Sheaf theory
on universal persistent homology spaces

JMM AMS Special Session on Topological Data Analysis:
Theory and Applications

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Jānis Lazovskis
University of Illinois at Chicago

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`math.uic.edu/~jlv`

Overview

Setting.

- Simplicial complexes are finite, abstract, and Čech.
- M is a connected manifold.
- Size of every point cloud $P \subseteq M$ is bounded by some fixed $n \in \mathbf{N}$.
- The Ran space of M is $\mathrm{Ran}^{\leqslant n}(M) := \{P \subseteq M : 0 < |P| \leqslant n\}$ and has topology induced by Hausdorff distance of subsets.

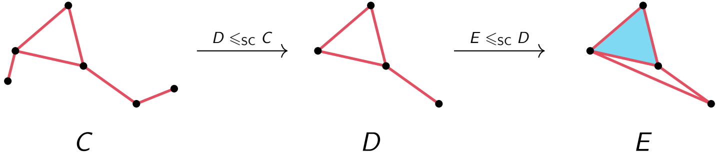
Results.

- For M Riemannian, the map that assigns a simplicial complex to every element of $\mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$ is continuous.
- For M piecewise linear, there exists a cosheaf on $\mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$ whose restriction to $P \times \mathbf{R}_{\geqslant 0}$ generates the persistence module of P.

Based on arXiv:1810.12358 "Stratifications and sheaves on the Ran space".

Posets and simplicial complexes

SC is the set of simplicial complexes.



$$\left(D \leqslant_{\mathsf{SC}} C\right) \iff \left(\text{there is a simplicial map } C \to D \text{ that is surjective on vertices}\right)$$

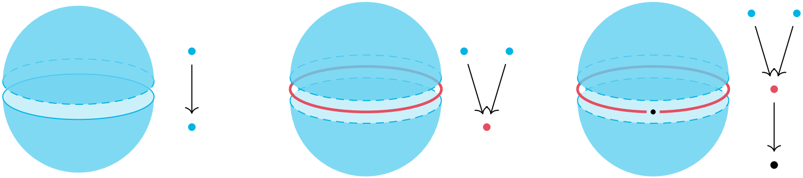
Lemma. The relation $\leqslant_{\mathsf{SC}}$ defines a

- preorder on simplicial complexes, and a
- partial order on isomorphism classes of simplicial complexes.

Let [SC] be the set of isomorphism classes of simplicial complexes.

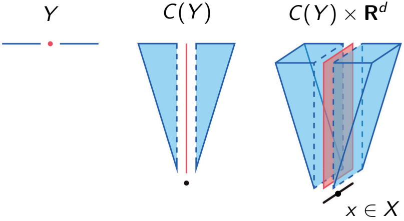
Stratifications

Definition. A *poset stratification* is a continuous map $f\colon X \to A$.



Stratifications can be *refined*. Equivalently, they are *compatible*.

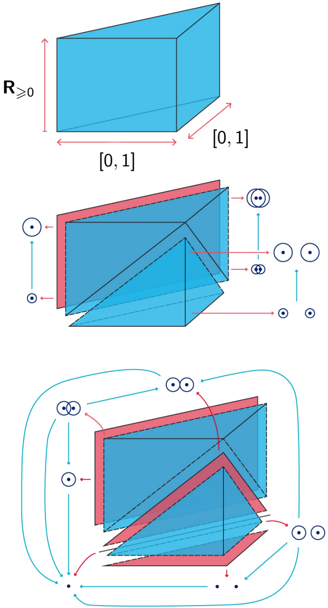
A poset stratification is *conical* if every $x \in X$ has a stratified neighborhood that looks like a cone.



Stratifying $\mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$

Let $\check{C} \colon \mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0} \to [\mathrm{SC}]$ be the map that assigns to (P, r) its simplicial complex isomoprhism class.

Example. $\mathrm{Ran}^{\leqslant 2}([0,1]) \times \mathbf{R}_{\geqslant 0}$.



Theorem. (L.)

