Constructible cosheaves over the Ran space

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Stratifying by simplical complex type

Let M be a smooth, compact, connected, manifold, and let SC be the category of finite simplicial complexes and simplicial maps. Order SC by setting $C \leqslant D$ whenever there is a simplicial map $D \to C$ that is surjective on vertices.

Definition: The **Ran space** of M is $\text{Ran}(M) := \{P \subseteq M \ : \ 0 < |P| < \infty\}$, with topology induced by Hausdorff distance of subsets of M.

Given a positive radius, every element of $\text{Ran}(M)$ has a simplicial complex associated to it. The Čech map $\check{C}\colon \text{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0} \to \text{SC}$ that produces this simplicial complex works well with the partial order on SC.

Theorem: The Čech map is continuous.

Hence the Čech map makes $\text{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$ into an SC-stratified space. This space is not conically stratified, because strata of the same dimension are next to each other (see Example 2 below), so we introduce new strata at such frontiers.

Definition: A **frontier simplicial complex** C is a triple $(V(C), S(C), F(C))$ where $(V(C), S(C))$ is a simplicial complex and $F(C) \subsetneq S(C)$ is closed under taking supersets in $S(C)$.

The category SCF is defined analogously to SC. Order SCF by setting $C \leqslant D$ whenever there is a simplicial map $D \to C$ that is surjective on vertices and injective on frontier simplices (elements of $F(C)$).

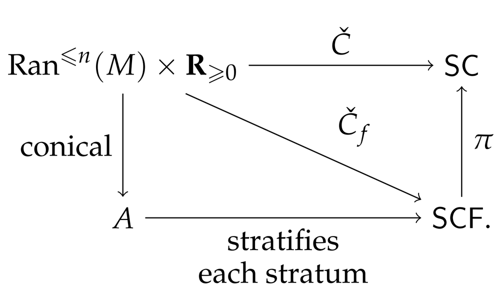
Let $\check{C}_f\colon \text{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0} \to \text{SCF}$ be the map with $V(\check{C}_f(P,r)) = P$ and

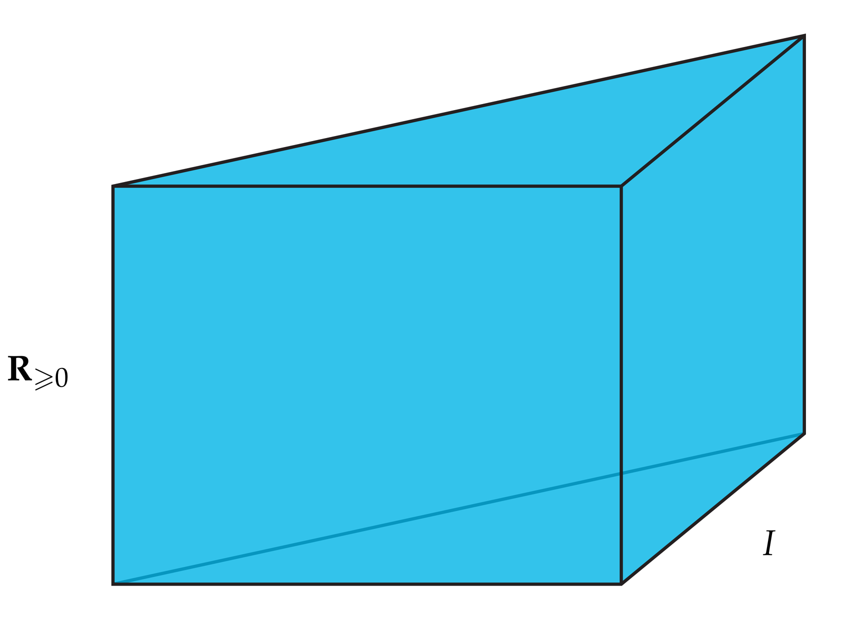
- $P' \in S(\check{C}_f(P,r))$ whenever $\bigcap_{p \in P'} B(p,r) \neq \varnothing$,
- $P' \in F(\check{C}_f(P,r))$ whenever $\bigcap_{p \in P'} B(p,r) \neq \varnothing$ and is of measure zero,

for every $P' \subseteq P$. The map $\check{C}_f$ is continuous by similar arguments as for $\check{C}$.

