Barcodes in persistence

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0.1 Motivation

Main TDA product is the *barcode*, a simple visual rep of the persistent homology of a space.
Example: Barcode of finite subset of $\mathbf{R}^2$. Dim 0 and dim 1.
This is just rank of homology groups. Note how some merge, some die.

We study not just objects, but functions between objects. What is a function between barcodes? Pipeline:

finite metric space	$\xrightarrow[\text{of balls}]{\text{SCs from nerve}}$	$\mathbf{R}$-indexed sequence of topological spaces	$\xrightarrow{\text{homology}}$	$\mathbf{R}$-indexed sequence of homology groups	$\xrightarrow{\text{rank}}$	barcode
(X, d)		$F\colon (\mathbf{R}, \leqslant) \to SC$		$F\colon (\mathbf{R}, \leqslant) \to Grp$		?

0.2 Formalizing barcodes

What do we want a morphism of barcodes to be? **Example.**
Definition: A multiset is set where the elements may repeat (a pair $\{S \in Set, m\colon S \to \mathbf{N}\}$).
Let $Int := \{[a, b] \subseteq \mathbf{R} : a < b\}$ be the set of intervals of $\mathbf{R}$.
Definition: A barcode is a multiset $B \subseteq Int$.
Theorem: The barcode of the TDA pipeline is uniquely determined.

Definition: A matching from a set A to a set B, written $\sigma\colon A \nrightarrow B$, is a bijection $\sigma\colon A' \to B'$, for some $A' \subseteq A$ and $B' \subseteq B$.

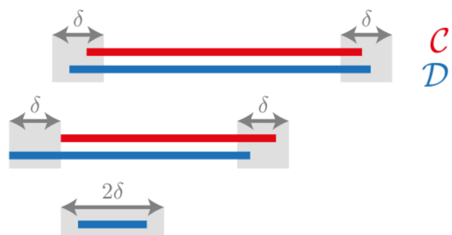
Hope is that morphisms earlier in the pipeline can be interpreted as matchings.
Problem: Even with slight shift, functoriality in step 3 would say we can't match "obvious" bars .
Solution 1: Define metrics on ambient spaces, "induced matching" for barcodes within ϵ of each other
Solution 2: Go back earlier in pipeline to induce "obvious" matching based on topological changes.
Pro 1: Stable under small perturbations. Con 1: May not reflect underlying changes.
Pro 2: Precisely reflects topological changes. Con 2: Carries too much information.

Example: Points at $(0, 0)$, $(2, 0)$, moving from $(2, \sqrt{3})$ to $(0, \sqrt{3})$

Definition: For $\delta \geqslant 0$ and $B \subseteq Int$, let $B^{\epsilon} = \{I \in B : [t, t+\delta] \subseteq I \text{ for some } t \in \mathbf{R}\}$. Note that $B^0 = B$ and $B^{\delta \gg 0} = \emptyset$.
Definition: For $\delta \geqslant 0$, a δ-matching from B to C is a matching $\sigma \colon B \nrightarrow C$ such that

- $B^{2\delta} \subseteq B'$,
- $C^{2\delta} \subseteq C'$,
- if $\sigma[a,b] = [x,y]$, then $[a,b] \subseteq [x-\delta, y+\delta]$ and $[x,y] \subseteq [a-\delta, b+\delta]$.

Example:



persitent homology of finite metric spaces
persistence module as a functor
maps between persistence modules as natty trans
category of barcodes
difficulty of barcode morphism, ways around by injectivity
reasons why fails: class can merge or die, barcode does not keep track
solutions: Reeb graph, merge tree. but these require knowing more
partial solution: what if know path between point samples?