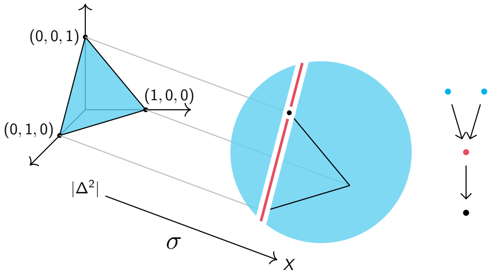
- If M is Riemannian, $\check{C}$ is continuous but not conical.
- If M is piecewise linear, there exists a conical stratification compatible with $\check{C}$.

Proof. Understand "thresholds" for (P, r).

Entrance paths and homotopies

Definition. An *entrance path* of a stratified space $f\colon X \to A$ is a continuous map $|\Delta^n| \to X$ that respects the stratification.

$\sigma \in \mathrm{Sing}_A(X)_2$:



New setting:

- M is piecewise linear.
- $\check{C}'\colon \mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0} \to [\mathrm{SC}]'$ is a conical refinement of $\check{C}$.
- $\mathrm{Ho}(\mathrm{Sing}_{[\mathrm{SC}]'}(\mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}))$ is the homotopy category of entrance paths.

Lemma. Every morphism in $\mathrm{Ho}(\mathrm{Sing}_{[\mathrm{SC}]'}(\mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}))$ induces functorially a unique simplicial map in SC.

Cosheaves

A *cosheaf* on X is a functor $\mathcal{F}\colon \mathrm{Open}(X) \to \mathcal{C}$ for which the natural map $\mathrm{colim}_{U \subseteq V} \mathcal{F}(V) \to \mathcal{F}(U)$ is an isomorphism, for all $U \in \mathrm{Open}(X)$.

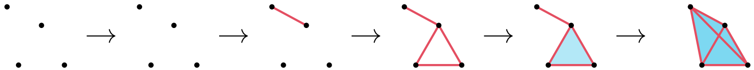
Definition. Let $\mathcal{F}\colon \mathrm{Open}(\mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}) \to \mathrm{Cat}_{/\mathrm{SC}}$ be the functor

$$\mathcal{F}(U) = \Big(\mathrm{Ho}(\mathrm{Sing}_{[\mathrm{SC}]'}(U)) \to \mathrm{SC}\Big)$$

from the previous slide.

Theorem. (L.) The functor $\mathcal{F}$ is a cosheaf. The restriction of $\mathcal{F}$ to $\{P\} \times \mathbf{R}_{\geqslant 0}$ is also a cosheaf, isomorphic to the persistence module of P.

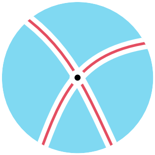
$\check{C}(P, r)$:



$$H_0: \quad \mathbf{Z}^4 \longrightarrow \mathbf{Z}^4 \longrightarrow \mathbf{Z}^3 \longrightarrow \mathbf{Z} \longrightarrow \mathbf{Z} \longrightarrow \mathbf{Z}$$

$$H_1: \quad 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbf{Z} \longrightarrow 0 \longrightarrow 0$$

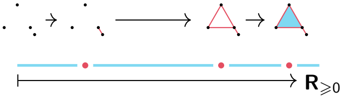
The universal persistence cosheaf $\mathcal{F}$



stratified open set
$U \subseteq \mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$

The **objects** of $\mathrm{Ho}(\mathrm{Sing}_{[\mathrm{SC}]'}(U))$ are the simplicial complexes produced by U through $\check{C}$.

The **morphisms** are homotopy classes of entrance paths with the same endpoints.



stratified closed set
$\{P\} \times \mathbf{R}_{\geqslant 0} \subseteq \mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$

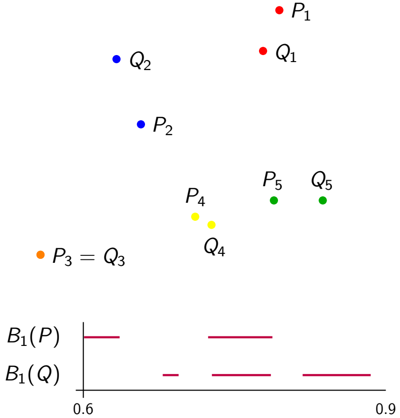
The restriction cosheaf $\mathcal{F}|_{\{P\} \times \mathbf{R}_{\geqslant 0}}$ produces a **zigzag** diagram whose backward arrows are all the identity.

Application: Comparing barcodes

Question. How can we compare / match two barcodes?

