the $\mathcal{C}$-valued sheaf $\mathcal{H}$ on X defined as the pullback, or fiber product, of $i_*\mathcal{F}$ and $j_*\mathcal{G}$ over $j_*j^*i_*\mathcal{F}$ in the diagram below.

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & i_*\mathcal{F} \\ \downarrow & & \downarrow \text{restriction} \\ j_*\mathcal{G} & \xrightarrow{\;j_*\varphi\;} & j_*j^*i_*\mathcal{F} \end{array}$$

Note the definition requires a choice of sheaf map $\varphi : \mathcal{G} \to j^*i_*\mathcal{F}$. In the proof below, this sheaf map will be the morphism of simplicial sets from Lemma 6.3.1 through the functor $\mathrm{Hom}_{\mathrm{Set}}(\Delta^\bullet, -) = f_2(-)$.

Recall the ordering of SC_n described by Definition 5.1.3 in a previous post ("Exit paths, part 2," 2017-09-28). Fix a cover $\{A_i\}_{i=1}^N$ of SC_n by nested open subsets (so $N = |SC_n|$), with $B_i := f_1^{-1}(A_i)$ and $B_{\leqslant i} := \bigcup_{j=1}^i B_i$. We now have an induced order on and cover of $\mathrm{im}(f) = \mathrm{sSet}'$, as a full subcategory of sSet. Even more, we now have an induced total order on $\mathrm{sSet}' = \{S_1, \ldots, S_N\}$, with S_i the unique simplicial set in $A_i \setminus A_{i-1}$. For example, $S_1 = \mathrm{Hom}_{\mathrm{Set}}(\Delta^\bullet, \Delta^n)$ and $S_N = \mathrm{Hom}_{\mathrm{Set}}(\Delta^\bullet, \bigcup_{i=1}^n \Delta^0)$.

For ease of notation, we let $B_0 = \emptyset$ and write $S_\emptyset = \mathrm{Hom}(\Delta^\bullet, \emptyset)$, $S_0 = \mathrm{Hom}(\Delta^\bullet, \Delta^0)$.

Definition 6.3.3. Let $\mathcal{F}_i : \mathrm{Op}(B_i)^{op} \to \mathrm{sSet}$ be the locally constant sheaf given by $\mathcal{F}_i(U_x) = S_i$, where U_x is a subset of the maximal neighborhood of $x \in B_i$. In general,

$$\mathcal{F}_i(U) = \begin{cases} S_i & \text{if } \;U \neq \emptyset, \\ & \quad\; U \text{ is path connected,} \\ & \quad\; \text{every loop } \gamma : I \to U \text{ induces id} : f(\gamma(0)) \to f(\gamma(1)), \\ S_\emptyset & \text{else if } U \neq \emptyset, \\ S_0 & \text{else.} \end{cases}$$

In general, we say $U \subseteq X$ is *good* if it is non-empty, path connected, and every loop $\gamma : I \to U$ induces the identity morphism on simplicial sets.

Proposition 6.3.4. Let $\mathcal{F}_{\leqslant 1} = \mathcal{F}_1$, and $\mathcal{F}_{\leqslant i}$ be the sheaf on $B_{\leqslant i}$ obtained by Artin gluing $\mathcal{F}_i$ onto $\mathcal{F}_{\leqslant i-1}$, for all $i = 2, \ldots, N$. Then $\mathcal{F} = \mathcal{F}_{\leqslant N}$ is the SC_n-constructible sheaf on X described by

$$\mathcal{F}(U) = \begin{cases} S_{\max\{1 \leqslant \ell \leqslant N \,:\, U \cap B_\ell \neq \emptyset\}} & \text{if } U \text{ is good,} \\ S_\emptyset & \text{else if } U \neq \emptyset, \\ S_0 & \text{else.} \end{cases} \tag{12}$$

Proof: We proceed by induction. Begin with the constant sheaf $\mathcal{F}_1$ on B_1 and $\mathcal{F}_2$ on B_2, which we would like to glue together to get a sheaf $\mathcal{F}_{\leqslant 2}$ on $B_{\leqslant 2}$. Since f_1 is continuous in the Alexandrov topology on the poset $SC_{\leqslant n}$, $B_1 \subseteq B_{\leqslant 2}$ is open and $B_2 \subseteq B_{\leqslant 2}$ is closed. Let $i : B_1 \hookrightarrow B_{\leqslant 2}$ and $j : B_2 \hookrightarrow B_{\leqslant 2}$ be the inclusion maps. The sheaf $j^*i_*\mathcal{F}_1$ has support $\mathrm{cl}(B_1) \cap B_2 \neq \emptyset$ with

$$j^*i_*\mathcal{F}_1(U) = \operatorname*{colim}_{V \supseteq j(U)} [i_*\mathcal{F}_1(V)] = \operatorname*{colim}_{V \supseteq U} [\mathcal{F}_1(V \cap B_1)] = \begin{cases} S_1 & \text{if } U \cap \mathrm{cl}(B_1) \text{ is good,} \\ S_\emptyset & \text{else,} \end{cases}$$

