Stratifying configuration spaces for persistent homology

Jānis Lazovskis

MSU Math / Computer Science seminar
2019-04-15

Abstract: The collection of all finite subsets of a given metric space is a natural starting point to understand the foundations of persistent homology. We consider the product of this collection with the non-negative reals as a domain for the Čech construction of a simplicial complex. The stability of this construction stratifies the domain and allows us, among other things, to describe paths in configuration spaces as morphisms of persistent homology.

0.1 Intro

Motivation. Can abstractly compare PH of two point samples. What if know more?
Want to tell you about two things. A function and a cosheaf.

M is a nice space (compact, connected manifold / compact metric space / semialgebraic set)
The Ran space of M. The set SC of finite abstract simplicial complexes.

0.2 The Čech map

The Čech map on $\mathrm{Ran}(M) \times \mathbf{R}_{\geqslant 0} \to \mathsf{SC}$. visualization: random path in $\mathrm{Ran}(M)$ with simplices
Isomorphism classes of SC. Partial order on $[\mathsf{SC}]$. Čech map $[\check{C}]$.
Thm. (L)

1. The Čech map $[\check{C}]$ is continuous.
2. The stratification of $\mathrm{Ran}(M) \times \mathbf{R}_{\geqslant 0}$ induced by $[\check{C}]$ is not conical.
3. If M is semialgebraic, there exists a conical stratification of $\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$ that refines $[\check{C}]$

A stratification is a continuous map $f\colon X \to (A, \leqslant)$. Then X is called A-stratified.
A stratification $g\colon X \to B$ refines $f\colon X \to A$ if for every $a \in A$ there is a subset $B' \subseteq B$ such that $X_a = \bigcup_{b \in B'} X_b$.
A stratification is conical if every $x \in X$ has a neighborhood that looks like the cone of a stratified space.
diagram: stratification examples

Conjecture: New strata at intersections of closures makes $[\check{C}]$ conical.

0.3 The Čech cosheaf

Let M be semialgebraic and $n \in \mathbf{N}$.
Thm. (L) There exists a cosheaf $\mathcal{F}$ on the basic opens of $\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$ for which:

1. The costalk of $\mathcal{F}$ at (P, r) is $\check{C}(P, r)$.
2. $\mathcal{F}$ is $[\mathsf{SCC}]$-constructible.
3. The restriction of $\mathcal{F}$ to $\{P\} \times \mathbf{R}_{\geqslant 0}$ is a cosheaf and generates the persistence module of P.

Let $f\colon X \to A$ be a conically stratified space. The homotopy category of entrance paths of X, written $\mathrm{Ho}(\mathrm{Sing}_A(X))$, is:

- object are points of X
- morphisms are homotopy classes of paths that "respect the stratification"

For every morphism $[\sigma]$, there is a pair $a_0 \leqslant a_1$ in A for which $\sigma(t \neq 0) = a_1$, $\sigma(1) = a_0$. The path "enters" a lower stratum.

Let $[\check{C}C]\colon \mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0} \to [\mathsf{SCC}]$ be a conical stratification that refines $[\check{C}]$.
Let $\underline{\mathsf{SC}}$ be the category of finite simplicial complexes and simplicial maps.
Lem. Every morphism $[\sigma]$ of $\mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}))$ induces a unique simplicial map $\check{\sigma}$.
Def. Let $\mathcal{F}\colon \mathrm{Op}(\mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}) \to \mathsf{Cat}_{/\underline{\mathsf{SC}}}$ be given by diagram: the cosheaf $\mathcal{F}$

$$\mathcal{F}(U) = \begin{pmatrix} \mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(U)) & \to & \underline{\mathsf{SC}} \\ (P, r) & \mapsto & \check{C}(P, r) \\ [\sigma] & \mapsto & \check{\sigma} \end{pmatrix}$$

and $\mathcal{F}(V \subseteq U)$ the inclusion.

Produces a diagram of simplicial complexes and simp maps.

0.4 Implications for TDA and persistent homology

Consider a path $\gamma\colon I \to \mathrm{Ran}_{\leqslant n}(M)$. Image is naturally stratified. visualization: complicated path

Do we get a morphism left to right? No. Basic open problem. Times $a_1 < \cdots < a_m$ are 0-dim associated strata, get zigzag

$$PH(0) \to PH(a_1) \leftarrow PH(\tfrac{a_1+a_2}{2}) \to PH(a_2) \leftarrow PH(\tfrac{a_2+a_3}{2}) \to \cdots \leftarrow PH(1).$$

How do we follow homology classes? Right filtration functor.

More on paths:
- when contractible? Stay in stratum? visualization: contractible path

If time, visualizations:
- monodromy induces monodromy in space of PDiags
- Difference between merge death and true death