Bauer, Lesnick (2015): *Induced Matchings and the Algebraic Stability of Persistence Barcodes.*

Take two data sets P, Q and a pairing of their elements (e.g. time series).



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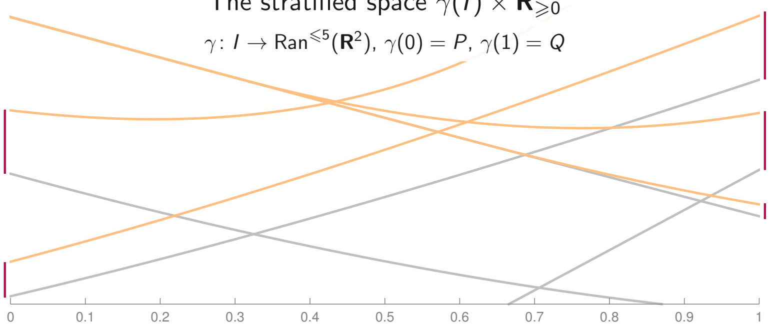
The stratified space $\gamma(I) \times \mathbf{R}_{\geqslant 0}$

$\gamma\colon I \to \mathrm{Ran}^{\leqslant 5}(\mathbf{R}^2)$, $\gamma(0) = P$, $\gamma(1) = Q$



$B_1(P)$ $B_1(Q)$

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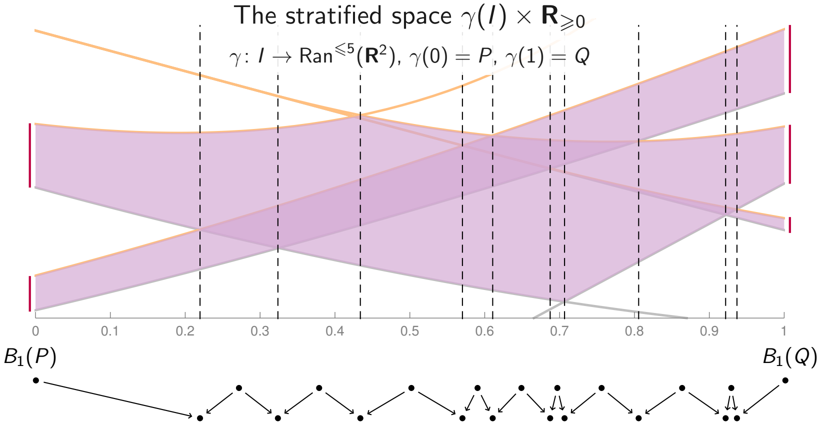

Extension: Constructible (co)sheaves

Question. Can $\mathcal{F}$ be described as a constructible cosheaf?

Curry, Patel (2016): *Classification of Constructible Cosheaves.*
MacPherson, Patel (2018): *Persistent Local Systems.*

- **No.** For small enough basic opens $V \subseteq U$ associated to a common stratum $\mathcal{F}(V \subseteq U)$ is an isomorphism, but not for all.
- **Yes.** If ordered configuration space is used and the stratification is refined to separate "swaps."

Question. Can $\mathcal{F}$ be described as a constructible sheaf?

Lurie (2017): *Higher Algebra, Appendix A.*

"The category of constructible sheaves over X is equivalent to the category of functors $\mathrm{Sing}_A(X) \to \mathcal{S}$*."*

- **Maybe.** Every $\sigma \in \mathrm{Sing}_A(\mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0})_k$ induces a unique commutative diagram in SC. Extend as a functor into $N(\mathrm{SC})$.

End

Thank you for your attention.

References.

1. Bauer, Ulrich and Michael Lesnick. *Induced Matchings and the Algebraic Stability of Persistence Barcodes,* 2015.
2. Curry, Justin and Amit Patel. *Classification of Constructible Cosheaves,* 2016.
3. Lazovskis, Jānis. *Stratifications and sheaves on the Ran space,* 2018.
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5. MacPherson, Robert and Amit Patel. *Persistent Local Systems,* 2018.
6. Shiota, Masahiro. *Geometry of Subanalytic and Semialgebraic sets,* 1997.
7. Topaz, Ziegelmeier, Halverson. *Topological Data Analysis of Biological Aggregation Models,* 2015.