0.1 What is the Ran space?

Let M be a compact manifold.

Definition: The *Ran space* of M is $\mathrm{Ran}(M) := \{S \subset M \ : \ 0 < |S| < \infty\}$.
mention n, $\leqslant n$

Topology: Let $\{U_i\}$ be a collection of open subsets of M. Set

$$\mathrm{Ran}(\{U_i\}) = \{P \in \mathrm{Ran}(M) \ : \ P \subseteq \textstyle\bigcup_i U_i,\ P \cap U_i \neq \emptyset \ \forall \ i\} \subseteq \mathrm{Ran}(M).$$

A *neighborhood* of $P = \{P_1, \dots, P_n\} \in \mathrm{Ran}(M)$ is

$$\mathrm{Ran}(\{U_i\}_{i=1}^n \ : \ U_i \text{ is an open neighborhood of } p_i,\ U_i \cap U_j = \emptyset \iff i \neq j\}.$$

The topology on $\mathrm{Ran}(M)$ is the coarsest topology for which neighborhoods of all $P \in \mathrm{Ran}(M)$ are open.

Example: $\mathrm{Ran}^{\leqslant 2}(I)$ draw square, diagonal for Ran^1

Theorem: (Beilinson–Drinfeld, 1995) If M is path-connected, then $\mathrm{Ran}(M)$ is weakly contractible.

Now fix an embedding of M into $\mathbf{R}^N$, for large enough N.

Extension: Consider the space $X = \mathrm{Ran}(M) \times \mathbf{R}_{\geqslant 0}$. There is a natural map from X to the space of simplicial complexes by $(P, t) \mapsto \check{C}ech(P, t)$, the Čech complex of radius t. Recall that for a k-simplex σ,

$$\sigma = \underbrace{\{\sigma_1, \dots, \sigma_k\}}_{\subseteq P = \{P_1, \dots, P_n\}} \in \check{C}ech(P, t) \quad \iff \quad B(\sigma_i, t) \cap B(\sigma_j, t) \neq \emptyset \ \forall \ 1 \leqslant i, j \leqslant k.$$

Usually take Euclidean distance in $\mathbf{R}^N$. draw simple example

0.2 Stratifying $\mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$

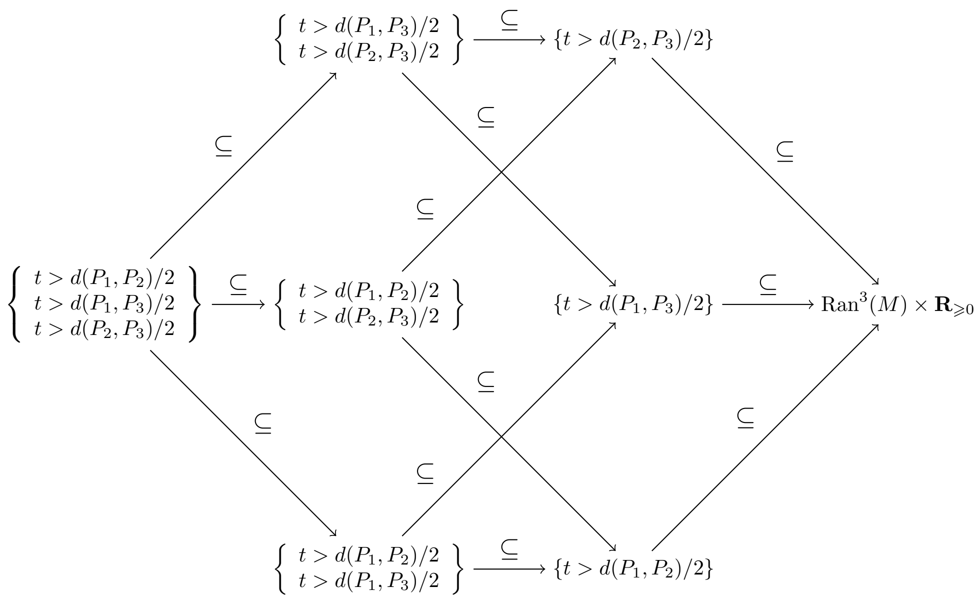
Motivation: What types of simplicial complexes do we get from X? Types up to homotopy?

Definition: A *stratification* on a topological space X is a continuous map $f : X \to (A, \leqslant)$.

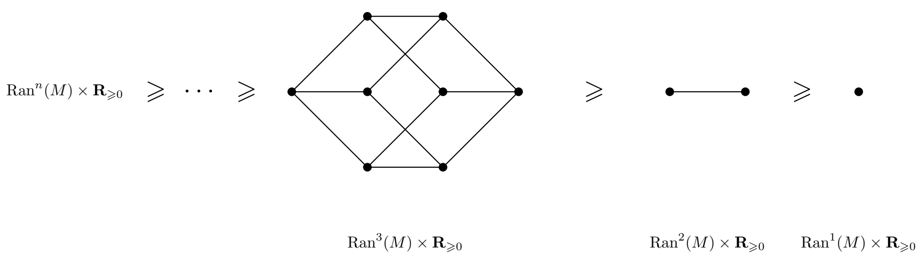
Definition: A subset $U \subseteq A$ is open in the *upset topology* on $(A, \leqslant)$ if $x \in U$ and $x \leqslant y$, then $y \in U$.
tree example, cube example, disconnected example. draw circles for sets.

Claim: $\mathrm{Ran}^{\leqslant n}(M) \times \mathbf{R}_{\geqslant 0}$ is stratified (in the product topology).

Proof: First stratify $\text{Ran}^k(M) \times \mathbf{R}_{\geqslant 0}$ for all $1 \leqslant k \leqslant n$. For example, when $k = 3$:



$\text{Ran}^k(M)$ has $2^{1+\cdots+k}$ nodes. All edges are open. Note $\text{Ran}^n(M)$ is open in $\text{Ran}^{\leqslant n}(M)$ (points can't split, can only merge). Hence $\text{Ran}^{\geqslant k}(M)$ is open in $\text{Ran}^{\leqslant n}(M)$. Make image into poset as below.



Preimages of opens are open, so cts. □

0.3 Exit paths on stratifications

Motivation: Classify all (A-constructible) sheaves on X. But also, more geometric structure?

Embed simplices into a stratified space.

Definition: An *exit path* in an A-stratified space X is a continuous map $\gamma : [0,1] \to X$ for which there exists a pair of chains $a_1 \leqslant \cdots \leqslant a_n$ in A and $0 = t_0 \leqslant \cdots \leqslant t_n = 1$ in $[0,1]$ such that $f(\gamma(t)) = a_i$ whenever $t \in (t_{i-1}, t_i]$.

This really is a path, and so gives good intuition for what is happening. Recall that the *geometric realization* of the n-simplex Δ^n is $|\Delta^n| = \{(t_0,\ldots,t_n) \in \mathbf{R}^{n+1} \ : \ t_0 + \cdots + t_n = 1\}$. Oserving that $[0,1] \cong |\Delta^1|$, this definition may be generalized by instead considering maps from $|\Delta^n|$.

Definition: An *exit path* in an A-stratified space X is a continuous map $\gamma : |\Delta^n| \to X$ for which there exists a chain $a_0 \leqslant \cdots \leqslant a_n$ in A such that $f(\gamma(t_0,\ldots,t_i,0,\ldots,0)) = a_i$ for $t_i \neq 0$.

Example: Consider a particular $\gamma : |\Delta^2| \to \mathrm{Ran}^{\leqslant 2}(M)$.