The entrance path category

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Abstract: The infinity category of entrance paths of a stratified space contains information about the space itself, as well as the structures we can build on the space. I will talk about several uses of this category, by working with a simplified version and by considering what the full generality has to offer.

0.1 Motivation

Recall the Čech functor $\mathcal{F}\colon \mathsf{Bsc}(\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}) \to \mathsf{Cat}_{/\underline{\mathsf{SC}}}$.
Question: Can the Čech functor be simplified?

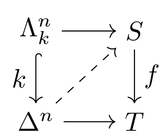
Observation: Vast majority of morphisms in the diagram $\mathcal{F}(U)$ are identity.
Instead make diagram with only one non-trivial map for every such homotopy class.

0.2 Simplicial sets and ∞-categories

A simplicial set is a set of sets $S = \{S_0, S_1, \dots\}$ with compatible maps among the S_i:
  face maps $s_i\colon S_{n-1} \to S_n$
  degeneracy maps $d_i\colon S_{n+1} \to S_n$
The compatible maps come from the category Δ.
  $\mathrm{Obj}(\Delta) = \{[n] = (0, 1, \dots, n) : n \in \mathbf{Z}_{\geqslant 0}\}$
  $\mathrm{Hom}_\Delta([n], [m]) = \{\text{order-preserving set maps } [n] \to [m]\}$

Definition: A simplicial set is a functor $\Delta^{\mathsf{op}} \to \mathrm{Set}$.
Example: The nerve of a category. $\mathrm{Sing}(X)$ for a topological space X. The standard k-simplex Δ^k.

Let Λ^n_k be Δ^n without the kth face. It is a union of n copies of Δ^{n-1}, as all but the kth face of Δ^n.
Definition: A morphism $S \to T$ of simplicial sets is a fibration if whenever the solid maps exist, the dashed map exists.



Definition: A simplicial set S is a Kan complex if $S \to *$ is a fibration.
It is a weak Kan complex, or ∞-category, if $S \to *$ is a fibration for $0 < k < n$.
Slogan: All inner k-horns of S can be filled.

0.3 Stratified spaces

Let $f\colon X \to (A, \leqslant)$ be a stratification.
Definition. An entrance path of X is a continuous map $\sigma\colon \Delta^n_{top} \to X$ such that there exists a chain $a_0 \leqslant \cdots \leqslant a_n$ in A with

$$f(\sigma(0, \dots, t_i, \dots, t_n)) = a_{n-i},\ t_i \neq 0,$$

for all $i = 0, \ldots, n$.
$\mathrm{Sing}_A(X)$ is the category of entrance paths of X. If f is conical, $\mathrm{Sing}_A(X)$ is an ∞-category.

Recall: Stratification by the Čech map $[\check{C}] \colon \mathrm{Ran}(M) \times \mathbf{R}_{\geqslant 0} \to [\mathsf{SC}]$
Conical stratification, refining $[\check{C}]$, by $[\check{C}C] \colon \mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0} \to [\mathsf{SCC}]$

0.4 A Čech map of simplicial sets

Recall the cosheaf $U \mapsto \Big(\mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(U)) \to \mathsf{Cat}_{/\underline{\mathsf{SC}}}\Big)$ for every $U \in \mathsf{Bsc}(\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0})$.

Goal: Try to construct a fibration $\mathrm{Sing}_{[\mathsf{SCC}]}(\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}) \to N(\underline{\mathsf{SC}})$
Image of n-simplex is clear.

Motivation: Fibering by vector bundles:

$$
\begin{array}{ccccc}
\text{category of vector bundles} & & \text{category of manifolds} & & \text{category of small categories} \\
\mathrm{Bun} & \longrightarrow & \mathrm{Mfld} & \longrightarrow & \mathsf{Cat} \\
(E \to M) & \longmapsto & M & \longmapsto & \mathrm{Vect}_M
\end{array}
$$

Given a morphism $f \colon M \to N$ of manifolds and a vector bundle $E \to N$, we can construct the pullback bundle $f^*E \to M$ over M. This is filling the horn Λ^1_0. How does this extend?

Straightening: Given $A \to B$, does it classify the fibers of some $B \to C$?
Unstraightening: Given a fibration $B \to C$, can the fibers be arranged into A to get $A \to B$?
Answers to both are yes (in the appropriate categories).

Goal: $\mathsf{sSet} \to \mathrm{Sing}_A(X) \to \mathcal{S}$.
Question: What is in place of $E \to M$? Can we fill Λ^1_0? Vector bundle example has $f^*E \subseteq \times E$.
Implication: Do we need products / limits ? Maybe over C is a sub-simplicial complex? Maybe sub-cell?

Start with inner horns.
Given the simplicial map $\mathrm{Sing}_{[\mathsf{SCC}]}(\mathrm{Ran}_{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}) \to N(\underline{\mathsf{SC}})$, can Λ^2_1 be filled?