Pipeline: Data → simplicial complex (family) → vector spaces (family) → barcodes
The "barcode" of data is easier to interpret.

0.1 Persistent homology

Setup: (embedded) space X, map $f : X \to \mathbf{R}$
Calculate: *sublevel sets* $X_t = \{x \in X \ : \ f(x) \leqslant t\}$, homology $\{H_n(X_t)\}_{t\in\mathbf{R}}$
Simplify: over field k and if fin-dim, Krull–Remak–Schmidt–Azumaya says there is a unique decomposition

$$H_n(X_t; k) = \bigoplus_{i=1}^{n_t} k \qquad \text{and} \qquad H_n(X_t; k) \to H_n(X_{t+\epsilon}; k)$$

is either identity or 0 on k components of source. Length of $\mathrm{id} : k \to k$ is a bar in the barcode

Example (height function, distance from subset of $\mathbf{R}^N$)

currently: "persistence module" functor $(\mathbf{R}, \leqslant) \to \mathrm{Vect}$, map (no functor) $\mathrm{Vect} \to \{(I_j, \ell_j)\}_j \subseteq \mathbf{R} \times \mathbf{Z}_{>0}$
goal 1: define cat of barcodes, make map Vect → Barc functorial
goal 2: replace $(\mathbf{R}, \leqslant) \to \mathrm{Vect}$ with Spaces → Vect or at least FinSpaces → Vect

0.2 Functoriality

Start with goal 1.
Problem: no functor because not keeping track of death / combining difference
Solution: keep track with bases

BVect, FPmod, BPVect, functor $\mathcal{B}$

How does it fit in to existing picture? Need to interpret objects of $\mathcal{B}(\mathrm{BPVect})$ as collections $\{(I_j, \ell_j)\}_j$.
Let $S \in \mathrm{Set}_*$ in the image of $\mathcal{B}$. There is a partial order on

$$Pairs := \bigcup_{t\in\mathbf{R}} \ \bigcup_{i\in\mathcal{B}(V_t,B_t)} \{(t, i)\}$$

Goal 2.
Finite spaces are finite point samples, or $\mathrm{Ran}(\mathbf{R}^N) = \{P \subseteq \mathbf{R}^N \ : \ 0 < |P| < \infty\}$
Morphisms between them are paths in Ran space