for any non-empty $U \subseteq B_2$. Let the sheaf map $\varphi : \mathcal{F}_2 \to j^*i_*\mathcal{F}_1$ be the inclusion simplicial set morphism on good sets (it can be thought of as induced through Lemma 6.3.1 by a path starting in $U \cap B_2$ and ending in $V \cap B_1$, for V a small enough set in the colimit above). Note that $S_2 = \mathrm{Hom}_{\mathrm{Set}}(\Delta^\bullet, \Delta^n \setminus \Delta^1)$, where $\Delta^n \setminus \Delta^1$ is the simplicial complex resulting from removing an edge from the complete simplicial complex on n vertices. Let $\mathcal{F}_{\leqslant 2}$ be the pullback of $i_*\mathcal{F}_1$ and $j_*\mathcal{F}_2$ along $j_*j^*i_*\mathcal{F}_1$, and $U \subseteq B_{\leqslant 2}$ a good set. If $U \subseteq B_1$, then $\mathcal{F}_{\leqslant 2}(U) = \mathcal{F}_1(U) = S_1$, and if $U \subseteq B_2$, then $\mathcal{F}_{\leqslant 2}(U) = \mathcal{F}_2(U) = S_2$. Now suppose that $U \cap B_1 \neq \emptyset$ but also $U \cap B_2 \neq \emptyset$, which, since U is good, implies that $U \cap \mathrm{cl}(B_1) \cap B_2 \neq \emptyset$. Then we have the pullback square

$$\begin{array}{ccc} & & S_1 \\ & & \| \\ \mathcal{F}_{\leqslant 2}(U) & \longrightarrow & i_*\mathcal{F}_1(U) \\ \downarrow & & \downarrow \text{restriction} \\ j_*\mathcal{F}_2(U) & \xrightarrow{j_*\varphi} & j_*j^*i_*\mathcal{F}_1(U) \\ \| & & \| \\ S_2 & & S_1. \end{array}$$

If U is not good, then the simplicial sets are $S_\emptyset$ or S_0, with nothing interesting going on. The pullback over a good set U can be computed levelwise as

$$\mathcal{F}_{\leqslant 2}(U)_m = \{(\alpha, \beta) \in (S_1)_m \times (S_2)_m \ : \ \alpha = j_*\varphi(\beta)\}. \tag{13}$$

Since $j_*\varphi$ is induced by the inclusion φ, it is the identity on its image. So $\alpha = j_*\varphi(\beta)$ means $\alpha = \beta$, or in other words, $\mathcal{F}_{\leqslant 2}(U) = S_2$. Hence for arbitrary $U \subseteq B_{\leqslant 2}$, we have

$$\mathcal{F}_{\leqslant 2}(U) = \begin{cases} S_{\max\{\ell=1,2 \ : \ U \cap B_\ell \neq \emptyset\}} & \text{if } U \text{ is good,} \\ S_\emptyset & \text{else if } U \neq \emptyset, \\ S_0 & \text{else.} \end{cases}$$

For the inductive step with $k > 1$, let $\mathcal{F}_{\leqslant k}$ be the sheaf on $B_{\leqslant k}$ defined as in (12), but with k instead of N. We would like to glue $\mathcal{F}_{\leqslant k}$ to $\mathcal{F}_{k+1}$ on B_{k+1} to get a sheaf $\mathcal{F}_{\leqslant k+1}$ on $B_{\leqslant k+1}$. As before, $B_k \subseteq B_{\leqslant k+1}$ is open and $B_{k+1} \subseteq B_{\leqslant k+1}$ is closed. For $i : B_k \hookrightarrow B_{\leqslant k+1}$ and $j : B_{k+1} \hookrightarrow B_{\leqslant k+1}$ the inclusion maps, the sheaf $j^*i_*\mathcal{F}_{\leqslant k}$ has support $\mathrm{cl}(B_{\leqslant k}) \cap B_{k+1}$, with

$$j^*i_*\mathcal{F}_{\leqslant k}(U) = \operatorname*{colim}_{V \supseteq j(U)} [i_*\mathcal{F}_{\leqslant k}(V)] = \operatorname*{colim}_{V \supseteq U} [\mathcal{F}_{\leqslant k}(V \cap B_{\leqslant k})] = \begin{cases} S_{\max\{1 \leqslant \ell \leqslant k \ : \ U \cap \mathrm{cl}(B_\ell) \neq \emptyset\}} & \text{if } U \cap \mathrm{cl}(B_{\leqslant k}) \text{ is good,} \\ S_\emptyset & \text{else,} \end{cases}$$

