0.1 Setup

Recall the following terms:

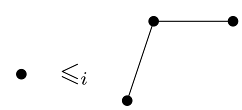
- **(weak) partial order:** A relation $\leqslant$ on a set A that is symmetric, anti-reflexive, transitive
  - Write $(A, \leqslant)$ for a partially ordered set
  - If $f\colon A \to B$ is monotonic, the order on A induces an order on B. That is, if the condition
  $$\forall\ b \in B,\ \forall\ a, a' \in f^{-1}(b),\ a \leqslant a'' \leqslant a' \implies f(a'') = b,$$
  is satisfied, then f induces a partial order on B, via $a \leqslant_A a' \implies f(a) \leqslant_B f(a')$.
- **simplicial complex:** A pair (V, S) for V a set and $S \subseteq P(V)$ (of *simplices*) closed under power sets
- **simplicial map:** A map of sets $f\colon V_1 \to V_2$

Remark: A simplicial map is usually defined as a map $: S_1 \to S_2$ with the condition that the images of the vertices of a simplex always span a simplex. This construction is equivalent to choosing a set map between the vertices of the simplicial complex.

Let SC be the category of simplicial complexes and simplicial maps. We will describe partial orders on the objects of SC.

Inclusion partial order: Let $C \leqslant_i D$ iff $S(C) \subseteq S(D)$, up to renaming of vertices.

Example:



$C = (\{x\}, \{\{x\}, \emptyset\})$ $\qquad$ $D = (\{a, b, c\}, \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \emptyset\})$

Note there are three distinct ways to rename the vertex on the left to make the relation hold. This means there are three distinct simplicial maps $C \to D$. Goal: describe relations that induce unique simplicial maps.

0.2 Induced complexes and maps

Definition: Let M be a smooth, compact, connected, embedded manifold in $\mathbf{R}^N$. The *Ran space* of M is the space of all finite subsets of M, written $\mathrm{Ran}(M) := \{P \subseteq M\ :\ 0 < |P| < \infty\}$.

- For distance / metric problems, we may assume M is PL or even $M = \mathbf{R}^N$.
- Natural quotient map $M^n \to \mathrm{Ran}^n(M) = (M^n \setminus \Delta)/S^n$, less obvious quotient $M^n \to \mathrm{Ran}^{\leqslant n}(M)$.
- Recall a *clique* of a graph is subgraph that is a complete graph.

Definition: Let $VR\colon \mathrm{Ran}(M) \times \mathbf{R}_{>0} \to \mathrm{Obj}(SC)$ be the map defined by
$$VR(P, t) := \text{clique complex}\left(\{\text{distinct } \{p, q\} \in P \times P\ :\ d(p, q) \leqslant 2t\}\right).$$
Let $\check{C}\colon \mathrm{Ran}(M) \times \mathbf{R}_{>0} \to \mathrm{Obj}(SC)$ be the map defined by
$$V(\check{C}(P, t)) = P, \qquad S(\check{C}(P, t)) = \bigcup_{k=2}^{|V|} \left\{\left\{\text{distinct } \{p_1, \ldots, p_k\} \subseteq P^k\ :\ \bigcap_{i=1}^k B(p_i, t) \neq \emptyset\right\}\right\}$$

Remark: The VR construction seems simpler than the Č construction. Also:

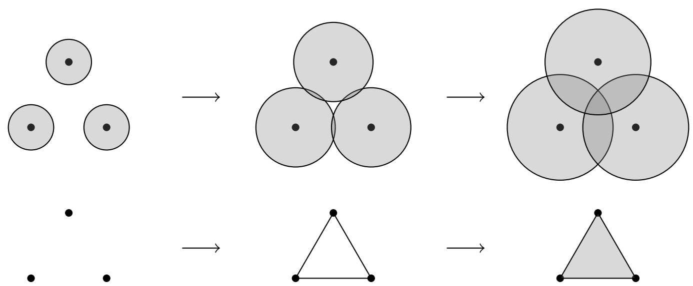
- The VR construction is also a special case of the Č construction, where the union is only for $k = 2$.
- $\check{C}(P, t) \leqslant_i VR(P, t)$, and the clique complex of $\check{C}(P, t)$ is $VR(P, t)$.

