Quasi locally constant functions

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1 Motivation

Let X, Y be topological spaces and $(A, \leqslant)$ a poset. Let $\varphi\colon X \to Y$ be a function.

Def: φ is locally constant if for every $x \in X$, there is $U \ni x$ open so that $\varphi|_U$ is constant.
Def: φ is A-constructible if there is a continuous function $f\colon X \to (A, \leqslant)$ such that $\varphi|_{f^{-1}(a)}$ is locally constant, for all $a \in A$.

The pair (X, f), or just X when f is clear, is called a "A-stratified space." f is an "A-stratification" of X. Often we have $Y = A$. Continuity can be hard to check. Extending this def to sheaves causes issues with restriction sheaf.

Ex 1: If X is connected, φ locally constant $\implies$ φ constant.
Ex 2: $\varphi(x) = \lceil x \rceil$ is not locally constant $\mathbf{R} \to \mathbf{R}$, but is $\mathbf{Z}$-stratified by $f(x) = \lceil x \rceil$, as function $\mathbf{R} \to (\mathbf{Z}, \leqslant)$.

Question: What happens to φ as a boundary is crossed? How do φ and f hold this data?

2 Simplicical complexes and partial orders

Def: An abstract simplical complex is a pair $C = (V(C), S(C))$, where $V(C)$ is a set and $S(C) \subseteq P(V(C))$ is closed under taking subsets. A simplicial map $C \to C'$ is a set map $V(C) \to V(C')$ whose natural extension $S(C) \to S(C')$ is well-defined.

Let SC be the set of abstract simplicial complexes. Put a binary relation $\leqslant$ on SC by $C \leqslant C'$ if there exists a simplicial map $C' \to C$ that is surjective on vertices.

Prop: The relation $\leqslant$ is a partial order.
Uses partial order of set containment.

3 The Ran space

Let M be a manifold.
Def: The Ran space of M is $\mathrm{Ran}^{\leqslant n}(M) := \{P \subseteq M : 0 < |P| \leqslant n\}$.
Prop: There natural map $\mathrm{Ran}^{\leqslant n}(M) \to \mathbf{Z}$ is continuous.

Topologize $\mathrm{Ran}^{\leqslant n}(M)$ by the topology induced by the Hausdorff distance of sets.
Def: The Čech map $\check{C}\colon \mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0} \to (\mathsf{SC}, \leqslant)$ is defined by:
  $V(\check{C}(P, r)) = P$
  $P' \in S(\check{C}(P, r))$ iff $\bigcap_{p' \in P'} \overline{B}_M(p', r) \neq \emptyset$.
Thm: The Čech map is continuous.

Thm: Every path $\gamma\colon I \to \mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$ satisfying $\check{C}(\gamma(t)) \leqslant \check{C}(\gamma(t'))$ whenever $t \leqslant t'$, induces a unique simplicial map $\check{C}(\gamma(0)) \to \check{C}(\gamma(1))$.

Paths like this are called entry paths.
Ex: For every $P \in \mathrm{Ran}^{\leqslant n}(M)$, the (infinite) path $\{P\} \times \mathbf{R}_{\geqslant 0}$ is an entry path.

Taking the homology of $\check{C}|_{\{P\}\times\mathbf{R}_{\geqslant 0}}$ gives the persistent homology of P.
Goal: Extend an entry path (in the $\mathbf{Z}$-stratification) from P to Q to a morphism of diagrams

$$H_k(\check{C}|_{\{P\}\times\mathbf{R}_{\geqslant 0}}) \to H_k(\check{C}|_{\{Q\}\times\mathbf{R}_{\geqslant 0}}).$$