for any non-empty $U \subseteq B_{k+1}$. Let the sheaf map $\varphi : \mathcal{F}_{k+1} \to j^*i_*\mathcal{F}_{\leqslant k}$ be the inclusion simplicial set morphism on good sets (it can be thought of as induced through Lemma 6.3.1 by a path starting in $U \cap B_{k+1}$ and ending in $V \cap B_{\leqslant k}$, for V a small enough set in the colimit above). For $U \subseteq B_{\leqslant k+1}$ a good set, if $U \subseteq B_{\leqslant k}$, then $\mathcal{F}_{\leqslant k+1}(U) = \mathcal{F}_{\leqslant k}(U)$, and if $U \subseteq B_{k+1}$, then $\mathcal{F}_{\leqslant k+1}(U) = \mathcal{F}_{k+1}(U) = S_{k+1}$. Now suppose that $U \cap B_{\leqslant k} \neq \emptyset$ but also $U \cap B_{k+1} \neq \emptyset$, which, since U is good, implies that $U \cap \mathrm{cl}(B_{\leqslant k}) \cap B_{k+1} \neq \emptyset$. Then we have the pullback square

$$\begin{array}{ccc} & & S_{\max\{1 \leqslant \ell \leqslant k \ : \ U \cap B_\ell \neq \emptyset\}} = S_k \\ & & \| \\ \mathcal{F}_{\leqslant k+1}(U) & \longrightarrow & i_*\mathcal{F}_{\leqslant k}(U) \\ \downarrow & & \downarrow \text{restriction} \\ j_*\mathcal{F}_{k+1}(U) & \xrightarrow{j_*\varphi} & j_*j^*i_*\mathcal{F}_{\leqslant k}(U) \\ \| & & \| \\ S_{k+1} & & S_{\max\{1 \leqslant \ell \leqslant k \ : \ U \cap \mathrm{cl}(B_\ell) \neq \emptyset\}} = S_k. \end{array}$$

If U is not good, then the simplicial sets are $S_\emptyset$ or S_0, with nothing interesting going on. Again, as in (13), the pullback $\mathcal{F}_{\leqslant k+1}$ on a good set U is

$$\mathcal{F}_{\leqslant k+1}(U)_m = \{(\alpha, \beta) \in (S_\ell)_m \times (S_{k+1})_m \ : \ \alpha = j_*\varphi(\beta)\},$$

and as before, this implies that $\mathcal{F}_{\leqslant k+1}(U) = S_{k+1}$. Hence $\mathcal{F}_{\leqslant k+1}$ is exactly of the form (12), with $k+1$ instead of N, and by induction we get the desired description for $\mathcal{F}_{\leqslant N} = \mathcal{F}$. ■

Remark 6.3.5. The statements given in this post do not extend to $\mathrm{Ran}^{\leqslant n}(M)$, at least not as stated. Lemma 6.3.1 fails if somewhere along the path γ a point splits in two or more points, as there is no canonical choice which of the "new" points should be the image of the "old" point. This means that the proof of Proposition 6.3.4 will also fail, because we relied on a uniquely defined sheaf map φ between strata.

Next, we hope to use this approach to describe classic persistent homology results, and maybe link this to the concept of *persistence modules*.

References: Milne (Étale cohomology, Chapter 2.3)

6.4 Artin gluing a sheaf 3: the Ran space

2018-02-05

Keywords: *Ran space, constructible sheaf, Artin gluing, symmetric group*

The goal of this post is to extend earlier ideas, of a sheaf defined on $\mathrm{Conf}_n(M) \times \mathbf{R}_{\geqslant 0}$, to a family of sheaves defined on $\bigcup_{k=1}^n \mathrm{Conf}_n(M) \times \mathbf{R}_{\geqslant 0} = \mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$.

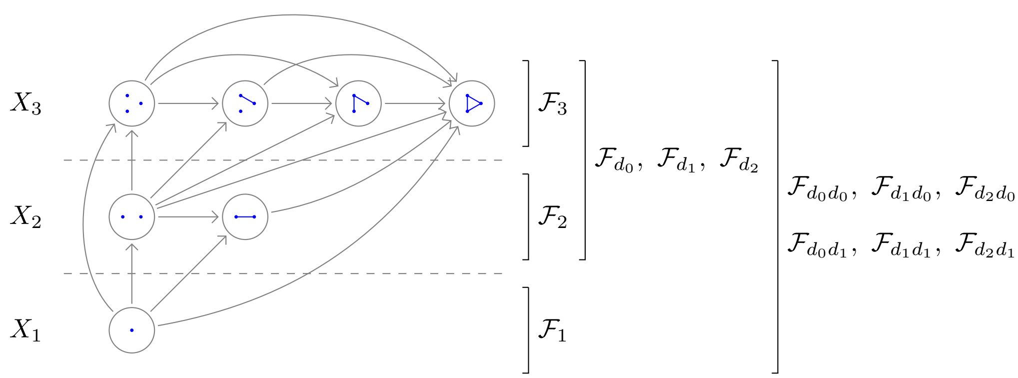
Recall our main map $f : \mathrm{Conf}_n(M) \times \mathbf{R}_{\geqslant 0} \xrightarrow{VR(-)} SC_n \xrightarrow{\mathrm{Hom}(\Delta^\bullet,-)} \mathrm{sSet}$. Following Definition 6.3.3 and Proposition 6.3.4 in a previous post ("Artin gluing a sheaf 2: simplicial sets and configuration spaces," 2018-01-31), define a sheaf $\mathcal{F}_k$ on X_k by

