① Higgs bundles:

bundle of k-differential forms

(operations on vector bundles)

holomorphic v.b. $\quad E \quad \mathrm{End}(E) \quad \Omega^k_M = \Gamma(\Lambda^k T^*M)$

$p \downarrow \quad \swarrow q \quad \swarrow r$

cpx manifold $\quad M \xleftarrow{\pi} \mathrm{End}(E) \otimes \Omega^1_M$

smooth global sections (maps $f: M \to \Lambda^k T^*M$ s.t. $r \circ f = \mathrm{id}_M$)

- also $f(M)$

(picture)

$q^{-1}(x) = \mathrm{End}(p^{-1}(x))$

$\pi^{-1}(x) \ni f \otimes g\,dz$ (if rank $E=1$, else matrix valued), $\quad f: \pi^{-1}(x) \to \pi^{-1}(x)$

$g: M \to \mathbb{C}$

z is a local coordinate

Def: A Higgs bundle is a holomorphic vector bundle $E \to M$ and a holomorphic section θ of $\mathrm{End}(E) \otimes \Omega^1_M$ s.t. $\theta \wedge \theta = 0$.

$\uparrow$ Higgs field

② Example (canonical bundle)

$K = \Omega^n_{M^n}$ is a 1-dimensional holomorphic vector bundle

Let M be a Riemann surface (1 dim cpx mani)

Let $E = K^{1/2} \oplus K^{-1/2}$ (dual) where $K = (K^{1/2})^{\otimes 2}$ $\longrightarrow$ How do we know $K^{1/2}$ exists?

constant fncs ± 1

SES: $0 \to \{\pm 1\} \to K^\times \xrightarrow{(\ )^2} K^\times \to 0$ $\quad$ sequence of sheaves on X

cocycles mod coboundaries

LES: $\cdots \to H^1(X, \{\pm 1\}) \to H^1(X, K^\times) \xrightarrow{(\ )^{\otimes 2}} H^1(X, K^\times) \to H^2(X, \{\pm 1\}) \to \cdots$

$H^1(X, K^\times) =$ all line bundles on X

What does a Higgs field look like? $\quad \mathrm{End}(E) \otimes K = \mathrm{Hom}(E,E) \otimes K$

$\cong E^* \otimes E \otimes K$

$\cong \mathrm{Hom}(E, E \otimes K)$

$$\Theta : E \longrightarrow E \otimes K$$

$$E = K^{1/2} \oplus K^{-1/2}, \qquad E \otimes K = (K^{1/2} \oplus K^{-1/2}) \otimes K = K^{3/2} \oplus K^{1/2}$$

$$\Theta_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{for } 1 \in \mathrm{Hom}(K^{1/2}, K^{1/2}) \cong K^{-1/2} \otimes K^{1/2} \cong \mathbb{C}$$

$$\Theta_1 \wedge \Theta_1 = 0$$

$$\Theta_2 = \begin{pmatrix} 0 & w \\ 1 & 0 \end{pmatrix} \quad \text{for } w \in \mathrm{Hom}(K^{-1/2}, K^{3/2}) \cong K^{1/2} \otimes K^{3/2} \cong K^{\otimes 2}$$

"quadratic differentials"

$\Theta_2 \wedge \Theta_2 = 0$ because on Riemann surface

③ Moduli space

$(H^1(L))^* \sim H^0(L^* K)$

Moduli of Higgs bundles and Hitchin map

$T_E \mathcal{N}(d,m) = H^1(\Sigma, \underbrace{End(E)}_{E^* E}) \quad \varphi \in H^0(\Sigma, End(E) \otimes K)$

Consider the problem of deforming a given vector bundle $E \to C$ (meaning that we deform its complex structure, keeping the complex structure on C fixed). The *tangent space* at E to the space of such deformations can be identified with $H^1(C, End(E))$. Assuming C is compact, by Serre duality the latter space is dual to $H^0(C, End(E) \otimes \Omega^1_C)$. Hence we can identify the *cotangent space* at E with the space of all Higgs fields on E. Let $Bun_{GL(n)}$ be the moduli space (in fact stack) of vector bundles over C. Varying E in the above consideration we can identify $T^*Bun_{GL(n)}$ with the moduli space (stack) $\mathcal{M}_{Higgs}$ of Higgs bundles consisting of pairs (E, Φ). The map h which assigns to a pair (E, Φ) the spectral curve of Φ is called *Hitchin map*. It can be thought of as assignment to (E, Φ) coefficients $Tr(\Phi^k), 1 \le k \le n$ of the characteristic polynomial. Hence the Hitchin map $h : \mathcal{M}_{Higgs} \to B$ is in fact a holomorphic map to a vector space B (although the vector structure is not canonical).