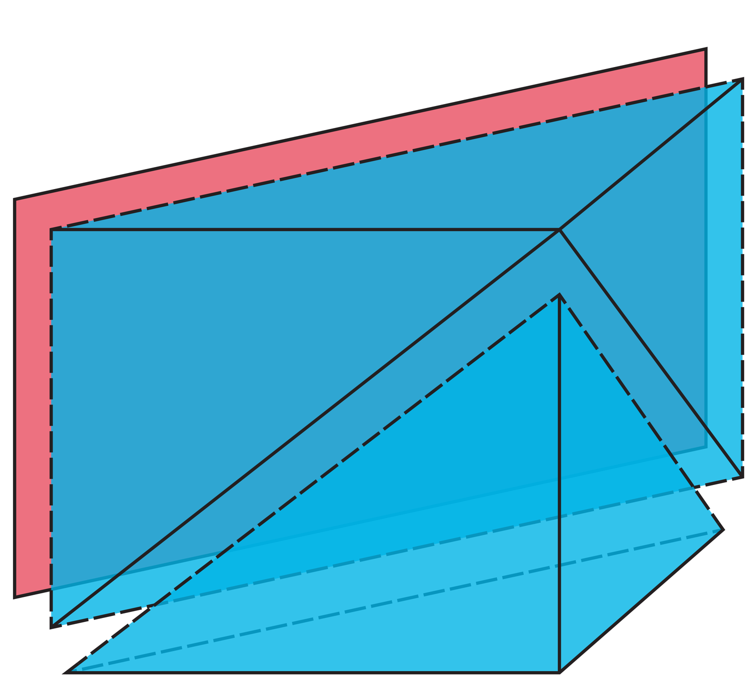
Theorem: If M is piecewise-linear, there exists a conical stratification of $\text{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$ compatible with $\check{C}_f$.

A stratification of a space is compatible with another stratification if the restriction of the first to strata of the second is a stratification. We now get a commutative diagram of continuous functions