$$\mathcal{F}_k(U) = \begin{cases} S_{k,\max\{1\leqslant \ell \leqslant N_k \;:\; U\cap B_\ell \neq \emptyset\}} & \text{if } U \text{ is good,} \\ S_\emptyset & \text{else if } U \neq \emptyset, \end{cases} \tag{14}$$

for all $k = 1, \ldots, n$. We have assumed a total order on all simplicial complexes on k vertices, induced by a cover $U_k, \ldots, U_{k,N_k}$ of nested opens of X_k. This induces a total order $S_{k,1}, \ldots, S_{k,N_k}$ on the image of $\mathrm{Ran}^k(M) \times \mathbf{R}_{\geqslant 0}$ in sSet, and by the product order, a total order on all of $\mathrm{sSet}' := f(\mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0})$.

A small example

Let $n = 3$, so $X = \mathrm{Ran}^{\leqslant 3}(M) \times \mathbf{R}_{\geqslant 0}$. We already have $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ on X_1, X_2, X_3, respectively, and we will extend them from the top down to sheaves over all of X, as in the diagram below.



The map i will be the inclusion of an open set into a larger one, and j the inclusion of a closed set into a larger one. Recall that the pullback of two sheaves is defined equivalently by a map of sheaves on the boundary of the open nd closed sets. With that in mind, for $U \subseteq X_2 \cup X_3$ good, the pullback square

$$\begin{array}{ccc} \mathcal{F}_{d_0}(U) & \longrightarrow & i_*\mathcal{F}_3(U) = S_{3,\max\{\ell=1,2,3,4 \;:\; U\cap U_{3,\ell}\neq\emptyset\}} \\ \big\downarrow & & \big\downarrow \text{restriction} \\ S_{2,\max\{\ell=1,2 \;:\; U\cap U_{2,\ell}\neq\emptyset\}} = j_*\mathcal{F}_2(U) & \xrightarrow{d_0} & j_*j^*i_*\mathcal{F}_3(U) = S_{3,\max\{\ell=1,2,3,4 \;:\; U\cap U_{3,\ell}\neq\emptyset\}} \end{array}$$

defines $\mathcal{F}_{d_0}$, where the d_0 indicates the face map that skips the 0th spot. The sheaf $\mathcal{F}_{d_1}$ is defined similarly, but by the face map d_1, and $\mathcal{F}_{d_2}$ by the face map d_2. For each of these three sheaves on $X_3 \cup X_2$, we have two other sheaves, based on where the single point maps to. However, we note that for $U \subseteq X$ good and $U \cap X_1 \neq \emptyset$,

$$((i_*\mathcal{F}_{d_0} \times j_*\mathcal{F}_1)(U) \text{ defined by } d_0) \quad = \quad ((i_*\mathcal{F}_{d_1} \times j_*\mathcal{F}_1)(U) \text{ defined by } d_0),$$

where $\times$ denotes the pullback over the appropriate sheaf, and similarly for the other sheaves on good sets intersecting X_1. We now have 6 unique shaves on all of X.

Generalizing

Now let n be any positive integer, and $X = \mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$. We reverse the indexation of the $\mathcal{F}_k$ and X_k above to make notation less cumbersome (so now $\mathcal{F}_k$ is $\mathcal{F}_{n-k+1}$ from (14), over $X_k = \mathrm{Ran}^{n-k+1}(M) \times \mathbf{R}_{\geqslant 0}$). Define pullback sheaves $\mathcal{F}_{d_{\ell_1}}$ for $\ell_1 = 0, \ldots, n$ on $X_2 \cup X_2$ by the diagram

$$\begin{array}{ccc} \mathcal{F}_{d_{\ell_1}} & \longrightarrow & i_*\mathcal{F}_1 \\ \downarrow & & \downarrow \text{restriction} \\ j_*\mathcal{F}_2 & \xrightarrow{d_{\ell_1}} & j_*j^*i_*\mathcal{F}_1. \end{array}$$

At the kth step, for $1 < k < n$, we have sheaves $\mathcal{F}_{d_{\ell_1}\cdots d_{\ell_{k-1}}}$ over $\bigcup_{m=1}^{k} X_m$, defined by sequences of face maps $d_{\ell_{k-1}}$ when going from X_k to X_{k-1} and so on, where $\ell_m \in \{0, \ldots, n-m+1\}$. Define pullback sheaves $\mathcal{F}_{d_{\ell_1}\cdots d_{\ell_{k-1}} d_{\ell_k}}$, for $\ell_k = 0, \ldots, n-k+1$ on $\bigcup_{m=1}^{k+1} X_k$ by the diagram

$$\begin{array}{ccc} \mathcal{F}_{d_{\ell_1}\cdots d_{\ell_k}} & \longrightarrow & i_*\mathcal{F}_{d_{\ell_1}\cdots d_{\ell_{k-1}}} \\ \downarrow & & \downarrow \text{restriction} \\ j_*\mathcal{F}_{k+1} & \xrightarrow{d_{\ell_k}} & j_*j^*i_*\mathcal{F}_{d_{\ell_1}\cdots d_{\ell_{k-1}}}. \end{array}$$

