Stratifications and sheaves for persistent homology

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Abstract: I will describe a stratification of $\mathrm{Ran}(M) \times \mathbf{R}_{\geqslant 0}$, the space of all finite subsets of a manifold M and all non-negative radii, to give a unifying perspective on persistence modules. The stratification uses a particular partial order on the set of isomorphism classes of simplicial complexes, chosen so that homotopy classes of entrance paths of the stratified space can be uniquely interpreted as simplicial maps.

0.1 Motivation

Bauer–Lesnick 2015, "Induced matchings and the algebraic stability of persistence barcodes"
Question: How can algebra help compare two barcodes?
Answer: When all bars move in the same direction, algebra gives a good description.

Question: Under what conditions will algebra compare any two barcodes?
Partial answer: A "morphism" between the underlying spaces gives a "morphism" between barcodes.

Setting: Barcodes induced by point samples. Morphism between samples is a path.

0.2 Tools for persistence

M is a connected Riemannian manifold.
Usually takes in a function $f\colon M \to \mathbf{R}$ and computes homology of $f^{-1}((-\infty, r])$.
We have a function $f_P\colon M \to \mathbf{R}$ given by $x \mapsto d(x, P)$ for every $P \in \mathrm{Ran}(M)$.

Instead of homology, only compute space. Instead of two steps, uncurry and combine into one step:

$$\mathrm{Ran}(M) \times \mathbf{R} \to \mathsf{Top}$$

Definition. The Ran space of M is $\mathrm{Ran}(M) = \{P \subseteq M : 0 < |P| < \infty\}$. Distance is

$$d_H(P, Q) := \max\left\{\max_{p\in P}\min_{q\in Q} d(p,q), \max_{q\in Q}\min_{p\in P} d(p,q)\right\}.$$

Topology on $\mathrm{Ran}(M)$ is the metric topology associated to d_H.
Topology on $\mathrm{Ran}(M) \times \mathbf{R}_{\geqslant 0}$ is the metric topology associated to the sup norm:

$$d_\infty((P, r), (Q, s)) := \max\{d_H(P, Q), |r - s|\}.$$

Definition. A simplicial complex is (V, S) where V is countable and $S \subseteq P(V)$ is closed under taking faces.
Let SC be the set of simplicial complexes.
Let $[\mathsf{SC}]$ be the set of isomorphism classes of simplicial complexes.

Definition. Write $[C] \succcurlyeq [D]$ if there is a simplicial map $C \to D$ that is surjective on vertices.
Observation. This defines a partial order on $[\mathsf{SC}]$ (and a preorder on SC).

0.3 Stratifications

Whitney stratifications are classical approach (submanifolds, sequences of tangent spaces)
Definition. A space M is A-stratified if there is a continuous map $f\colon M \to (A, \leqslant)$.
This is a poset stratification. Posets have the upset topoogy.

Definition. Let $f\colon \mathrm{Ran}(M) \to \mathbf{Z}$ be the point counting map.
Observation. f is a stratification, as ball in $\mathrm{Ran}(M)$ around P is bounded by balls in M around the P_i.
Theorem (Ayala-Francis-Tanaka 2017) $f|_{\mathrm{Ran}_{\leqslant n}(M)}$ is a conical stratification, for every finite n.

Definition Let $\check{C}\colon \mathrm{Ran}(M) \times \mathbf{R}_{\geqslant 0} \to \mathsf{SC}$ given by
$V(\check{C}(P,r)) = P$,
$S(\check{C}(P,r)) \ni P'$ whenever $\bigcap_{p \in P'} B(p,r) \neq \emptyset$, for every $P' \subseteq P$.
Let $[\check{C}]\colon \mathrm{Ran}(M) \times \mathbf{R}_{\geqslant 0} \to [\mathsf{SC}]$ be the composition of $\check{C}$ with the projection.

Theorem (L.)
1. The Čech map $[\check{C}]$ is continuous.
2. $[\check{C}]$ is not conical, but when M is semialgebraic, it has a conical refinement.

0.4 Sheaves and simplicial sets

Presheaf is a functor $\mathsf{Op}(X) \to \mathcal{C}$ for some category $\mathcal{C}$. same for precosheaf
Presheaf is sheaf if it satisfies limit condition for every open cover of every open set. analogous for cosheaf.

Open sets cause problems.
Problem 1: Local data of paths may not be enough - point-swapping paths need large open set.
Problem 2: Using (co)limit arguments to show inverse image pre(co)sheaves are (co)sheaves is hard.
Solution: Use $\mathsf{Bsc}(X)$ instead of $\mathsf{Op}(X)$. A basic is the image of a stratified open embedding $Z \times C(L)$.

$\mathrm{Sing}(M)$. $\mathrm{Sing}_A(M)$. $\mathrm{Ho}(\mathcal{C})$. Unique simplicial map lemma.
Definition. Let $\mathcal{F}\colon \mathsf{Op}(\mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}) \to \mathsf{Cat}_{/\underline{\mathsf{SC}}}$ be given by

$$\mathcal{F}(U) = \left(\begin{array}{rcl} \mathrm{Ho}(\mathrm{Sing}_{[\mathsf{SCC}]}(U)) & \to & \underline{\mathsf{SC}} \\ (P,r) & \mapsto & \check{C}(P,r) \\ [\sigma] & \mapsto & \check{\sigma} \end{array} \right)$$

and $\mathcal{F}(V \subseteq U)$ the inclusion.

Theorem (L.) On basics, $\mathcal{F}$ is a cosheaf. For every $P \in \mathrm{Ran}(M)$, $\mathcal{F}|_{\{P\} \times \mathbf{R}_{\geqslant 0}}$ is also a cosheaf.

0.5 Persistent homology

Can be viewed as a functor $(\mathbf{R}, \leqslant) \to \mathsf{Vect}$. The image of this diagram is the *persistence module*.
This is induced by a functor $(\mathbf{R}, \leqslant) \to \mathsf{Top}$, a diagram of simplicial complexes.
Observation: By construction, $\mathcal{F}|_{\{P\} \times \mathbf{R}_{\geqslant 0}}$ is the diagram inducing the persistence module.

Motivating problem: How does $\mathcal{F}$ help to compare the barcodes of two point samples P, Q?
Take $\gamma\colon I \to \mathrm{Ran}_{\leqslant n}(M)$ with $\gamma(0) = P$, $\gamma(1) = Q$.
Get a zigzag of diagrams of simplicial complexes.

Example: H_0 and H_1 bars. Best when homology classes don't merge.