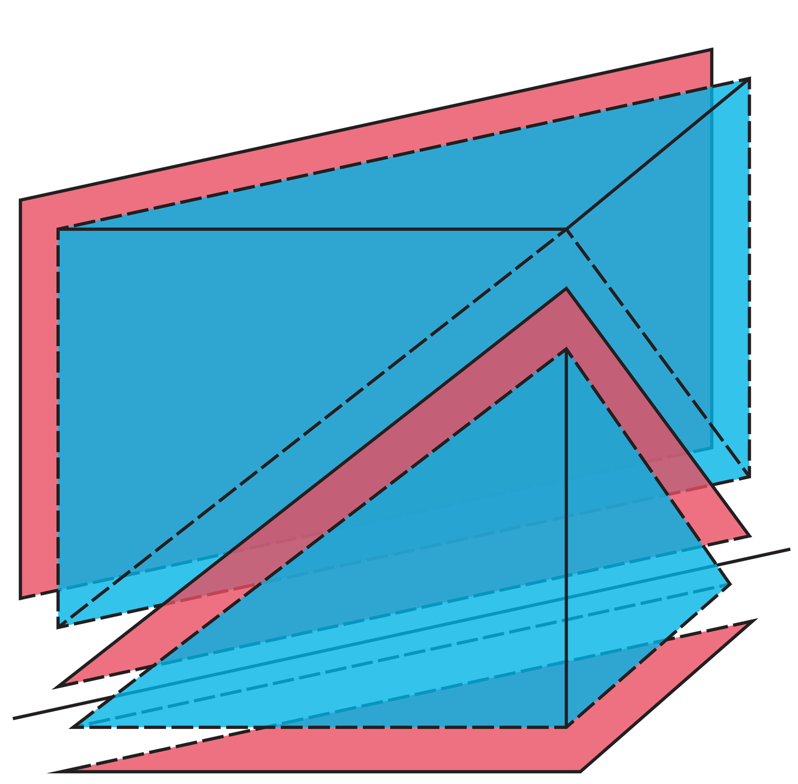




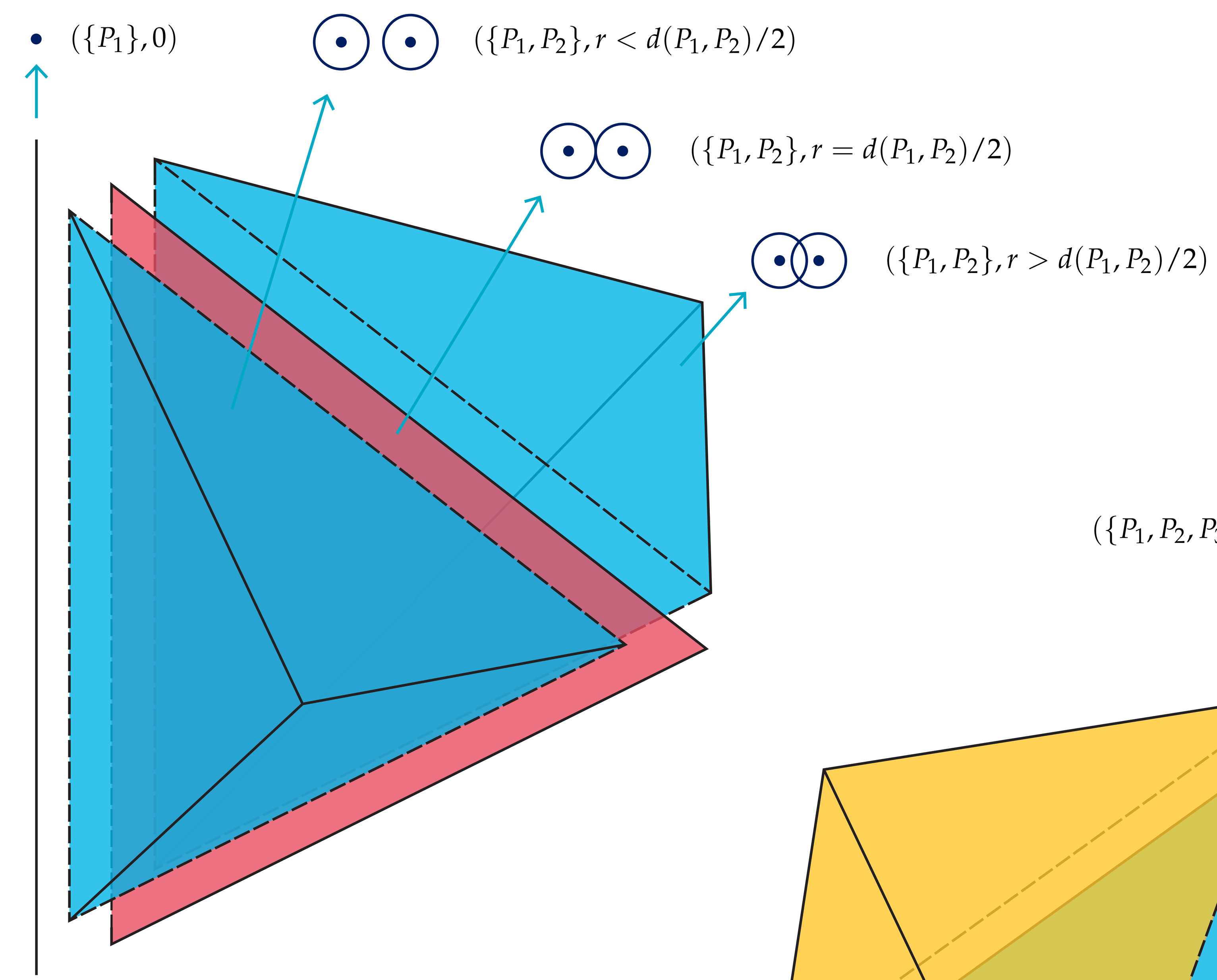
$\text{Ran}^{\leqslant 2}(I) \times \mathbf{R}_{\geqslant 0}$