At the end of this inductive process, we have $n!$ distinct sheaves $\mathcal{F}_{d_{\ell_1}\cdots d_{\ell_{n-1}}}$ on all of X. Note there is a sheaf map $\mathcal{F}_{d_{\ell_1}\cdots d_{\ell_i}\cdots d_{\ell_{n-1}}} \to \mathcal{F}_{d_{\ell_1}\cdots d_{\ell_i'}\cdots d_{\ell_{n-1}}}$, given on U good by

$$\mathcal{F}_{d_{\ell_1}\cdots d_{\ell_i}\cdots d_{\ell_{n-1}}}(U) = S \mapsto \begin{cases} S & \text{if } |S_0| \leqslant n-i, \\ (\ell_i \ \ell_i')(S) & \text{else,} \end{cases}$$

where $(\ell_i \ \ell_i') \in \mathfrak{S}_n$ (the symmetric group on the numbers $0, \ldots, n-1$) is the transposition swaps the ℓ_i and ℓ_i' indices of S_0, the 0-cells of S, inducing a map of simplicial sets. If the two sheaves differ in only two indices $\ell_i \neq \ell_i'$ and $\ell_j \neq \ell_j'$, with $i < j$, then we get $S \mapsto (\ell_j \ \ell_j')_{d_{\ell_{i-1}}\cdots d_{\ell_j}}(\ell_i \ \ell_i')(S)$. Here $(\ell_j \ \ell_j')_{d_{\ell_{i-1}}\cdots d_{\ell_j}}$ is the element of $\mathfrak{S}_{n-i}$ found by taking $(\ell_j \ \ell_j')$ from $\mathfrak{S}_{n-j}$ to $\mathfrak{S}_{n-i}$ by the sequence of group inclusion maps induced by the face maps $d_{\ell_j}, \ldots, d_{\ell_{i-1}}$.

Remark 6.4.1. This construction is not the most satisfying for several reasons:

- we do not have a single sheaf, rather a family of sheaves, and
- the use of "good" sets leaves something to be desired, as we should be able to consider larger sets.

Both will hopefully be remedied in a later post.

6.5 Artin gluing a sheaf 4: a single sheaf in two ways

2018-02-10

Keywords: *Ran space, constructible sheaf, simplicial complex, simplicial set, ordering, product order, colimit*

The goal of this post is to give an alternative perspective on making a sheaf over $X = \mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$, alternative to that of a previous post ("Artin gluing a sheaf 3: the Ran space," 2018-02-05). We will have one unique sheaf on all of X, valued either in simplicial complexes or simplicial sets.

Remark 6.5.1. Here we straddle the geometric category SC of simplicial complexes and the algebraic category sSet of simplicial sets. There is a functor $[\,\cdot\,] : SC \to \text{sSet}$ for which every n-simplex in S gets $(n+1)!$ elements in $[S]$, representing all the ways of ordering the vertices of S (which we would like to view as unordered, to begin with).

Recall from previous posts:

- maps $f : X \to SC$ and $g = [f] : X \to \text{sSet}$,
- the SC_k-stratification of $\text{Ran}^k(M) \times \mathbf{R}_{\geqslant 0}$,
- the point-counting stratification of $\text{Ran}^{\leqslant n}(M)$,
- the combined (via the product order) $SC_{\leqslant n}$-stratification of $\text{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$,
- an induced (by the SC_k-stratification) cover by nested open sets $B_{k,1}, \dots, B_{k,N_k}$ of $\text{Ran}^k(M) \times \mathbf{R}_{\geqslant 0}$,
- a corresponding induced total order $S_{k,1}, \dots, S_{k,N_k}$ on $f(\text{Ran}^k(M) \times \mathbf{R}_{\geqslant 0})$.

The product order also induces a cover by nested opens of all of X and a total order on $f(X)$ and $g(X)$. We call a path $\gamma : I \to X$ a *descending path* if $t_1 < t_2 \in I$ implies $h(\gamma(t_1)) \geqslant h(\gamma(t_2))$ in any stratified space $h : X \to A$. Below, h is either f or g.

Lemma 6.5.2. A descending path $\gamma : I \to X$ induces a unique morphism $h(\gamma(0)) \to h(\gamma(1))$.

