0.1 Background and motivation

Let X be a topological space, $f : X \to \mathbf{R}$ a function, and $X_{\leqslant t} = \{x \in X \ : \ f(x) \leqslant t\}$.

Definition: A function $f : X \to \mathbf{R}$ is an *$\mathbf{R}$-filtration* if $X_{\leqslant t} \subseteq X_{\leqslant s}$ whenever $t \leqslant s$.

Example: Here are the main examples of $\mathbf{R}$-filtrations:

- Let M be a manifold embedded in $\mathbf{R}^N$, ℓ a line in $\mathbf{R}^N$, and $\pi : M \to \ell$ the projection map.
- Let X be a simplicial complex and $\dim : X \to \mathbf{R}$ the map that gives the dimension of an input simplex.
- Let P be a finite subset of $\mathbf{R}^N$, $\mathbf{P}(P)$ its set of subsets, and $\operatorname{diam} : \mathbf{P}(P) \to \mathbf{R}$ the map that gives the diameter of an input set.

Definition: Let f be an $\mathbf{R}$-filtration of X. A *persistence module* is a functor

$$\begin{array}{rcl} PM(f : X \to \mathbf{R}) \ : \ (\mathbf{R}, \leqslant) & \to & R\text{Mod}, \\ t & \mapsto & M_{\leqslant t}, \\ (t \leqslant s) & \mapsto & (M_{\leqslant t} \to M_{\leqslant s}) \, . \end{array}$$

(categories are clear)

Example: The main example of persistence modules are the homology functors $t \mapsto H_k(M_{\leqslant t}, A)$. Their image is called the *(kth) persistent homology* of X.

Question: What happens if $f : X \to \mathbf{R}$ changes? Let's fix f and vary X, because it's easier to classify topological spaces rather than functions on them. Persistence module is now:

$$\begin{array}{rcl} PM(f : - \to \mathbf{R}) \ : \ \mathcal{S} \times (\mathbf{R}, \leqslant) & \to & R\text{Mod}, \\ (X, t) & \mapsto & M_{f^{-1}(\leqslant t)}, \\ (X, t) \leqslant (Y, s) & \mapsto & ? \end{array}$$

What should $\mathcal{S}$ be? Which f should be chosen? What is a morphism, or order, in $\mathcal{S}$?

0.2 Classification (of SCs)

Instead of $\mathcal{S} \times (\mathbf{R}, \leqslant) \to R\text{Mod}$, consider $\mathcal{S} \times (\mathbf{R}, \leqslant) \to SC \to R\text{Mod}$ (side note: SC is a strange category. We actually use simplicial sets, but will not introduce them here).

Definition: A *k-simplex* is the convex hull of $k + 1$ vertices in general position (they span a k-subspace) in $\mathbf{R}^N$. A *face* of a simplex is the convex hull of a proper subset of its vertices. A *simplicial complex* is a collection S of simplices such that

- if $K \in S$, then every face of K is in S,
- if $K, L \in S$, then $K \cap L \in S$.

A *simplicial map* $K \to L$ is defined by its action on the vertices - the image of vertices of a simplex in K must span a simplex in L.

Think of $\mathcal{S} \times \mathbf{R}_{\geqslant 0}$ as the space of simplicial complexes, via $(P, t) \mapsto VR(P, t)$. Note

$$\sigma_k \in VR(P,t) \quad \Longleftrightarrow \quad d(P_i, P_j) < t \ \forall\ 1 \leqslant i < j \leqslant k,$$

where $\sigma_k = \{P_0, \ldots, P_k\}$ is a k-simplex in $VR(P, t)$. Contrast with Čech complex (more restrictive), which contains same k-simplex iff $\bigcap_{i=0}^{k} B(P_i, t) \neq \emptyset$. We work with simplicial complexes because they are easy to work with.

Hence an element of $\mathcal{S}$ is a finite subset of $\mathbf{R}^N$. What does "the space of finite subsets of $\mathbf{R}^N$" look like?

Definition: The Ran space of $\mathbf{R}^N$ is $\mathrm{Ran}(\mathbf{R}^N) = \{P \subseteq \mathbf{R}^N \ :\ 0 < |P| < \infty\}$ with Hausdorff distance.

$$d_H(P,Q) = \max_{p\in P} \min_{q \in Q} d_{\mathbf{R}^N}(p,q) + \max_{q\in Q} \min_{p \in P} d_{\mathbf{R}^N}(p,q)$$

Can use a manifold M instead of $\mathbf{R}^N$. Can fix a finite n to cap Ran off at, for $\mathrm{Ran}^{\leqslant n}(M)$. Can fix an n, then $\mathrm{Ran}^n(M) = \mathrm{Conf}_n(M)$, the *configuration space* of n points.

Unanswered Q: How is $\mathrm{Ran}(\mathbf{R}^N)$ ordered (for functor)? Natural:

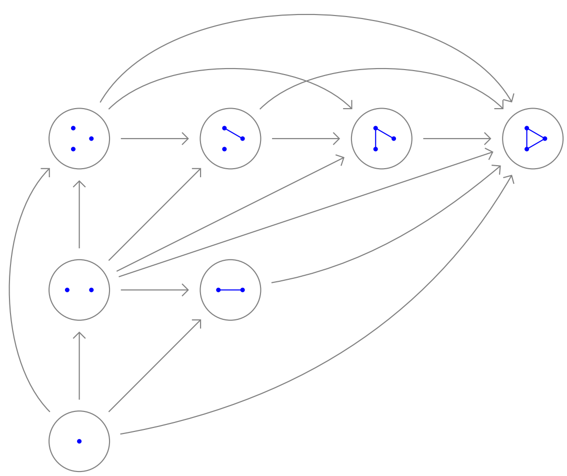
$$(P,t) \leqslant (Q,r) \quad \Longleftrightarrow \quad VR(P,t) \hookrightarrow VR(Q,r).$$

Choose $f : \mathrm{Ran}(\mathbf{R}^N) \to \mathbf{R}$ to be the diameter function.

Question: What does an "injection" of simplicial complexes look like in $\mathrm{Ran}(\mathbf{R}^N) \times \mathbf{R}$? Are there different ways to inject? How exactly do they differ?

0.3 Stratification

Motivation 1: What does $\mathrm{Ran}(M) \times \mathbf{R}_{\geqslant 0}$ look like? It is naturally divided up into pieces where $VR(P, t)$ is the same. What are these pieces? How do they border with each other?



Motivation 2: Can a sheaf be made over this space? What happens on open sets?