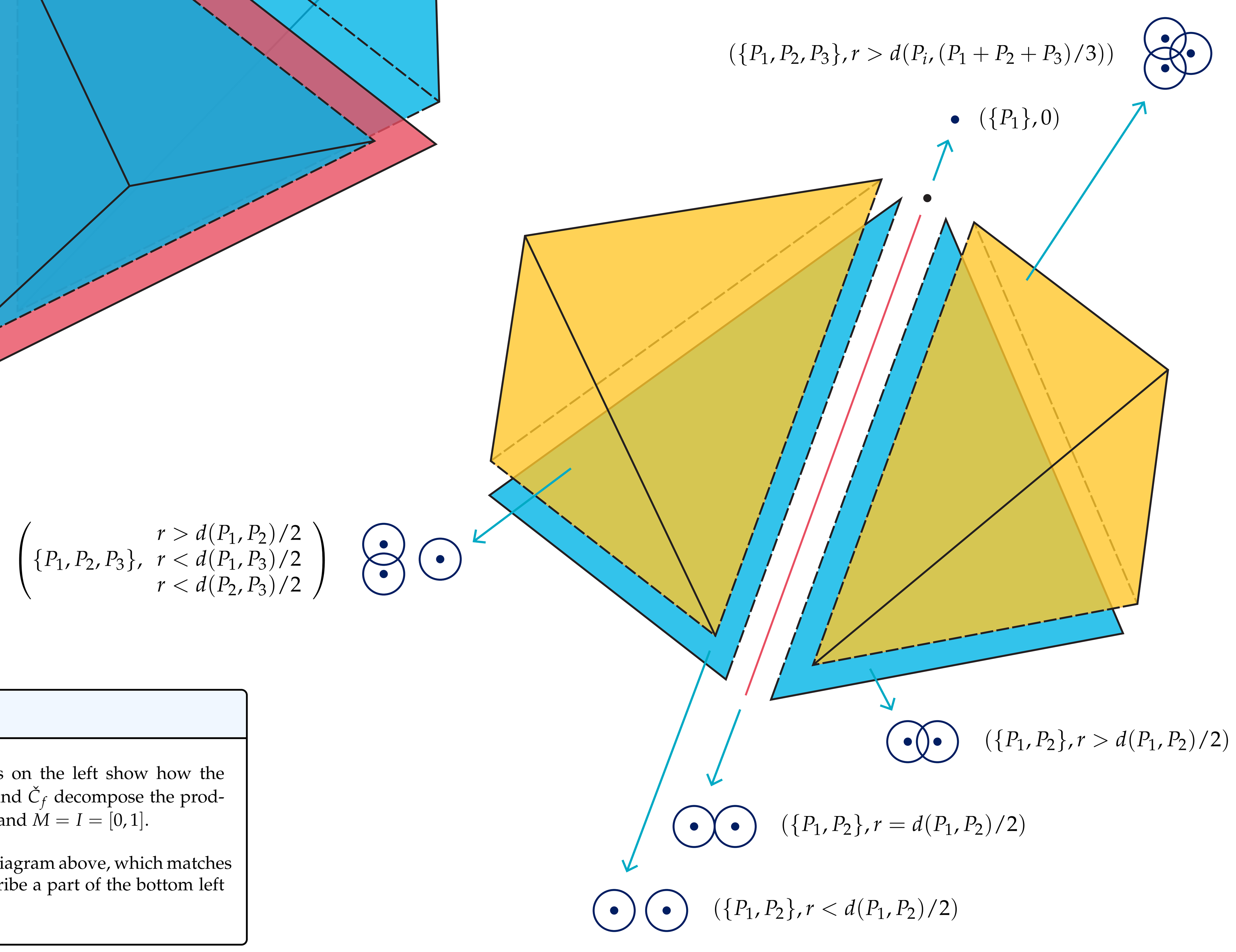


$\check{C}\left(\text{Ran}^{\leqslant 2}(I) \times \mathbf{R}_{\geqslant 0}\right)$



$\check{C}_f\left(\text{Ran}^{\leqslant 2}(I) \times \mathbf{R}_{\geqslant 0}\right)$





Example 1

The two diagrams on the left and below show some of the low-dimensional strata representing 1, 2, and 3 point combinations on a manifold.

The diagrams intersect in a 2-dimensional subspace, which matches up by color.

Example 2

The three diagrams on the left show how the two Čech maps $\check{C}$ and $\check{C}_f$ decompose the product space for $n = 2$ and $M = I = [0,1]$.

Compare with the diagram above, which matches up by color to describe a part of the bottom left diagram.

The homotopy category of exit paths

Definition: An **exit path** in an A-stratified space X is a continuous map $\sigma\colon |\Delta^k| \to X$ for which there exists a chain $a_0 \leqslant \cdots \leqslant a_k$ in A such that $f(\sigma(t_0,\dots,t_i,0,\dots,0)) = a_i$ for $t_i \neq 0$. An **entry path** is an exit path with the opposite indexing of coordinates.

The full subcategory of $\text{Sing}(X)$ of exit paths respecting the A-stratification is denoted $\text{Sing}^A(X)$. We can consider homotopy classes of exit paths, forming the homotopy category $\text{Ho}(\text{Sing}^A(X))$.

Definition: Two exit paths $\sigma, \sigma' \in \text{Sing}^A(X)_1$ with common endpoints are **homotopic** if there exists a 2-simplex $\tau \in \text{Sing}^A(X)_2$ for which $d_2\tau = \sigma$, $d_1\tau = \sigma'$, and $d_0\tau = c_{\sigma(0)}$ is the constant path at $\sigma(0)$.

Proposition: Every equivalence class of paths $[\gamma] \in \text{Ho}(\text{Sing}^{\text{SC}}(\text{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}))$ defines a unique simplicial map $\check{\gamma}\colon \check{C}(\gamma(1)) \to \check{C}(\gamma(0))$.