Proof: Write $\gamma(0) = \{P_1, \dots, P_n\}$ and $\gamma(1) = \{Q_1, \dots, Q_m\}$, with $m \leqslant n$. Since the path is descending, points can only collide, not split. Hence γ induces n paths $\gamma_i : I \to M$ for $i = 1, \dots, n$, with γ_i the path based at P_i. This induces a map $h(\gamma(0))_0 \to h(\gamma(1))_0$ on 0-cells (vertices or 0-objects), which completely defines a map $h(\gamma(0)) \to h(\gamma(1))$ in the desired category. ∎

Our sheaves will be defined using colimits. Fortunately, both SC and sSet have (small) colimits. Finally, we also need an auxiliary function $\sigma : \text{Op}(X) \to SC$ that finds the correct simplicial complex. Define it by

$$\sigma(U) = \begin{cases} S_{k,\ell} & \text{if } U \neq \emptyset,\ \text{for } k = \max\{1 \leqslant k' \leqslant n \ :\ U \cap \text{Ran}^k(M) \times \mathbf{R}_{\geqslant 0} \neq \emptyset\}, \\ & \qquad\qquad\qquad \ell = \max\{1 \leqslant \ell' \leqslant N_k \ :\ U \cap B_{k,\ell'} \neq \emptyset\}, \\ * & \text{if } U = \emptyset. \end{cases}$$

Proposition 6.5.3. Let $\mathcal{F}$ be the function $\text{Op}(X)^{op} \to SC$ on objects given by

$$\mathcal{F}(U) = \text{colim}\left(\sigma(U) \rightrightarrows S \ :\ \text{every } \sigma(U) \to S \text{ is induced by a descending } \gamma : I \to U\right).$$

This is a functor and satisfies the sheaf gluing conditions.

Proof: We have a well-defined function, so we have to describe the restriction maps and show gluing works. Since $V \subseteq U \subseteq X$, every S in the directed system defining $\mathcal{F}(V)$ is contained in the directed system defining $\mathcal{F}(U)$. As there are maps $\sigma(V) \to \mathcal{F}(V)$ and $S \to \mathcal{F}(V)$, for every S in the directed system of V, precomposing with any descending path we get maps $\sigma(U) \to \mathcal{F}(V)$ and $S \to \mathcal{F}(V)$, for every S in the directed system of U. Then universality of the colimit gives us a unique map $\mathcal{F}(U) \to \mathcal{F}(V)$. Note that if there are no paths (decending or otherwise) from U to V, then the colimit over an empty diagram still exists, it is just the initial object $\emptyset$ of SC.

To check the gluing condition, first note that every open $U \subseteq X$ must nontrivially intersect $\text{Ran}^n(M) \times \mathbf{R}_{\geqslant 0}$, the top stratum (in the point-counting stratification). So for $W = U \cap V$, if we have $\alpha \in \mathcal{F}(U)$ and $\beta \in \mathcal{F}(V)$ such that $\alpha|_W = \beta|_W$ is a k-simplex, then α and β must have been k-simplices as well. This is because a simplicial takes a simplex to a simplex, and we cannot collide points while remaining in the top stratum. Hence the pullback of $S \ni \alpha$ and $T \ni \beta$ via some induced maps (by descending paths) from U to W and V to W, respectively, will restrict to the identity on the chosen k-simplex. Hence the gluing condition holds, and $\mathcal{F}$ is a sheaf. ∎

Functoriality of $[\,\cdot\,]$ allows us to extend the proof to build a sheaf valued in simplicial sets.

Proposition 6.5.4. Let $\mathcal{G}$ be the function $\mathrm{Op}(X)^{op} \to \mathrm{sSet}$ on objects given by

$$\mathcal{G}(U) = \mathrm{colim}\,([\sigma(U)] \rightrightarrows S \ : \ \text{every } [\sigma(U)] \to S \text{ is induced by a descending } \gamma : I \to U)\,.$$

This is a functor and satisfies the sheaf gluing conditions.

Remark 6.5.5. The sheaf $\mathcal{G}$ is non-trivial on more sets. For example, any path contained within one stratum of X induces the identity map on simplicial sets (though not on simplicial complexes). Hence $\mathcal{G}$ is non-trivial on every open set contained within a single stratum.

References: nLab (article "Simplicial complexes"), n-category Cafe (post "Simplicial Sets vs. Simplicial Complexes," 2017-08-19)

7 Persistent homology - functoriality

7.1 Functorial persistence

2018-02-28

Keywords: *persistence module, barcode, persistence diagram, filtration, induced matching, functor, natural transformation, pointed set*

The goal of this post is to overcome some hurdles encountered by Bauer and Lesnick. In their approach, some geometric information is lost in passing from persistence modules to matchings. Namely, if an interval ends, we forget if the k-cycle it represents becomes part of another k-cycle or goes to 0. Recall:

- $(\mathbf{R}, \leqslant)$ is the category of real numbers and unique morphisms $s \to t$ whenever $s \leqslant t$,
- Vect (BVect) is the category of (based) finite dimensional vector spaces, and
- Set_* is the category of pointed sets.

We begin by recalling all the classical notions in the TDA pipeline.

