Let M be a compact smooth m-manifold embedded in $\mathbf{R}^N$. Let X be a topological space. Recall the following concepts:

- $\mathrm{Ran}^{\leqslant n}(M) = \{P \subset M \ : \ 0 < |P| \leqslant n\}$
- $\mathrm{Ran}^{\leqslant n}(\{U_i\}_{i\in I}) = \{P \in \mathrm{Ran}^{\leqslant n}(M) \ : \ P \subset \bigcup_{i\in I} U_i, \ P \cap U_i \neq \emptyset \ \forall \ i\}$
- the topology on $\mathrm{Ran}^{\leqslant n}(M)$ is the coarsest topology for which all $\mathrm{Ran}(\{U_i\}_{i\in I})$ are open, for every nonempty finite collection of pairwise disjoint open sets
- $2d(P,Q) = \sup\limits_{p\in P} \inf\limits_{q\in Q} d(p,q) + \sup\limits_{q\in Q} \sup\limits_{p\in P} d(p,q)$. Hausdorff distance is max of two terms

We also have some categories:

- $\mathrm{Sing}(X)$ is the category of continuous functions $\gamma \ : \ \Delta^n_{top} \to X$ and face / degeneracy maps
  - subcategory $\mathrm{Sing}^A(X)$
- $\mathrm{Shv}(X)$ is the category of sheaves and sheaf morphisms
  - subcategory $\mathrm{Shv}^A(X)$

0.1 Stratifying the Ran space

Definition: A (*poset*) *stratification* of X is a continuous map $f : X \to A$, where A is a poset. A *constructible sheaf* over $f : X \to A$ is a sheaf over X that is locally constant on every stratum $X_a = \{x \in X \ : \ f(x) = a\}$.

Motivation: Consider the space $X = \mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$ and SC, the collection of all ordered simplicial complexes (so $\{\{1,2,3\},\{(1\ 2)\}\}$ is not the same as $\{\{1,2,3\},\{(2\ 3)\}\}$). There is a natural map

$$\begin{array}{rcl} f \ : \ X & \to & SC, \\ (P,t) & \mapsto & VR(P,t), \end{array}$$

where $VR(P,t)$ is the Vietoris–Rips complex of radius t around the points of P. It seems like there should be a constructible sheaf over X valued in simplicial complexes. Let's try to build it!

Construction 1: We begin by defining a stratification. Let $A = \{S \in SC \ : \ |V(S)| \leqslant n\}$ and define a relation $\leqslant$ on A by

$$(S \leqslant T) \iff \left(\begin{array}{c} \exists \ \sigma \in \mathrm{Sing}(X)_1 \text{ such that} \\ f(\sigma(0)) = S, \ f(\sigma(t > 0)) = T. \end{array} \right)$$

Let $(A, \leqslant)$ be the poset generated by relations of the type given above, which makes $f : X \times \mathbf{R}_{\geqslant 0} \to A$ a stratification. To see this, take the open set $U_S = \{T \in A \ : \ S \leqslant T\}$ in the basis of the upwards directed topology of A, for any $S \in A$, and consider $(P,t) \in f^{-1}(U_S)$. If for all $\epsilon > 0$ we have $B^X_\epsilon(P,t) \cap f^{-1}(U_S)^C \neq \emptyset$, then for any such ϵ there exists $T_\epsilon \in A$ with $B^X_\epsilon(P,t) \cap f^{-1}(T_\epsilon) \neq \emptyset$, for $S \not\leqslant T_\epsilon$ (as $T_\epsilon \notin U_S$). This means there exists $\sigma \in \mathrm{Sing}(X)_1$ with $\sigma(0) = (P,t)$ and $\sigma(t > 0) \in f^{-1}(T_\epsilon)$, which in turn implies $S \leqslant T_\epsilon$, a contradiction. Hence f is continuous, so $f : X \to A$ is a stratification.

Problem: This defines what an SC-valued constructible sheaf could be on X by giving the value at all the stalks, but the extension to open sets is not clear. Comparing simplices is hard, because there is no vertex order.

Partial solution 1: Instead use f on $(M^{\times k}\setminus\Delta_k)\times\mathbf{R}_{\geqslant 0}$, and define $\mathcal{F}(U)$ to be the subset of Δ^k_{top} containing a simplex σ if there is at least one $(P,t)\in U$ with $\sigma\in VR(P,t)$ (note the vertices must be ordered for this to be well-defined). Then we can push the sheaf forward through the quotient map

$$(M^{\times k}\setminus\Delta_k)\times\mathbf{R}_{\geqslant 0}\xrightarrow{S_k}\mathrm{Ran}^k(M)\times\mathbf{R}_{\geqslant 0}.$$

But this gives sheaf only on one piece of $\mathrm{Ran}^{\leqslant n}(M)\times\mathbf{R}_{\geqslant 0}$, not the whole thing.

