Neighborhoods of embedded manifolds 30.11.2016

0.1 Definitions

M is a d-manifold embedded in $\mathbf{R}^n$. We will conflate spaces and their embeddings.

$$T_pM \oplus N_pM \cong \mathbf{R}^n \qquad \qquad B^d_\epsilon \longrightarrow N^\epsilon M$$

$$(N^\epsilon M)_p = N_pM \cap B^n_{\epsilon,p} \qquad \qquad N^\epsilon M \longrightarrow M$$

The *conditioning number* of M is

$$\tau = \sup_{\substack{\text{embeddings} \\ N^\epsilon M}} \epsilon.$$

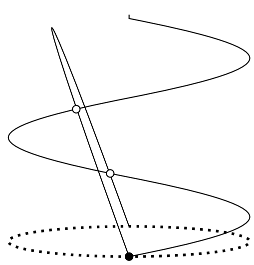
0.2 How to find τ (mfld known)

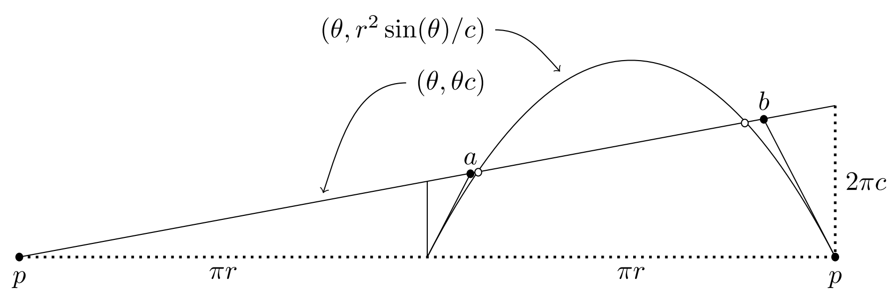
0.2.1 Examples

- circle: $\tau = r$
- n-sphere: $\tau = r$
- torus: $\tau = \min\{b, a-b\}$
- helix: $M = \{(r\cos(z/c), r\sin(z/c), z) \ : \ z \in \mathbf{R}\}$ has radius r and period $2\pi c$

1. Mathematica helix visualization
2. Locally: intersection of normal planes $\tau^\ell_p = \frac{r^2+c^2}{r}$
3. Globally: only need to consider local and $\tau = \min\limits_{\substack{p\in M \\ q\in N_pC\cap C}} \{\tau^\ell_p, d(p,q)/2\}$

To find intersection of normal plane with curve, unfold cylinder with intersection of normal plane:





Approximate with tangent lines. Too big when $c > r/\sqrt{3}$.

0.2.2 Generalize to d-manifolds

1. The curvature of $\gamma : \mathbf{R} \to \mathbf{R}^n$ at $\gamma(t)$ is $\kappa(t) = \sqrt{|\dot\gamma|^2|\ddot\gamma|^2 - (\dot\gamma\ddot\gamma)^2}/|\dot\gamma|^3 = 1/\rho(t)$. When p.by a.l, $\kappa(t) = |\ddot\gamma|$
2. This was $\tau^\ell_{\gamma(t)}$ above. Another way: take $x, y, z \in \mathbf{R}^3$ in gen pos,

$$r(x,y,z) = (\text{radius of unique circle through } x,y,z)$$

$$\lim_{y,z\to x} r(x,y,z) = \rho(x) = \tau^\ell_x \qquad \text{(easy to calc)}$$

$$\inf_{\substack{x\in C \\ x\neq y\neq z\neq x}} r(x,y,z) = \tau \qquad \text{(difficult to calc)}$$

3. Generalize this approach. Take $x_1, \ldots, x_{d+2}$ in gen pos (gives naturally copy of $\mathbf{R}^d$),

$$r(x_1, \ldots, x_{d+2}) = (\text{radius of unique } d\text{-sphere through } x_1, \ldots, x_{d+2})$$
$$\inf_{\substack{p \in M \\ x_i \text{ disctinct}}} r(p, x_1, \ldots, x_{d+1}) = \tau$$

4. Can we take limit for $\approx$ d-curvature? No $\to$ saddle point
   4.5. Possible sol: Take curvature of all smooth paths on M through point. Still not clear

0.3 How to find τ (mfld unknown)

Definition 0.3.1. Let $\epsilon > 0$. The *Vietoris–Rips* complex of X of radius ϵ is a simplicial complex V for which a k-tuple of points $\{x_1, \ldots, x_k\}$ defines a $(k-1)$-simplex in V iff $d(x_i, x_j) < \epsilon$ for all $1 \leqslant i, j \leqslant k$.

Assumptions:
1. $X = \{x_1, \ldots, x_N\}$ points sampled on unknown d-manifold M
2. Every simplex $V' \subset V$ is inside a d-simplex
   a. So resembles a d-manifold
   b. Needs appropriate choice of ϵ

Local cond num is min radius of d-spheres nearby:

$$\tau_p^\ell = \min_{\substack{x_{i_j} \in X \\ (p, x_{i_j}) \subset V}} r(p, x_{i_1}, \ldots, x_{i_{d+1}})$$

Global cond num is min radius over all d-spheres:

$$\tau = \min_{\substack{X' \subset X \\ |X'| = d+2}} r(X')$$

Easy to calculate. Equation of d-sphere through $x_1, \ldots, x_{d+2}$:

$$\det \begin{bmatrix} \sum_{i=1}^{d+1} x_i^2 & x_1 & x_2 & \cdots & x_{d+1} & 1 \\ \sum_{i=1}^{d+1} p_{1,i}^2 & p_{1,1} & p_{1,2} & \cdots & p_{1,d+1} & 1 \\ \sum_{i=1}^{d+1} p_{2,i}^2 & p_{2,1} & p_{2,2} & \cdots & p_{2,d+1} & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \sum_{i=1}^{d+1} p_{d+2,i}^2 & p_{d+2,1} & p_{d+2,2} & \cdots & p_{d+2,d+1} & 1 \end{bmatrix} = 0.$$

View x_i as lying in natural copy of $\mathbf{R}^{d+1} \subset \mathbf{R}^n$ they define.