Stratifications and factorization

Jānis Lazovskis

UIC Algebraic K-theory seminar
2018-12-05

0.1 Poset stratifications

poset stratification of a topological space. stratum
eg 1: SCs with subset partial order
non-eg 2: SCs with $C \leqslant D$ if there is a simplicial map $D \to C$ surj on vert (preorder, no anti-symmetry)
eg 3: iso classes of SCs (unlabeled SCs) with above relation. is partial order
Let X be top space, $f\colon X \to A$ strat. $\mathrm{Sing}(X)$, $\mathrm{Sing}_A(X)$, $\mathrm{Sing}^A(X)$. These are ssets.

0.2 Application

Piecewise linear manifold M, integer $n \in \mathbf{Z}_{>0}$, Ran space $\mathrm{Ran}^{\leqslant n}(M)$ has topology on it.
Main space $X = \mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$, main map $u\check{C}\colon X \xrightarrow{\check{C}} SC \xrightarrow{\sim} uSC$, unlabeled Cech. Is continuous.
$\mathrm{Sing}_{uSC}(X)$ is not necessarily ∞-cat. Since PL, exists compatible strat uSC' (comm sq)
Take ho cat $\mathrm{Ho}(\mathrm{Sing}_{\mathsf{uSC'}}(X))$, objects as in $\mathrm{Sing}_{\mathsf{uSC'}}(X)$, homotopy rel between objects.
Lemma 1: every $\sigma \in \mathrm{Sing}_{\mathsf{uSC'}}(X)_1$ induces a unique simplicial map $\check{C}(\sigma(0)) \to \check{C}(\sigma(1))$ in SC.
Proof: geometric argument from continuity. Get induced paths $\sigma_i\colon I \to X$ on vertices.
Lemma 2: every $[\sigma] \in \mathrm{Ho}(\mathrm{Sing}_{\mathsf{uSC'}}(X))$ induces a unique simplicial map $\check{C}(\sigma(0)) \to \check{C}(\sigma(1))$ in SC.
Proof: existence by Lemma 1. Uniqueness by homotopy on induced $\tau_i \in \mathrm{Sing}(M)_2$ with $d_0\tau_i = \sigma_i$.
Def: Define functor $\mathrm{Ho}(\mathrm{Sing}_{\mathsf{uSC}}(X)) \to \mathsf{SC}$ in the natural way. Is also functor for all $U \subseteq X$ open.
Theorem: the functor $\mathrm{Op}(X) \to \mathrm{Cat}_{/\mathsf{SC}}$ with $U \mapsto (\mathrm{Sing}_{\mathsf{uSC'}}(U) \to \mathsf{SC})$ is a cosheaf.
Proof: Construct inverse to natural map $\mathrm{colim}\, \mathcal{F}(U_i) \to \mathcal{F}(U)$ for cover $\{U_i\}$ of U.

0.3 Factorization

Ran space has natural operad structure on it. product.
define different cosheaf (as Lurie) on it.