Partial solution 2: Use Lurie's equivalence $\mathrm{Shv}^A(X)\cong\mathrm{Fun}(\mathrm{Sing}^A(X),\mathcal{S})$. A first hurdle is all the new terminology. Also, there are conditions for this to work, in increasing order of restrictiveness:

- A satisfies the ascending chain condition.
- X is paracompact (compact is sufficient),
- X is locally of singular shape (locally contractible is sufficient), and
- the A-stratification of X is *conical*.

The first three hold, but f is not conical, as strata change without changing dimension.

Simplification: Try a simpler space which may have a nice stratification. Let $X=\mathrm{Ran}^{\leqslant n}(M)$ instead.

Construction 2: Let $A=\{1,\dots,n\}$ with the natural order and $f:X\to A$ be given by $P\mapsto|P|$. To check that this is continuous, we need that $\mathrm{Ran}^{\geqslant k}(M)$ is open in $\mathrm{Ran}^{\leqslant n}(M)$ for all $0<k\leqslant n$. This is true:

$$\begin{aligned}
\mathrm{Ran}^n(M)\subseteq\mathrm{Ran}^{\leqslant n}(M)\text{ is open}\quad &\Longrightarrow\quad \mathrm{Ran}^{\leqslant n-1}(M)\subseteq\mathrm{Ran}^{\leqslant n}(M)\text{ is closed}\\
&\Longrightarrow\quad \mathrm{Ran}^{\leqslant n-2}(M)\subseteq\mathrm{Ran}^{\leqslant n}(M)\text{ is closed}\\
&\Longrightarrow\quad \mathrm{Ran}^{\leqslant k}(M)\subseteq\mathrm{Ran}^{\leqslant n}(M)\text{ is closed, for all }0<k\leqslant n\\
&\Longrightarrow\quad \mathrm{Ran}^{\geqslant k}(M)\subseteq\mathrm{Ran}^{\leqslant n}(M)\text{ is open, for all }0<k\leqslant n.
\end{aligned}$$

First three conditions satisfied. Need to check conical property.

0.2 Conical stratifications

Definition: A stratified space $f:X\to A$ is *conically stratified at* x if there exist:

- a topological space Z,
- a stratified space $g:Y\to A_{>f(x)}$,
- an open embedding $Z\times C(Y)\hookrightarrow X$ whose image contains x.

There is a natural stratification $g':C(Y)\to A_{\geqslant f(x)}$, given by $g'(Y,0)=f(x)$ and $g'(y,t\neq 0)=g(y)$. The product $Z\times C(Y)$ also has a natural $A_{\geqslant f(x)}$-stratifcation by ignoring the Z factor. Here "open embedding" means "embedding whose image is open".

Construction: We will check that $f:X\to A$ is conically stratified at every $P=\{P_1,\dots,P_k\}$. Set

$$\epsilon=\frac{1}{2}\min_{i<j}d(P_i,P_j),\qquad Z=\prod_{i=1}^k oB^{\mathbf{R}^m}_\epsilon(0),\qquad Y=\coprod_{\substack{\sum\ell_i=n\\ \sum t_i=\epsilon}}\prod_{i=1}^k\left\{Q\in\mathrm{Ran}^{\ell_i}(cB^{\mathbf{R}^m}_{t_i}(0))\ :\ \mathbf{d}(0,Q)=t_i,\ \sum Q_j=0\right\}.$$

Both Z,Y are topological spaces. The first condition on elements of Y is the *cone condition*, which ensures the right topology at the cone point in $C(Y)$. The second condition on Y is the *centroid condition*, which ensures that the point to which 0 maps to is the centroid of points splitting off it, so that we don't overcount when multiplying by Z. Define

$$\begin{aligned} \varphi \;:\; C(Y)\times Z &\to X, \\ \Big(\mathrm{Ran}^{\ell_i}(cB_{t_i}^{\mathbf{R}^m}(0)), t, R\Big) &\mapsto \mathrm{Ran}^{\ell_i}(cB_{tt_i}^{M}(R_i)), \end{aligned}$$

where $t \in [0,1)$ is the cone component and $R = \{R_1, \ldots, R_k\} \in Z$ is an element of $\mathrm{Ran}^k(M)$ near P. It is sufficient to describe where the Ran^{ℓ_i} map to, as every Q inside it is only scaled by t. That is, Q at a distance t_i from 0 maps to $\varphi(Q)$ at a distance tt_i from R_i, by scaling every component Q_j by t, then changing the center 0 to R_i.

The map φ is continuous by construction, injective by the centroid condition, and a homeomorphism onto its image by the cone condition. Hence φ is an embedding, and since the image is open, it is an open embedding.

0.3 Larger picture

Observation: The space $\mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$ was not conically stratified at the boundary between strata in the same dimension. Put in new stratum of one dim lower as boundary, representing when $t = d(P_i, P_j)$ in (P, t). But then:

- What will the stratifying poset be?
- Boundary has to be ordered lower than original strata (because of cone point), seem to lose structure.
  - Why should more edges be "higher" than less edges?
  - Is there a general order on simplicial complexes with unordered vertices? What is "more structure?"
- Maybe stratify $\mathbf{R}_{\geqslant 0}$ alone, then take product of stratified spaces?