Recall how simplicial complexes are special (nicer) cases of CW complexes. The goal of this talk is to create an analogous idea with categories:

CW complexes are to simplicial complexes like simplicial sets are to infinity categories.

Recall a CW complex is just a set of instructions of how to map n-spheres, increasing in n, by their boundary to what already has been constructed.

0.1 Simplicial complexes

Recall a *(topological) k-simplex* is

$$\Delta^n_{top} = \{a_0t_0 + \cdots + a_kt_k \in \mathbf{R}^{k+1} \ : \ t_0 + \cdots + t_k = 1\},$$

where the $a_i \in \mathbf{R}^{k+1}$ are the *vertices* of the simplex and $t_i \in [0,1]$. This may also be viewed as the convex hull of the set $A = \{a_0, \dots, a_k\}$. A *face* of Δ^n_{top} is the convex hull of any proper subset of A, with the *ith face* being the convex hull of $\{a_0, \dots, \widehat{a_i}, \dots, a_k\}$. Then a *simplicial complex* S is a collection of simplices for which

- $\sigma \in S \implies \tau \in S$ for all faces τ of σ, and
- for all $\sigma, \tau \in S$, either $\sigma \cap \tau \in S$ or $\sigma \cap \tau = \emptyset$.

A *simplicial map* from a simplicial complex S to a simplicial complex T is a map $f : S \to T$ such that the images of the vertices of S span simplices in T. Hence f is determined by f on the vertices of S.

Remark. For σ a k-simplex and τ a $(k+1)$-simplex, there are natural maps

$$\begin{array}{rcccl} s_i & : & \sigma & \hookrightarrow & \tau, \\ & & \sigma & \mapsto & (i\text{th face of } \tau), \end{array} \qquad \begin{array}{rcccl} d_i & : & \tau & \twoheadrightarrow & \sigma, \\ & & (i\text{th face of } \tau) & \mapsto & \sigma, \end{array}$$

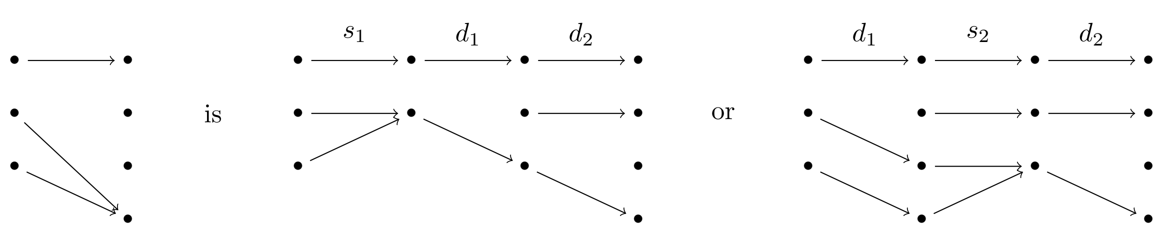
for all $0 \leqslant i \leqslant k+1$. The vertices follow the ordering in the images.

0.2 Simplicial sets + examples

Now we will define a category Δ. The objects are $[n] = (0, 1, \dots, n)$ and the morphisms are non-decreasing (equivalently order-preserving) maps $[n] \to [m]$. Note that every morphism is a composition of:

$$\begin{array}{rl} \textit{coface maps} & s^i : [n] \to [n-1], \text{ hits } i \text{ twice } (i, i+1 \mapsto i) \\ \textit{codegeneracy maps} & d^i : [n] \to [n+1], \text{ skips } i \end{array}$$

For example:



Remark: Every object $[n]$ of Δ may be viewed as a category by itself. It contains $n+1$ objects, and $|\mathrm{Hom}_{[n]}(a,b)| = 1$ iff $a \leqslant b$, and is empty otherwise.

Definition: Let sSet $=$ Fun$(\Delta^{op}, \mathrm{Set})$ be the category of *simplicial sets*. An object (functor) may be described as $S = \{S_n = S([n])\}_{n \geqslant 0}$ with

$$\begin{array}{rl} \textit{face maps} & S(s^i) = s_i : S_n \to S_{n+1} \\ \textit{degeneracy maps} & S(d^i) = d_i : S_n \to S_{n-1}. \end{array}$$

Morphisms $f : S \to T$ are natural transformations.

Now we give some standard examples of simplicial sets.