Definition 7.1.1. A *persistence module* is a functor $F : (\mathbf{R}, \leqslant) \to \text{Vect}$. The *barcode* of a persistence module F is a collection of pairs (I, k), where $I \subseteq \mathbf{R}$ is an interval and $k \in \mathbf{Z}_{>0}$ is a positive integer.

Crawley-Boevey describes how to find the decomposition of a persistence module into interval modules. The k for each I is usually 1, but is 2 (and more) if the same interval appears twice (or more) in the decomposition. A barcode contains the same information as a *persistence diagram*, though the former is drawn as horizontal bars and the latter is presented on a pair of axes.

Definition 7.1.2. A *matching* χ of barcodes $\{(I_i, k_i)\}_i$ and $\{(J_j, \ell_j)\}_j$ is a bijection $I' \to J'$, for some $I' \subseteq \{(I_i, k_i)\}_i$ and $J' \subseteq \{(J_j, \ell_j)\}_j$.

We write matchings as $\chi\colon \{(I_i, k_i)\}_i \nrightarrow \{(J_j, \ell_j)\}_j$.

Definition 7.1.3. A *filtered persistence module* is a functor $F : (\mathbf{R}, \leqslant) \to \text{BVect}$ for which $F(s \leqslant t)(e_i) = f_j$ or 0, for every e_i in the basis of $F(s)$ and f_j in the basis of $F(t)$.

The notion of filtered persistence module is used for a stronger geometric connection. Indeed, for every filtered space X the persistence module along this filtration is also filtered (once interval modules have been found), as then inclusions $X_s \hookrightarrow X_t$ will induce isomorphisms in homology onto their image. That is, a pair of homology classes from the source may combine in the target, but if the classes come from interval modules, a class from the source can not be in two non-homologous classes of the target.

Remark 7.1.4. The above dicussion highlights that choosing a basis in the definition of a persistence module already uses the decomposition of persistence modules into interval modules.

It is immediate that a morphism of persistence modules is a natural transformation. Let BPVect be the full subcategory of BVect consisting of elements in the image of some filtered persistence module (the objects are the same, we just have a restriction of allowed morphisms).

Definition 7.1.5. Let $\mathcal{B}$ be the functor defined by

$$\begin{aligned} \mathcal{B}\colon \text{BPVect} &\to \text{Set}_*, \\ (V, \{e_1, \dots, e_n\}) &\mapsto \{0, 1, \dots, n\}, \\ (\varphi : (V, \{e_i\}) \to (W, \{f_j\})) &\mapsto \left(i \mapsto \begin{cases} j & \text{if } \varphi(e_i) = f_j, \\ 0 & \text{if } \varphi(e_i) = 0 \text{ or } i = 0. \end{cases} \right) \end{aligned}$$

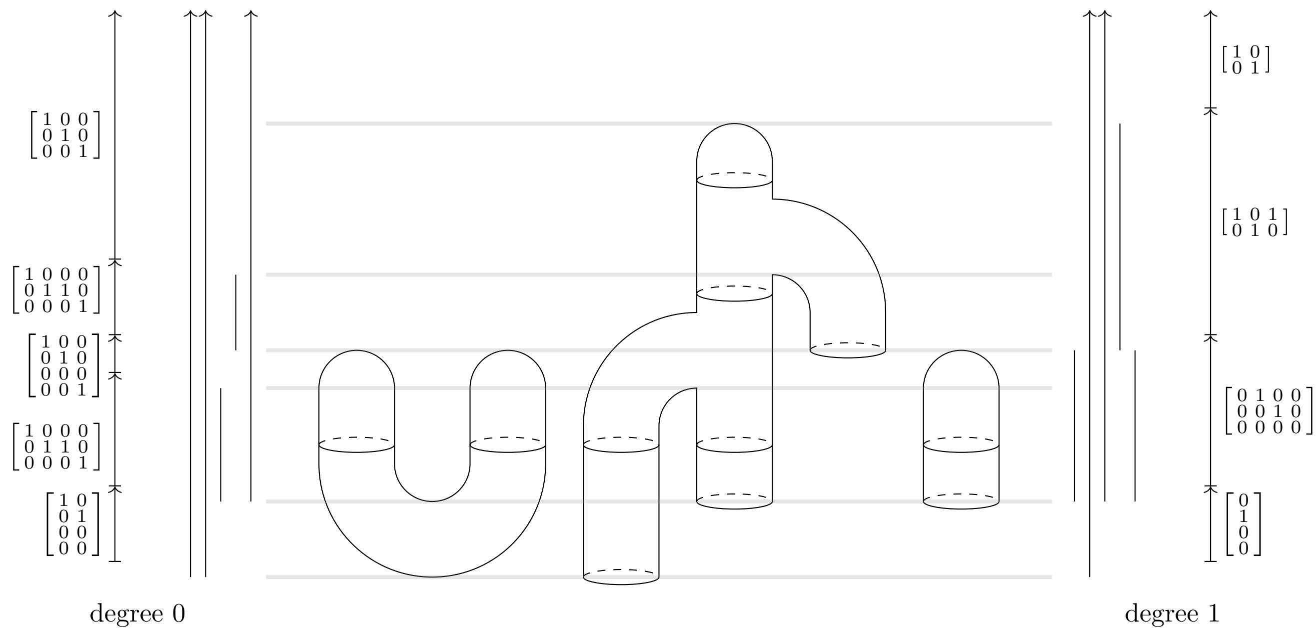
The basepoint of every set in the image of $\mathcal{B}$ is 0.