Remark: The space $\mathrm{Ran}(M) \times \mathbf{R}_{>0}$ has the Hausdorff topology on it, so it has a notion of "path" in it. Through the VR and Č maps, we can interpret a path $\gamma\colon I \to \mathrm{Ran}(M) \times \mathbf{R}_{>0}$ as a map of simplices $f(\gamma(0)) \to f(\gamma(1))$, for $f = VR, \check{C}$. However, this may not always work in the way we want:

0.3 A partial order that induces simplicial maps

For simplicity, fix $n \in \mathbf{Z}_{>0}$. Define a partial order on SC by considering the preimage of $\check{C}$ in $M^n \times \mathbf{R}_{>0}$, rather than $\mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{>0}$.

Motivation: When a collection of points moves closer together, or the radius increases, we have induced unique simplicial maps. For example:



Let $A := \{1 < 2 < 3\}$. The product A^N has the product order. Let T be the set of all distinct 1-,2-,...,k-tuples in $\{1, \ldots, n\}$. As a set, this is

$$T := \bigcup_{k=2}^{n} \left(\{1, \ldots, n\}^k \setminus \Delta\right) /_{S_k}.$$

Every $v \in T$ induces a natural projection $\pi_v\colon M^n \to M^{|v|}$. This gives another map

$$\begin{aligned} \pi_v'\colon M^n \times \mathbf{R}_{>0} &\to A, \\ (P, t) &\mapsto \begin{cases} 1 & \forall\, i, j, \pi_v(P)_i = \pi_v(P)_j, \\ 2 & \exists\, i, j \text{ s.t. } \pi_v(P)_i \neq \pi_v(P)_j \text{ and } \bigcap_{i=1}^{|v|} B(\pi_v(P)_i, t) \neq \emptyset, \\ 3 & \exists\, i, j \text{ s.t. } \pi_v(P)_i \neq \pi_v(P)_j \text{ and } \bigcap_{i=1}^{|v|} B(\pi_v(P)_i, t) = \emptyset. \end{cases} \end{aligned}$$

This map is continuous on $(\pi_v')^{-1}(A) \cong M^{|v|} \times \mathbf{R}_{>0}$, as the balls are closed. Use this construction on all possible k-tuples to get a partial order on SC.

Proposition: $\exists$ a continuous map $f\colon M^n \times \mathbf{R}_{>0} \to A^{\sum_{k=2}^{n}\binom{n}{k}}$ and a monotonic map $g\colon \mathrm{im}(f) \to SC$.

Proof. (Sketch) Define the map f as

$$\begin{aligned} f\colon M^n \times \mathbf{R}_{>0} &\to A^{\sum_{k=2}^{n}\binom{n}{k}}, \\ (P,t) &\mapsto \prod_{v \in T} \pi_v'(P,t). \end{aligned}$$

For continuity, take $a \in A^{\sum_{k=2}^{n}\binom{n}{k}}$ and $(Q,s) \in f^{-1}(a) \neq \emptyset$. Since all points are a positive distance away from each other, we can find a ball small enough (in the Hausdorff topology) so that no new edges are created / lost within the ball (between exsiting points), and no points collide. We may get new points being "born," but this means we are moving up in the partial order on A^N, so we stay in the open set.

Define the map g as

$$\begin{aligned} g\colon \mathrm{im}(f) &\to \mathrm{Obj}(SC), \\ a &\mapsto \check{C}(a' \in f^{-1}(a)). \end{aligned}$$

This map is well-defined because $\check{C}$ is constant on $f^{-1}(a)$, as a indicates wether or not there is a k-simplex for every possible choice of k points from the vertex set. This map is monotone because for every distinct pair ... $\square$

Since product order on the As induces a partial order on SC.

Corollary: There is an induced continuous map $\mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{>0} \to SC$, given by h below.

$$\begin{array}{ccccc} M^n \times \mathbf{R}_{>0} & \xrightarrow{f} & (A, \leqslant) & \xrightarrow{g} & (SC, \leqslant) \\ \downarrow{\scriptstyle \pi \times \mathrm{id}} & & & \overset{h}{\nearrow} & \\ \mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{>0} & & & & \end{array}$$

Now, whenever we have a descending path $\gamma\colon I \to \mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{>0}$, that is, $h(\gamma(t)) \leqslant h(\gamma(s))$ whenever $t \geqslant s$, there is a unique induced simplicial map $\check{C}(\gamma(0)) \to \check{C}(\gamma(1))$.