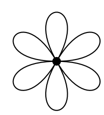
Example 1: The *standard k-simplex* Δ^k is a simplicial set, with $\Delta^n_k = \mathrm{Hom}_\Delta([k],[n])$. The collection of morphisms $[k] \to [n]$ is a set. A morphism $\varphi : [\ell] \to [k]$ induces a morphism $\Delta^n_k \to \Delta^n_\ell$, by $\alpha \in \mathrm{Hom}_\Delta([k],[n])$ becoming $\alpha \circ \varphi \in \mathrm{Hom}_\Delta([\ell],[n])$, so contravariance is also satisfied.

Example 2: Let C be a category. The *nerve* $N(C)$ of the category C is a simplicial set defined by $N(C)_n = \mathrm{Fun}([n], C)$. Note that

$$\begin{array}{rcl} N(C)_0 & = & \text{objects of } C \\ N(C)_1 & = & \text{morphisms of } C \\ N(C)_2 & = & \text{pairs of composable arrows of } C \\ & \vdots & \\ N(C)_n & = & \text{strings of } n \text{ composable arrows of } C \end{array}$$

Hence $N(C)_n$ can be thought of as a path of n segments in the category C.

Example 2.1: Consider the category $C = S_3$. This is a particular type of category, a *groupoid*, with a single object. The single object is the collection $\{e, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$ of the elements of S_3, and the morphisms, all invertible, are multiplication by these same elements. Elements of $N(S_3)_n$ should be viewed as all the ways to multiply by n elements of S_3. The nerve looks like this:

$N(S_3)_0 = \bullet$ $\qquad$ $N(S_3)_{k \geqslant 1} =$ 

Example 3: Let X be a topological space. Then $\mathrm{Sing}(X)$ is a simplicial set, with $\mathrm{Sing}(X)_n = \mathrm{Hom}_{\mathrm{Top}}(\Delta^n_{top}, X)$.

Example 4: A ordered directed graph $G = (V, E)$ is a simplicial set, by setting $G_n = \mathrm{Fun}([n], G)$, where G is interpreted as glued copies of $[1]$. If we call it G, then $G_0 = V$, $G_1 = V \cup E$ and G_k for $k \geqslant 2$ does not contain any non-degenerate maps

Definition: Let S be a simplicial set. The *geometric realization* of S is a topological space $|S|$ created in the following way:

$$\begin{array}{rcl} S_n \ni s & \mapsto & \Delta^n_{top}, \\ (s_i : S_n \to S_{n+1}) & \mapsto & \{\Delta^n_{top} \to \Delta^{n+1}_{top}\}, \\ (d_i : S_n \to S_{n-1}) & \mapsto & \{\Delta^n_{top} \to \Delta^{n-1}_{top}\}. \end{array}$$

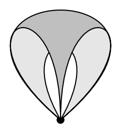
0.3 Infinity categories

Remark: Let X be a simplicial complex. Then $X = |\text{Sing}(X)|$. This called an *adjunction* between the geometric realization and Sing.

Definition: An ∞-category is a simplicial set S whose geometric realization $|S|$ is well-defined.

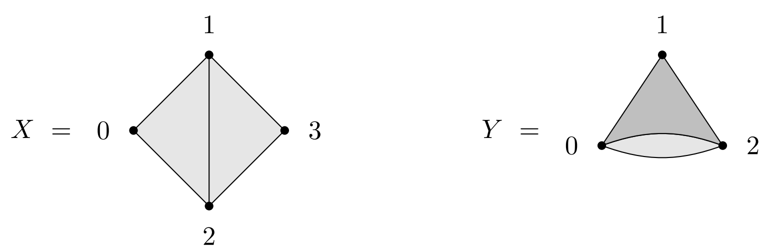
Example 1: The nerve of any category is an ∞-category. Consider $|N(S_3)|$ as before. A pair of elements has a unique composition, for example, (1 3)(1 2) = (1 3 2). Since this is unique, the geometric realization is well-defined.

$$|N(S_3)_2| \quad \supset$$



All 2-cells in $N(S_3)_2$ are glued by what the composition of two elements is. For example, we glue one with edges (1 2), (1 3), and (1 3 2).

Example 2: When X is not a simplicial complex, $|\text{Sing}(X)|$ doesn't make sense. Consider the two CW complexes below, only the one on the right is a simplicial complex.



On X, the map $s_0 : X_1 \to X_2$ takes the edge $(1, 2)$ to an edge of the 2-simplex $(0, 1, 2)$, and $s_3 : X_1 \to X_2$ takes it to an edge of $(1, 2, 3)$. However, in Y, the map $s_2 : Y_1 \to Y_2$ should take the edge $(0, 1)$ to the edges of both of the elements of Y_2, which is not a function (functions cannot be multi-valued).