Definition 7.1.6. Let F, G be persistence modules and η a morphism $F \to G$.

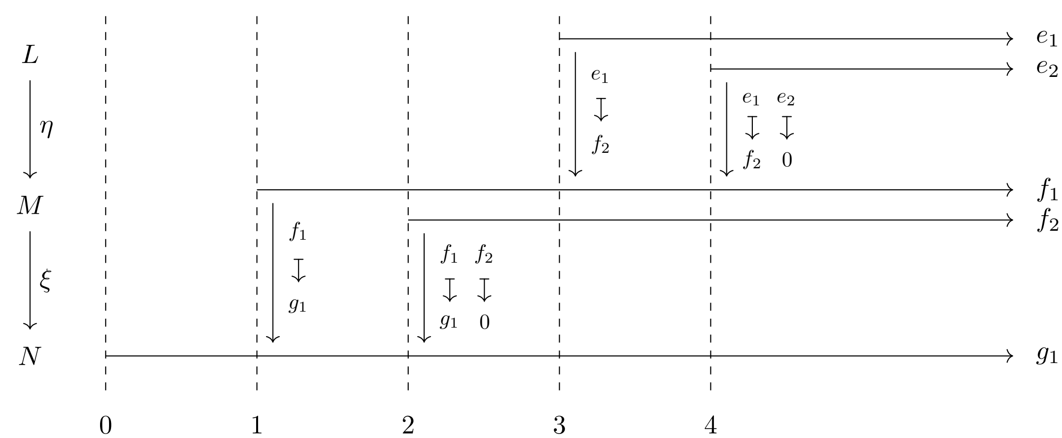
- The *persistence diagram* of F is the functor $\mathcal{B} \circ F$.
- The *matching* induced by η is the natural transformation $\mathcal{B}(\eta) : \mathcal{B} \circ F \to \mathcal{B} \circ G$.

Bauer and Lesnick's definition of "matching" allow for more freedom to mix and match barcode intervals, but this also restricts how much information of a persistence module morphism can be tracked.

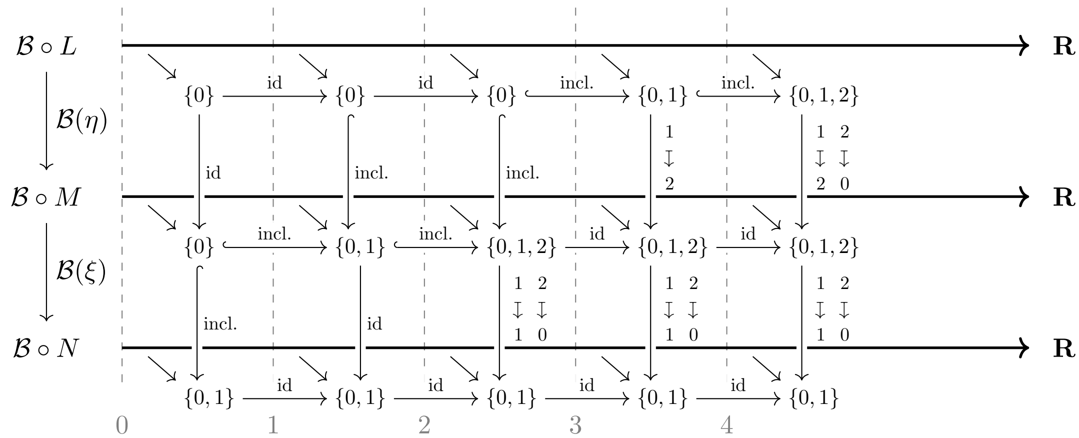
Example 7.1.7. The following example has a horizontal filtration with the degree 0 homology barcode on the left and the degree 1 homology barcode on the right. Linear maps of based vector spaces have also been shown to indicate how homology classes are born, die (column of zeros), and combine (row with more than one 1).



Example 7.1.8. Bauer and Lesnick present Example 5.6 to show that functoriality does not work in their setting. We reproduce their example and show that functoriality does work in our setting. Note that vertical ordering of the bars does not matter once they are named.



Apply the functor $\mathcal{B}$ to the whole diagram to get the matchings induced by η and ξ, as below.



Next we hope to understand how interleavings fit into this setup.

References: Bauer and Lesnick (Induced matchings and the algebraic stability of persistence barcodes), Crawley-Boevey (Decomposition of pointwise finite-dimensional persistence modules)

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