Definition: Let $F\colon \text{Ho}(\text{Sing}^{\text{SC}}(\text{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0})) \to \text{SC}$ be the functor defined on objects by $F(P,r) = \check{C}(P,r)$ and on morphisms by $F([\gamma]) = \check{\gamma}$.

The functor F has some nice properties:

- It is cofinal, so combines colimits
- Restriction induces functors $F_U\colon \text{Ho}(\text{Sing}^{\text{SC}}(U)) \to \text{SC}$ for all subsets U

Applications

Since F is cofinal, it works well with colimit structures.

Theorem: The functor $\mathcal{F}\colon \mathsf{Op}(\text{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}) \to \text{SC}$ given by $U \mapsto \text{colim}\, F_U$ is an SC-constructible cosheaf.

We can fix an element $P \in \text{Ran}^{\leqslant n}(M)$ and consider its lifetime over $\mathbf{R}_{\geqslant 0}$.

Proposition: The kth persistence module of P is the functor $PM_k\colon (\mathbf{R}, \leqslant) \to \text{Vect}$ given by $r \mapsto H_k(\mathcal{F}_{(P,r)})$ and $(r \leqslant s) \mapsto H_k(F_{\{P\} \times [r,s]})$.

The functor $F_{\{P\} \times [r,s]}$ may be viewed as a simplicial map, as there are finitely many times $r = t_0 < \cdots < t_N = s$ such that the natural path from (P, t_i) to (P, t_{i-1}) in $\text{Ho}(\text{Sing}^{\text{SC}}(\text{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}))$ is an exit path for all $i = 1, \dots, N$. The order on SC then gives a chain of simplicial maps.

The same construction of F may be repeated for the conical A-stratification, and then $\text{Sing}^A(\text{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0})$ is an ∞-category, by Lurie. As nerves are spaces, the homotopy category-nerve adjunction means we have a functor

$$F\colon \text{Sing}^A(\text{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}) \to N(\text{SC}).$$

Again by Lurie, we then get an A-constructible sheaf on $\text{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$.

Sheaves and cosheaves

Let $\mathcal{F}$ be a presheaf on X, $\mathcal{G}$ a precosheaf on X. For every open $U \subseteq X$ and every $\mathcal{U} = \{U_i\}$ an open cover of U, there are natural maps

$$\mathcal{F}(U) \longrightarrow \lim_{V \in \mathcal{U}} \mathcal{F}(V), \qquad \operatorname*{colim}_{V \in \mathcal{U}} \mathcal{G}(V) \longrightarrow \mathcal{G}(U).$$

If the first is an isomorphism, $\mathcal{F}$ is a sheaf on X. If the second is an isomorphism, $\mathcal{G}$ is a cosheaf on X.

Definition: Let $f\colon X \to A$ be an A-stratification and $\mathcal{F}$ a pre(co)sheaf on X. For every $a \in A$, there is another natural presheaf $\mathcal{F}_a$ on X_a and another natural precosheaf $\mathcal{F}^a$ on X_a, given by

$$\mathcal{F}_a(U) = \operatorname*{colim}_{V \supseteq U} \mathcal{F}(V), \qquad \mathcal{F}^a(U) = \lim_{V \supseteq U} \mathcal{F}(V).$$

We say $\mathcal{F}$ is ***A*-constructible** if $\mathcal{F}_a$ is a locally constant (co)sheaf, and ***A*-coconstructible** if $\mathcal{F}^a$ is a locally constant (co)sheaf, for all $a \in A$.

The usual definition of constructibility only requires the sheafification to be a locally constant sheaf, so this definition is more restrictive. We may attach the adjective "strongly" to emphasize the difference.

Definition: Let $\mathcal{F}$ be a a pre(co)sheaf on X. For every $x \in X$, the **stalk** $\mathcal{F}_x$ and **costalk** $\mathcal{F}^x$ are defined as

$$\mathcal{F}_x := \operatorname*{colim}_{U \ni x} \mathcal{F}(U), \qquad \mathcal{F}^x := \lim_{U \ni x} \mathcal{F}(U).$$

There are natural maps $\mathcal{F}(U) \to \mathcal{F}_x$ and $\mathcal{F}^x \to \mathcal{F}(U)$ whenever $x \in U$.

Further reading

[1] Jacob Lurie (2017), *Higher Algebra* (Section 5.5.1, Appendix A).
[2] Masahiro Shiota (1997), *Geometry of subanalytic and semialgebraic sets*.